13

Quantizing the Abraham model

Classical theories must emerge from quantum mechanics and there is no reason to expect a simple recipe which would yield the physically correct quantum theory from the classical input. On the other hand, at least in the nonrelativistic domain, the rules of canonical quantization have served well and it is natural to apply them to the Abraham model. There is one immediate difficulty. Canonical quantization starts from identifying the canonical variables of the classical theory. Thus we first have to rewrite the equations of motion for the Abraham model in Hamiltonian form. For this purpose we adopt the Coulomb gauge, as usual, so as to eliminate the constraints. In the quantized version we thereby obtain the Pauli–Fierz Hamiltonian which has an obvious extension to include spin.

We have to ensure that the Pauli–Fierz Hamiltonian generates a unitary time evolution on the appropriate Hilbert space of physical states. Mathematically this means that we have to specify conditions under which the Pauli–Fierz Hamiltonian is a self-adjoint operator, an issue which can be satisfactorily resolved. Still, the true physical situation is more subtle and in fact not so well understood. It is related to the abundance of very low-energy photons, i.e the infrared problem, and to the arbitrariness of the cutoff at high energies, i.e. the ultraviolet problem. There are many items of interest before these, and it will take us a while to start discussing these subtleties.

Some words on our notation: In the beginning we keep c, $\hbar$, and later set them equal to one, mostly without notice. The vector notation, like $\boldsymbol{x}$, tends to be a little heavy, in particular since some of the objects become either operators or random variables. Therefore we stick with x, whose vector character has to be inferred from the context.

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13.1 Lagrangian and Hamiltonian rewriting of the Abraham model

We consider N charges coupled to the Maxwell field. Their motion is governed by (11.1), (11.2), which we repeat with the only difference that the relativistic kinetic energy is replaced by its Galilean cousin.

(Inhomogeneous Maxwell–Lorentz equations)

$$c^{-1}\partial_t B = -\nabla \times E\,, \quad c^{-1}\partial_t E = \nabla \times B - c^{-1} j\,, \tag{13.1}$$

$$\nabla \cdot E = \rho\,, \quad \nabla \cdot B = 0\,, \tag{13.2}$$

where the charge and current density are given by

$$\rho(x,t) = \sum_{j=1}^{N} e_j \varphi(x - q_j(t))\,, \quad j(x,t) = \sum_{j=1}^{N} e_j \varphi(x - q_j(t)) v_j(t) \tag{13.3}$$

satisfying charge conservation by fiat.

(Newton's equations of motion)

$$m_j \frac{\mathrm{d}}{\mathrm{d}t} v_j(t) = e_j \big(E_\varphi(q_j(t),t) + c^{-1} v_j(t) \times B_\varphi(q_j(t),t)\big)\,, \tag{13.4}$$

$j = 1, \ldots, N$. φ is the charge distribution. It satisfies Condition (C), Eq. (2.38).

The Lagrangian for a charge subject to external potentials is discussed in every text on classical mechanics. The Lagrangian of the coupled system, charges plus Maxwell field, can almost be guessed on that basis. We introduce the electromagnetic potentials through

$$E = -c^{-1}\partial_t A - \nabla\phi\,, \quad B = \nabla \times A\,, \tag{13.5}$$

hence guaranteeing $\nabla \cdot B = 0$ and the first half of (13.1), and regard as position-like variables $\{q_j,\ j = 1, \ldots, N, \phi(x), A(x), x \in \mathbb{R}^3\}$. Let us define the Lagrange density

$$\mathcal{L}_0(x) = \frac{1}{2}\big(E(x)^2 - B(x)^2\big) + c^{-1} j(x) \cdot A(x) - \rho(x)\phi(x)\,, \tag{13.6}$$

where, according to (13.3), ρ, j depend on the positions and velocities of the charges. The Lagrangian of the Abraham model is then

$$L = \sum_{j=1}^{N} \frac{1}{2} m_j \dot{q}_j^2 + \int \mathrm{d}^3 x \mathcal{L}_0(x)\,. \tag{13.7}$$

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We only have to verify that the Euler–Lagrange equations for the action obtained from L yield (13.1), (13.2), and (13.4). Indeed

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \tag{13.8}$$

are Newton's equations of motion. Using ‘˙’ for ∂_t as concise notation, variation with respect to ϕ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta L}{\delta \dot{\phi}} - \frac{\delta L}{\delta \phi} = 0\,, \tag{13.9}$$

which is equivalent to

$$-\nabla \cdot (c^{-1}\dot{A} + \nabla\phi) = \rho \tag{13.10}$$

and which we recognize as the first half of (13.2). Finally

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta L}{\delta \dot{A}} - \frac{\delta L}{\delta A} = 0 \tag{13.11}$$

amounts to

$$c^{-1}(c^{-1}\ddot{A} + \nabla\dot{\phi}) = -\nabla \times (\nabla \times A) + c^{-1} j\,, \tag{13.12}$$

which is nothing but the second half of (13.1).

Since (13.9) represents only a constraint and is not an equation of motion, clearly we are using a redundant set of dynamical variables. Let us do the counting. We split the electromagnetic fields into longitudinal and transverse components,

$$E = E_\parallel + E_\perp\,, \quad B = B_\parallel + B_\perp\,. \tag{13.13}$$

Since $\nabla \cdot B = 0$, we have $B_\parallel = 0$. From $\nabla \cdot E = \rho$ we conclude

$$\widehat{E}_\parallel = -\mathrm{i}\widehat{\rho} k/k^2\,. \tag{13.14}$$

$E_\perp$ and $B_\perp$ satisfy a first-order evolution equation. Thus, in the sense of Lagrangian mechanics, there are two independent field degrees of freedom at every space point, while in (13.6) we employed four degrees of freedom.

We first eliminate ϕ through (13.10), i.e.

$$\widehat{\phi} = \frac{1}{k^2}\big(\mathrm{i}k \cdot c^{-1}\hat{\dot{A}}_\parallel + \widehat{\rho}\big)\,. \tag{13.15}$$

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Then, using Fourier transforms and Parseval's identity, (13.7) transforms to

$$L = \sum_{j=1}^{N} \frac{1}{2} m_j \dot{q}_j^2 + \frac{1}{2} \int \mathrm{d}^3k \big[c^{-2} \dot{\widehat{A}}_\perp^* \cdot \dot{\widehat{A}}_\perp + k^{-2} \widehat{\rho}^* \widehat{\rho} - (k \times \widehat{A}_\perp^*) \cdot (k \times \widehat{A}_\perp) \big]$$
$$+ \frac{1}{2} \int \mathrm{d}^3k \big[c^{-1} \widehat{j}^* \cdot \widehat{A} + c^{-1} \widehat{j} \cdot \widehat{A}^* - 2k^{-2} \widehat{\rho}^* \widehat{\rho}$$
$$- \mathrm{i} c^{-1} k^{-2} \big(\widehat{\rho}^* k \cdot \dot{\widehat{A}}_\parallel - \widehat{\rho} k \cdot \dot{\widehat{A}}_\parallel^* \big) \big] . \tag{13.16}$$

The term $\widehat{\rho}^* \widehat{\rho}$ depends only on the q_j's and is recognized as the Coulomb potential,

$$\frac{1}{2} \int \mathrm{d}^3k k^{-2} \widehat{\rho}^* \widehat{\rho} = \frac{1}{2} \sum_{i,j=1}^{N} e_i e_j \int \mathrm{d}^3y \mathrm{d}^3y' \varphi(y) (4\pi |q_i - q_j - y + y'|)^{-1} \varphi(y')$$
$$= V_{\varphi\mathrm{coul}}(q_1, \ldots, q_N) . \tag{13.17}$$

The Coulomb potential is smeared by φ, which as before is indicated by the subscript. φ appears twice, since both the i-th and the j-th particle carry a charge distribution. To simplify the last term of (13.16) we use the conservation law $\dot{\widehat{\rho}} + \mathrm{i} k \cdot \widehat{j} = 0$. Then

$$L = \sum_{j=1}^{N} \frac{1}{2} m_j \dot{q}_j^2 - V_{\varphi\mathrm{coul}} + \frac{1}{2} \int \mathrm{d}^3k \big[c^{-2} \dot{\widehat{A}}_\perp^* \cdot \dot{\widehat{A}}_\perp - k^2 \widehat{A}_\perp^* \cdot \widehat{A}_\perp \big]$$
$$+ \frac{1}{2} \int \mathrm{d}^3k \big[c^{-1} \widehat{j}^* \cdot \widehat{A}_\perp + c^{-1} \widehat{j} \cdot \widehat{A}_\perp^* \big]$$
$$+ c^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \Big(\frac{1}{2} \int \mathrm{d}^3k |k|^{-1} \mathrm{i} \big[\widehat{\rho} \widehat{A}_\parallel^* - \widehat{\rho}^* \widehat{A}_\parallel \big] \Big) . \tag{13.18}$$

Since $A_\parallel$ appears only inside a total time derivative, we have identified $A_\parallel$ as the second redundant field. To drop the redundant degrees of freedom, the simplest choice is to set $A_\parallel = 0$ by exploiting the gauge freedom, which means selecting the *Coulomb gauge* defined by

$$\nabla \cdot A = 0 . \tag{13.19}$$

The vector potential is purely transverse and we henceforth drop the subscript $\perp$. Transforming back to real space, the Lagrangian of the Abraham model reads

$$L = \sum_{j=1}^{N} \frac{1}{2} m_j \dot{q}_j^2 - V_{\varphi\mathrm{coul}} + \int \mathrm{d}^3x \mathcal{L}(x) \tag{13.20}$$

with the Lagrange density

$$\mathcal{L} = \frac{1}{2} \big[(c^{-1} \dot{A})^2 - (\nabla \times A)^2 \big] + c^{-1} j \cdot A . \tag{13.21}$$

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The transverse vector field $A(x)$, $x \in \mathbb{R}^3$, should be regarded as position-like variables.

The step from Lagrange to Hamilton is standard. One introduces the momentum p_j canonically conjugate to q_j by

$$p_j = m_j \dot{q}_j + c^{-1} e_j A_\varphi(q_j)\,. \tag{13.22}$$

For the momentum field canonically conjugate to A we obtain

$$\frac{\delta L}{\delta \dot{A}} = c^{-2}\dot{A} = -c^{-1}E_\perp\,. \tag{13.23}$$

Then the Hamiltonian corresponding to L reads

$$\begin{aligned} H = &\sum_{j=1}^{N} \frac{1}{2m_j}\big(p_j - c^{-1}e_j A_\varphi(q_j)\big)^2 + V_{\varphi\text{coul}} \\ &+ \frac{1}{2}\int \mathrm{d}^3x \big[E_\perp(x)^2 + (\nabla \times A(x))^2\big] \end{aligned} \tag{13.24}$$

with the canonically conjugate pairs q_j, p_j and $A(x)$, $-c^{-1}E_\perp(x)$.

13.2 The Pauli–Fierz Hamiltonian

In the form (13.24) we are ready to apply the rules of canonical quantization. The position and momentum of the j-th particle are elevated to algebraic objects (linear operators) which satisfy the commutation relations

$$[q_{i\alpha},\, p_{j\beta}] = \mathrm{i}\hbar\delta_{\alpha\beta}\delta_{ij}\,, \tag{13.25}$$

$\alpha, \beta = 1, 2, 3$, $i, j = 1, \ldots, N$. In the Schrödinger representation, which will be used throughout, the Hilbert space of wave functions is

$$\mathcal{H}_\mathrm{p} = L^2(\mathbb{R}^{3N})\,, \tag{13.26}$$

restricted to either the symmetric or antisymmetric subspace depending on whether the particles are bosons or fermions. Positions and momenta become

$$q_j = x_j\,, \quad p_j = -\mathrm{i}\hbar\nabla_{x_j} \tag{13.27}$$

as linear operators on $\mathcal{H}_\mathrm{p}$, i.e. if $\psi(x_1, \ldots, x_N) \in L^2(\mathbb{R}^{3N})$ is the wave function for the particles, then $q_j\psi(x_1, \ldots, x_N) = x_j\psi(x_1, \ldots, x_N)$, $p_j\psi(x_1, \ldots, x_N) = -\mathrm{i}\hbar\nabla_{x_j}\psi(x_1, \ldots, x_N)$.

For the fields $A(x)$, $-c^{-1}E_\perp(x)$ one is tempted to postulate commutation relations analogous to (13.25). The difficulty is that the quantization has to satisfy the transversality constraint (13.19) which is nonlocal. Fortunately it is linear and it

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becomes local in Fourier space as $k \cdot \widehat{A} = 0$. We thus introduce at each $k \in \mathbb{R}^3$ the standard dreibein

$$\widehat{k} = k/|k|\,, \quad e_1(k)\,, \quad e_2(k)\,, \tag{13.28}$$

which satisfies $\widehat{k} \cdot e_i(k) = 0$, $i = 1, 2$, $e_1(k) \cdot e_2(k) = 0$. There is some freedom of how to choose e_1, e_2, but the transverse projection $Q^{\perp}(k) = 1 - \widehat{k} \otimes \widehat{k} = 1 - |k|^{-2}|k\rangle\langle k|$ is unique. The two transverse components $e_1(k) \cdot \widehat{A}(k)$, $e_2(k) \cdot \widehat{A}(k)$ are regarded as independent variables, correspondingly for $-c^{-1}\widehat{E}_{\perp}$. Since A is real, we have $\widehat{A}(k)^* = \widehat{A}(-k)$. Therefore one has to restrict k to a half-space and take the real and imaginary parts of $\widehat{A}(k)$ as independent variables which are subject to the rules of canonical quantization. To achieve this goal it is helpful to introduce two standard Bose fields with creation and annihilation operators

$$a^*(k, \lambda)\,, \quad a(k, \lambda)\,, \quad k \in \mathbb{R}^3, \quad \lambda = 1, 2\,, \tag{13.29}$$

satisfying the canonical commutation relations

$$\begin{aligned} [a(k, \lambda), a^*(k', \lambda')] &= \delta_{\lambda\lambda'}\delta(k - k')\,, \\ [a(k, \lambda), a(k', \lambda')] = 0\,, \quad [a^*(k, \lambda), a^*(k', \lambda')] &= 0\,. \end{aligned} \tag{13.30}$$

◇ For a linear operator A, the adjoint operator is denoted by A^*. ◇

In terms of these Bose fields we set

$$\widehat{A}(k) = \sum_{\lambda=1,2} c\sqrt{\hbar/2\omega}\big(e_\lambda(k)a(k, \lambda) + e_\lambda(-k)a^*(-k, \lambda)\big)\,, \tag{13.31}$$

$$\widehat{E}_{\perp}(k) = \sum_{\lambda=1,2} \sqrt{\hbar\omega/2}\big(\mathrm{i}e_\lambda(k)a(k, \lambda) - \mathrm{i}e_\lambda(-k)a^*(-k, \lambda)\big) \tag{13.32}$$

with

$$\omega(k) = c|k|\,. \tag{13.33}$$

Then indeed $\widehat{A}$, $\widehat{E}_{\perp}$ are transverse, $\widehat{A}(k)^* = \widehat{A}(-k)$, $\widehat{E}(k)^* = \widehat{E}(-k)$, and

$$[e_\lambda(k) \cdot \widehat{A}(k), -c^{-1}e_{\lambda'}(k') \cdot \widehat{E}_{\perp}(k')^*] = \mathrm{i}\hbar\delta_{\lambda\lambda'}\delta(k - k') \tag{13.34}$$

which should be understood in analogy to (13.25).

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In physical space (13.31), (13.32) become

$$A(x) = \sum_{\lambda=1,2} \int \mathrm{d}^3k c\sqrt{\hbar/2\omega}\, e_\lambda(k)(2\pi)^{-3/2}\left(\mathrm{e}^{\mathrm{i}k\cdot x}a(k,\lambda) + \mathrm{e}^{-\mathrm{i}k\cdot x}a^*(k,\lambda)\right), \tag{13.35}$$

$$E_\perp(x) = \sum_{\lambda=1,2} \int \mathrm{d}^3k \sqrt{\hbar\omega/2}\, e_\lambda(k)(2\pi)^{-3/2}\mathrm{i}\left(\mathrm{e}^{\mathrm{i}k\cdot x}a(k,\lambda) - \mathrm{e}^{-\mathrm{i}k\cdot x}a^*(k,\lambda)\right). \tag{13.36}$$

Clearly $A^*(x) = A(x)$, $E_\perp^*(x) = E_\perp(x)$. The commutator (13.34) translates into

$$[A_\alpha(x), -c^{-1}E_{\perp\beta}(x')] = \mathrm{i}\hbar\delta^\perp_{\alpha\beta}(x - x') \tag{13.37}$$

with the transverse delta function

$$\delta^\perp_{\alpha\beta}(x) = (2\pi)^{-3}\int \mathrm{d}^3k \mathrm{e}^{\mathrm{i}k\cdot x}(\delta_{\alpha\beta} - \widehat{k}_\alpha\widehat{k}_\beta) = \frac{2}{3}\delta_{\alpha\beta}\delta(x) - \frac{1}{4\pi|x|^3}(\delta_{\alpha\beta} - 3\widehat{x}_\alpha\widehat{x}_\beta), \tag{13.38}$$

where $\widehat{x}_\alpha = x_\alpha/|x|$.

At this point we have left the classical world. $A(x)$, $E_\perp(x)$ and their Fourier transforms $\widehat{A}(k)$, $\widehat{E}_\perp(k)$ will now always stand for operator-valued fields. In the atomic and solid state physics literature by tradition one uses $a^\dagger$ as the boson creation operator adjoint to the annihilation operator a. We try to avoid such a profileration of symbols.

Next on the agenda should be the Fock representation of the Bose fields $a(k,\lambda)$ and the definition of $A(x)$, $E_\perp(x)$ as operator-valued fields acting on Fock space. But let us keep this for the beginning of the next section and proceed immediately to our goal, namely the Hamiltonian of the quantized Abraham model. All we have to do is to insert (13.35), (13.36) into the classical Hamiltonian. This results, after omitting the zero-point energy of photons, in the (*spinless*) *Pauli–Fierz Hamiltonian*

$$H = \sum_{j=1}^{N} \frac{1}{2m_j}\left(p_j - c^{-1}e_j A_\varphi(q_j)\right)^2 + V_{\varphi\mathrm{coul}} + H_\mathrm{f} \tag{13.39}$$

with the field Hamiltonian

$$H_\mathrm{f} = \sum_{\lambda=1,2} \int \mathrm{d}^3k \hbar\omega(k) a^*(k,\lambda)a(k,\lambda). \tag{13.40}$$

There is no ambiguity in the operator ordering, since $p_j \cdot A_\varphi(q_j) = A_\varphi(q_j)\cdot p_j$ by the transversality condition (13.19). We recall that the spherically symmetric

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form factor $\widehat{\varphi}$ cuts couplings to the field, more explicitly

$$A_\varphi(q) = \sum_{\lambda=1,2} \int \mathrm{d}^3k c\sqrt{\hbar/2\omega}\, e_\lambda(k)\big(\widehat{\varphi}(k)\mathrm{e}^{\mathrm{i}k\cdot q}a(k,\lambda) + \widehat{\varphi}^*(k)\mathrm{e}^{-\mathrm{i}k\cdot q}a^*(k,\lambda)\big)\,. \tag{13.41}$$

To simplify notation, $\widehat{\varphi}$ will be assumed to be real, which can always be achieved through a suitable canonical transformation of the form $a(k,\lambda) \to \mathrm{e}^{\mathrm{i}\theta(k)}a(k,\lambda)$.

Two immediate generalizations are noted. First of all it is convenient to add external potentials ϕ_{ex}, A_{ex}, where the abbreviation $e\phi_{\mathrm{ex}}(x) = V(x)$ will be employed frequently. This should be thought of as a limiting case of (13.39): we imagine that some charges are nailed down by letting their masses $m \to \infty$; then their kinetic term in (13.39) disappears and $V_{\varphi\mathrm{coul}}$ splits into an external potential plus an interaction potential for the movable charges. Similarly one can produce an external current which then generates A_{ex}. Thus the external potentials are not quantized and are added into the Hamiltonian as in the classical theory which yields

$$\begin{aligned} H = &\sum_{j=1}^{N} \frac{1}{2m_j}\big(p_j - c^{-1}e_jA_\varphi(q_j) - c^{-1}e_jA_{\mathrm{ex}}(q_j)\big)^2 \\ &+ V_{\varphi\mathrm{coul}} + \sum_{j=1}^{N} e_j\phi_{\mathrm{ex}}(q_j) + H_{\mathrm{f}}\,. \end{aligned} \tag{13.42}$$

Secondly, particles have spin. Of course, an electron has spin $\frac{1}{2}$. In our approximation nuclei are modeled as structureless particles carrying a nuclear spin, ranging from 0 to 9/2 according to experimental evidence. The classical theory is now of little help. The natural guess is to include spin as in the nonrelativistic one-particle Schrödinger theory. For a single electron in infinite space, no external potentials, the Hamiltonian then becomes

$$H = \frac{1}{2m}\big(\sigma\cdot(p - c^{-1}e_jA_\varphi(q))\big)^2 + H_{\mathrm{f}}\,, \tag{13.43}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli spin-$\frac{1}{2}$ matrices. If necessary, one could include higher terms in (13.43) as they emerge from the Foldy–Wouthuysen expansion of the Dirac equation.

Having introduced the Pauli–Fierz Hamiltonian as the major player of the quantum part of the treatise, we pause for a while with a few general remarks.

Zero-point energy. In the Pauli–Fierz Hamiltonian we have omitted the zero-point energy $\int \mathrm{d}^3k\hbar\omega$, which is infinite. The Heisenberg equations of motion remain unaltered by this reset in the zero of energy. However, one has to be careful. If one wants to compute the change in energy of the quantized Maxwell field through the

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insertion of a pair of perfectly conducting plates, then in this energy difference the zero-point energy has to be properly handled; compare with section 13.6. A further change in the zero of the energy scale comes from the Coulomb self-interaction, namely the diagonal part

$$\frac{1}{2}\sum_{j=1}^{N} e_j^2 \int \mathrm{d}^3k\, |\widehat{\varphi}(k)|^2 k^{-2} \tag{13.44}$$

in the sum (13.17), which is finite only because the form factor cuts off the high-frequency modes.

Range of validity, limiting cases. The claimed range of validity of the Pauli–Fierz Hamiltonian is flabbergasting. To be sure, on the high-energy side, nuclear physics and high-energy physics are omitted. On the long-distance side, we could phenomenologically include gravity on the Newtonian level, but anything beyond that is ignored. As the bold claim goes, any physical phenomenon in between, including life on Earth, is accurately described through the Pauli–Fierz Hamiltonian (13.39) (and a suitably chosen initial wave function). There have been speculations that quantum mechanics is modified roughly at the 10^{-5} m scale. But so far there seems to be no evidence in this direction. On the contrary, whenever a detailed comparison with the theory can be made, it reassuringly seems to work well. Of course, our trust is not based on strict mathematical deductions from the Pauli–Fierz Hamiltonian. This is too difficult a program. Our confidence comes from well-studied limit cases. In the static limit we imagine turning off the interaction to the quantized part of the Maxwell field. This clearly results in Schrödinger particles interacting through a purely Coulombic potential, for which many predictions are accessible to experimental verification. But beware, even there apparently simple questions remain to be better understood. For example, the size of atoms as we see them in nature remains mysterious if only the Coulomb interaction and the Pauli exclusion principle are allowed. Another limiting case is a region completely free of charges. At standard field strengths there are sufficiently many photons per unit volume for the predictions from the quantized Maxwell field to match with the ones of the classical Maxwell field. As will be discussed, radiation phenomena are well grasped by the Pauli–Fierz Hamiltonian. These and many other limiting cases are the reason for regarding (13.39) as an accurate description of low-energy phenomena.

Model parameters, renormalization. If we focus our attention on (13.43), there are four model parameters: the mass m, the charge e, the gyromagnetic ratio $g = 2$, and the form factor $\widehat{\varphi}$. c and $\hbar$, which also appear, are constants of nature. As discussed at length for the classical theory, what is observed experimentally is always the compound object consisting of the particle and its photon cloud. Thus m, e, g

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have to be regarded as bare parameters and their observed value must be computed from the theory. The bare values are renormalized through the interaction with the Maxwell field. As will be shown below, the charge e is not renormalized, since there is no vacuum polarization. One way to argue is to imagine two charged particles with a very large mass separated by a distance R. According to (13.39) their mutual force is then $e_1 e_2/4\pi R^2$ with the *bare* charges e_1, e_2. Further support is the response of a particle to slowly varying external potentials. In this adiabatic limit, e enters in the effective equation with its bare value while m and g are renormalized. The Pauli–Fierz model is not in a position to predict the experimental value of the mass, since the bare mass is unaccessible, in principle. The renormalized (effective) mass has to be given as an empirical input, to which the bare mass is correspondingly adjusted. On the other hand, the dimensionless gyromagnetic ratio g is a definite (though empirically slightly inaccurate) prediction of the theory; compare with sections 16.6 and 19.3.5. Perhaps the most unwanted feature of the Pauli–Fierz Hamiltonian is the form factor $\widehat{\varphi}$. The pragmatic attitude is to choose φ with some taste. On the classical level we concluded that the form factor cannot be removed. In the limit $\varphi(x) \to \delta(x)$ the particle-like objects become infinitely heavy. The simple structure of the energy–momentum relation (4.11) does not allow for compensation, since in a stable theory the bare mass has to be positive. The quantum theory has a richer structure and it seems that one can carry out the limit $\varphi(x) \to \delta(x)$ and at the same time take $m \to 0$ such that the observed mass remains fixed. We will come back to this point in due course.

The quest for a closed physical theory. We have commented on this point already. But let us expand on it in the present context. The static limit of the Pauli–Fierz Hamiltonian, i.e. Schrödinger particles interacting through the static Coulomb potential, is a closed theory for electrons and nuclei. The Hamiltonian is a self-adjoint linear operator and generates a unitary time evolution. This is also the case for the quantized Maxwell field without charges. Of course, this does not mean that we have solved any physical problem. It just assures us of a definite mathematical framework within which consequences can be explored. One would hope to have such a secure foundation also for the Pauli–Fierz model and it remains to be seen how much of this program can be realized.

We still have to complete the story of the Pauli–Fierz model. One defines the time-evolved linear operator $A(t)$ through

$$A(t) = \mathrm{e}^{\mathrm{i}Ht/\hbar} A \mathrm{e}^{-\mathrm{i}Ht/\hbar} \tag{13.45}$$

in the Heisenberg picture. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t) = \frac{\mathrm{i}}{\hbar}[H, \mathrm{e}^{\mathrm{i}Ht/\hbar} A \mathrm{e}^{-\mathrm{i}Ht/\hbar}] = \frac{\mathrm{i}}{\hbar}[H, A](t)\,. \tag{13.46}$$

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On this level, so to speak, as a control of the quantization prescription, we use the commutation relations (13.25) and (13.37) to verify that the operator-valued fields indeed satisfy the Maxwell equations and that the particles satisfy Newton's equations of motion. Computing the commutators as in (13.46) one obtains

$$\begin{aligned} &c^{-1}\partial_t B = -\nabla \times E\,, \quad c^{-1}\partial_t E = \nabla \times B - c^{-1} j\,, \\ &\nabla \cdot E = \rho\,, \quad \nabla \cdot B = 0\,, \end{aligned} \tag{13.47}$$

where now

$$\begin{aligned} \rho(x,t) &= \sum_{j=1}^{N} e_j \varphi(x - q_j(t))\,, \\ j(x,t) &= \sum_{j=1}^{N} \frac{1}{2} e_j \big(v_j(t)\varphi(x - q_j(t)) + \varphi(x - q_j(t))v_j(t)\big) \end{aligned} \tag{13.48}$$

with the velocity operator

$$v_j = \big(p_j - c^{-1} e_j A_\varphi(q_j)\big)/m_j\,. \tag{13.49}$$

Similarly, one obtains the symmetrized Lorentz force as

$$m_j \dot{v}_j(t) = e_j \Big(E_\varphi(q_j(t),t) + \frac{1}{2c}\big(v_j(t) \times B_\varphi(q_j(t),t) - B_\varphi(q_j(t),t) \times v_j(t)\big)\Big)\,. \tag{13.50}$$

If there are external fields, E_φ, B_φ is to be replaced by $E_\varphi + E_{\mathrm{ex}}$, $B_\varphi + B_{\mathrm{ex}}$. In (13.47)–(13.49), $q_j(t)$, $p_j(t)$, respectively $A(t)$, $-c^{-1}E_\perp(t)$, are operators satisfying the commutation relations (13.25), respectively (13.37), at all times.

Also of interest is to record the Heisenberg equations of motion for the Pauli–Fierz Hamiltonian (13.43) including spin. The Maxwell equations are as before. However, in the case of a single charged particle, the current density is now

$$j(x) = \frac{1}{2} e\big(v\varphi(x - q) + \varphi(x - q)v\big) + \frac{e\hbar}{2m}\sigma \times \nabla_q \varphi(x - q) \tag{13.51}$$

with the velocity operator $v = (p - c^{-1}eA_\varphi(q))/m$. The Schrödinger equation reads

$$m\ddot{q} = e\Big(E_\varphi(q,t) + \frac{1}{2c}\big(v \times B_\varphi(q,t) - B_\varphi(q,t) \times v\big)\Big) + \frac{e\hbar}{2mc}\sigma \cdot \nabla_q B_\varphi(q,t)\,, \tag{13.52}$$

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consistent with the general rule that the magnetic force equals $c^{-1}\int \mathrm{d}^3x\, j(x)\times B(x)$, and the Pauli equation for the spin reads

$$\dot{\sigma} = -\frac{e}{mc} B_\varphi(q,t)\times\sigma\,. \tag{13.53}$$

If one compares (13.52), (13.53) with the classical equations of motion of a spinning charge, cf. section 10.2, then one observes that in quantum mechanics the spin degrees of freedom couple somewhat differently to the Maxwell field than the classical internal angular momentum. Since φ is radial, in fact $\nabla_x\varphi_{\mathrm{r}}(|x|) = \varphi_{\mathrm{r}}'(|x|)\widehat{x}$ and the spin part of the current (13.51) has the effective charge distribution $\varphi_{\mathrm{r}}'(|x|)/|x|$. However, the evolution equation for σ has only superficial similarity with Eq. (10.20) for ω.

13.3 Fock space, self-adjointness

To define the Pauli–Fierz Hamiltonian as a linear operator, one has to introduce a suitable Hilbert space of wave functions. Provisionally we assume that the number of photons, either virtual or real in the usual parlance, is finite, though necessarily arbitrary, since H does not conserve the number of photons. This means that we will have to work in the Fock representation of the Bose fields $a(k,\lambda)$. We introduce the one-particle Hilbert space

$$\mathfrak{h} = L^2(\mathbb{R}^3)\otimes\mathbb{C}^2\,. \tag{13.54}$$

$\mathfrak{h}$ consists of wave functions $\psi(k,\lambda)$, with the photon wave number $k\in\mathbb{R}^3$ and the helicity $\lambda = 1,2$. The inner product in $\mathfrak{h}$ is $\langle\varphi,\psi\rangle_{\mathfrak{h}} = \sum_{\lambda=1,2}\int \mathrm{d}^3k\,\varphi^*(k,\lambda)\psi(k,\lambda)$. Out of $\mathfrak{h}$ we construct the Fock space $\mathcal{F}$ in the usual way

$$\mathcal{F} = \bigoplus_{n=0}^{\infty}\left(\mathfrak{h}^{\otimes n}\right)_{\mathrm{sym}}\,, \tag{13.55}$$

where $\mathfrak{h}^{\otimes n}$ denotes the n-fold tensor product and where "sym" means that we restrict to the subspace of wave functions symmetric under interchange of labels, i.e.

$$\psi_n(k_1,\lambda_1,\ldots,k_n,\lambda_n) = \psi_n(k_{\pi(1)},\lambda_{\pi(1)},\ldots,k_{\pi(n)},\lambda_{\pi(n)}) \tag{13.56}$$

for an arbitrary permutation π. By definition an element $\psi\in\mathcal{F}$ is of the form $(\psi_0,\psi_1,\ldots)$ and

$$\langle\varphi,\psi\rangle_{\mathcal{F}} = \sum_{n=0}^{\infty}\langle\varphi_n,\psi_n\rangle_{\mathfrak{h}^{\otimes n}}\,. \tag{13.57}$$

The Fock vacuum, ψ_0, will be denoted by Ω.

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◇ As the reader will have noticed, for the inner product in a Hilbert space we use the notation $\langle \varphi, \psi \rangle$, which is linear in the second and antilinear in the first argument. The standard physics notation would be the Dirac bracket $\langle \varphi | \psi \rangle$, which is also linear in the second argument. (A further widespread convention is a scalar product linear in the first argument.) The subscript in $\langle \varphi, \psi \rangle_{\mathcal{F}}$ is used to indicate the Hilbert space under consideration: it will be omitted if it is obvious from the context. The length of a vector is $\|\psi\| = \langle \psi, \psi \rangle^{1/2}$. ◇

For $f \in \mathfrak{h}$ one defines the smeared creation and annihilation operators

$$a(f) = \sum_{\lambda=1,2} \int \mathrm{d}^3k f^*(k,\lambda) a(k,\lambda)\,, \quad a^*(f) = \sum_{\lambda=1,2} \int \mathrm{d}^3k f(k,\lambda) a^*(k,\lambda)\,. \tag{13.58}$$

As operators in $\mathcal{F}$ they act through

$$\begin{aligned}(a(f)\psi)_n(k_1,\lambda_1,\ldots,k_n,\lambda_n) &= \sqrt{n+1} \sum_{\lambda=1,2} \int \mathrm{d}^3k f^*(k,\lambda) \\ &\quad \times \psi_{n+1}(k_1,\lambda_1,\ldots,k_n,\lambda_n,k,\lambda)\,, \qquad (13.59)\\ (a^*(f)\psi)_n(k_1,\lambda_1,\ldots,k_n,\lambda_n) &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(k_j,\lambda_j) \\ &\quad \times \psi_{n-1}(k_1,\lambda_1,\ldots,\widetilde{k}_j,\widetilde{\lambda}_j,\ldots,k_n,\lambda_n)\,, \qquad (13.60)\end{aligned}$$

where $\widetilde{\ }$ means that this variable is to be omitted. The field Hamiltonian

$$H_{\mathrm{f}} = \sum_{\lambda=1,2} \int \mathrm{d}^3k \hbar\omega(k) a^*(k,\lambda) a(k,\lambda) \tag{13.61}$$

acts as multiplication by $\sum_{j=1}^{n} \hbar\omega(k_j)$ on the n-particle subspace $\mathfrak{h}^{\otimes n}$. With all these definitions we see that the Pauli–Fierz Hamiltonian operates on the Hilbert space

$$\mathcal{H} = \mathcal{H}_{\mathrm{p}} \otimes \mathcal{H}_{\mathrm{f}} \tag{13.62}$$

with $\mathcal{H}_{\mathrm{p}} = L^2(\mathbb{R}^{3N})$ and $\mathcal{H}_{\mathrm{f}} = \mathcal{F}$. Physically the particle Hilbert space $\mathcal{H}_{\mathrm{p}}$ is too large, since in nature only symmetric, respectively antisymmetric, wave functions are realized. Still mathematically it is convenient to work with all of $L^2(\mathbb{R}^{3N})$.

In any dynamical theory, usually the first step is to establish the existence of solutions of the evolution equations. In our case this means to prove that H is a self-adjoint operator on a suitable domain of functions, where for concreteness we consider the Pauli–Fierz operator of (13.43) for a single electron. If not even the self-adjointness question can be resolved, there is little hope of rigorously handling qualitative properties of interest.

We observe that $\langle\psi, H\psi\rangle_{\mathcal{H}} \geq 0$, clearly. This means that H has equal defect indices and therefore at least one self-adjoint extension. Amongst those there is a distinguished extension, called the Friedrichs extension, which is obtained through the closure of the quadratic form $\langle\psi, H\psi\rangle_{\mathcal{H}}$ with smooth wave functions of a finite number of photons. The Friedrichs extension gives no information on the domain of self-adjointness and, in principle, there could be other extensions. A more concrete approach is to prove that, for the purpose of the existence of dynamics, the interaction can be regarded as small. We decompose H as

$$\begin{aligned} H &= H_0 + H_1 \\ &= \frac{1}{2m}p^2 + H_{\mathrm{f}} - \frac{e}{2mc}\big(p \cdot A_\varphi(x) + A_\varphi(x) \cdot p\big) \\ &\quad + \frac{e^2}{2mc^2}A_\varphi(x)^2 - \frac{e\hbar}{2mc}\sigma \cdot B_\varphi(x)\,, \end{aligned} \tag{13.63}$$

$B_\varphi(x) = \nabla \times A_\varphi(x)$, $p_1 = p$ for the momentum, and $q_1 = x$ for the position of the particle, and want to prove that H_1 is small compared to H_0, $H_0 = (p^2/2m) + H_{\mathrm{f}}$.

Abstractly one uses the Kato–Rellich theorem. We consider the densely defined linear operators A, B on a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$ and suppose that

(i) for the domains $D(B) \supset D(A)$,

(ii) for some constants a, b and all $\psi \in D(A)$

$$\|B\psi\| \leq a\|A\psi\| + b\|\psi\|\,. \tag{13.64}$$

Then B is said to be A-bounded. The smallest a is called the relative bound. Usually a can be made smaller at the expense of b.

Theorem 13.1 (Kato–Rellich theorem). *Suppose A is self-adjoint, B is symmetric, and B is A-bounded with relative bound $a < 1$. Then $A + B$ is self-adjoint on $D(A)$ and essentially self-adjoint on any core of A.*

For multiparticle Schrödinger operators of the form $-\frac{1}{2}\Delta + V$ the Kato–Rellich theorem is a standard technique and yields the existence of dynamics for a very large class of potentials V including the Coulomb potential. For the Pauli–Fierz operator the form version of Theorem 13.1 is more convenient.

Theorem 13.2 (KLMN theorem). *Let A be a positive self-adjoint operator. Let $\beta(\psi, \varphi) = \langle\psi, B\varphi\rangle$ be a symmetric quadratic form defined for all $\psi, \varphi \in D(A^{1/2})$ such that for some constants $a < 1$, $b < \infty$*

$$|\langle\psi, B\psi\rangle| \leq a\langle\psi, A\psi\rangle + b\langle\psi, \psi\rangle \tag{13.65}$$

for all $\psi \in D(A^{1/2})$. Then there exists a unique self-adjoint operator C with

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$D(C) \subset D(A^{1/2})$ *such that*

$$\langle \psi, C\psi \rangle = \langle \psi, A\psi \rangle + \langle \psi, B\psi \rangle. \tag{13.66}$$

Moreover, C is bounded from below by $-b$.

Let us see how the KLMN theorem works in the case of the Pauli–Fierz Hamiltonian H, which means that one has to establish

$$|\langle \psi, H_1\psi \rangle_{\mathcal{H}}| \leq a\langle \psi, H_0\psi \rangle_{\mathcal{H}} + b\langle \psi, \psi \rangle_{\mathcal{H}} \tag{13.67}$$

with $a < 1$. We set $\hbar = c = m = 1$ and, following the convention (13.58), put

$$A_\varphi(x) = a(f_x) + a^*(f_x) \tag{13.68}$$

with

$$f_x(k, \lambda) = \widehat{\varphi}(k)\sqrt{1/2\omega}\, e_\lambda(k)\mathrm{e}^{-\mathrm{i}k\cdot x}\,. \tag{13.69}$$

The creation and annihilation operators are bounded through $(H_\mathrm{f})^{1/2}$ as

$$\begin{aligned} \|a^*(f)\psi\|_{\mathcal{F}} &\leq \|f/\sqrt{\omega}\|_{\mathfrak{h}}\|(H_\mathrm{f})^{1/2}\psi\|_{\mathcal{F}} + \|f\|_{\mathfrak{h}}\|\psi\|_{\mathcal{F}}\,, \\ \|a(f)\psi\|_{\mathcal{F}} &\leq \|f/\sqrt{\omega}\|_{\mathfrak{h}}\|(H_\mathrm{f})^{1/2}\psi\|_{\mathcal{F}} \end{aligned} \tag{13.70}$$

and by the Schwarz inequality

$$\begin{aligned} |\langle \psi, (a(f) + a^*(f))^2\psi \rangle_{\mathcal{F}}| &\leq 2\langle \psi, a^*(f)a(f)\psi \rangle_{\mathcal{F}} + \|f\|_{\mathfrak{h}}^2\|\psi\|_{\mathcal{F}}^2 \\ &\quad + 2|\langle \psi, a^*(f)a^*(f)\psi \rangle_{\mathcal{F}}| \\ &\leq 5\|f/\sqrt{\omega}\|_{\mathfrak{h}}^2\langle \psi, H_\mathrm{f}\psi \rangle_{\mathcal{F}} + 3\|f\|_{\mathfrak{h}}^2\|\psi\|_{\mathcal{F}}^2\,. \end{aligned} \tag{13.71}$$

Therefore the A_φ^2-term has a relative bound less than 1 only if e is sufficiently small.

We do not attempt to optimize the constants and thus write

$$|\langle \psi, p \cdot A_\varphi(x)\psi \rangle_{\mathcal{H}}| \;\leq\; \frac{1}{2}\langle \psi, p^2\psi \rangle_{\mathcal{H}} + \frac{1}{2}\langle \psi, A_\varphi(x)^2\psi \rangle_{\mathcal{H}}\,, \tag{13.72}$$

$$|\langle \psi, \sigma \cdot B_\varphi(x)\psi \rangle_{\mathcal{H}}| \leq \frac{1}{2}\langle \psi, B_\varphi(x)^2\psi \rangle_{\mathcal{H}} + \frac{3}{2}\|\psi\|_{\mathcal{H}}^2\,. \tag{13.73}$$

Also, by using (13.69), (13.71),

$$\langle \psi, A_\varphi(x)^2\psi \rangle_{\mathcal{H}} \leq 5\|\widehat{\varphi}/\omega\|_{\mathfrak{h}}^2\langle \psi, H_\mathrm{f}\psi \rangle_{\mathcal{H}} + 3\|\widehat{\varphi}/\sqrt{\omega}\|_{\mathfrak{h}}^2\|\psi\|_{\mathcal{H}}^2\,, \tag{13.74}$$

$$\langle \psi, B_\varphi(x)^2\psi \rangle_{\mathcal{H}} \leq 5\||k|\widehat{\varphi}/\omega\|_{\mathfrak{h}}^2\,\langle\, \psi, H_\mathrm{f}\psi \rangle_{\mathcal{H}} + 3\||k|\widehat{\varphi}/\sqrt{\omega}\|_{\mathfrak{h}}^2\|\psi\|_{\mathcal{H}}^2\,. \tag{13.75}$$

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Thus if

$$\int \mathrm{d}^3k|\widehat{\varphi}(k)|^2(\omega^{-2}+\omega) < \infty\,, \tag{13.76}$$

one can find a constant e_0 such that for $|e| \leq e_0$ the operator H_1 is H_0 form-bounded with a bound less than 1. By a similar reasoning form-bounded can be replaced by bounded. From Theorem 13.2 we conclude

Theorem 13.3 (Self-adjointness, Kato–Rellich). *If $|e| \leq e_0$ with suitable e_0 and if the form factor $\widehat{\varphi}$ satisfies the condition (13.76), then the Pauli–Fierz operator H of (13.63) is self-adjoint on the domain $D(\frac{1}{2m}p^2 + H_\mathrm{f})$.*

Since $\widehat{\varphi}(0) = (2\pi)^{-3/2}$, the condition (13.76) is satisfied if, as assumed, $\widehat{\varphi}$ cuts off ultraviolet wave numbers.

$\diamond$ We denote constants by $c_0, c_1, \ldots, e_0$, etc., depending on the context. The numerical value of these constants may change from equation to equation. Since we always work with computable bounds, in principle these constants can be expressed through the parameters of the Pauli–Fierz Hamiltonian. To do so actually would overburden the notation. $\diamond$

The restriction on e is intrinsic to the method, since only then is $e^2 A_\varphi(x)^2$ small compared to H_f. To go beyond one needs a completely different technique which is based on functional integration, as will be explained in chapter 14.

Theorem 13.4 (Self-adjointness, functional integration). *If (13.76) holds, then the N-particle Pauli–Fierz Hamiltonian H of (13.39) is self-adjoint on the domain $D(\sum_{j=1}^N (p_j^2/2m_j) + H_\mathrm{f})$. Furthermore H is bounded from below.*

Proof: Hiroshima (2002).

Theorem 13.4 remains valid under the inclusion of spin and the addition of external potentials with very mild conditions on their regularity.

In summary, the Pauli–Fierz Hamiltonian uniquely generates the unitary time evolution $\mathrm{e}^{-\mathrm{i}Ht/\hbar}$ on $\mathcal{H}$ provided the condition (13.76) holds. Under a suitable ultraviolet cutoff the quantum dynamics of charges and photons is well defined.

13.4 Energy and length scales

The characteristic energy and length scales will depend on the physical situation. In our context two distinct cases are of particular importance. For the point-charge (= ultraviolet) limit relativistic units are used, which means that lengths are measured in units of the Compton wavelength

$$\lambda_\mathrm{c} = \hbar/m_\mathrm{e}c \tag{13.77}$$

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and energies in units of the rest energy $m_e c^2$ of the electron. For applications in atomic physics and quantum optics, atomic units are more appropriate, where the size of an atom is set by the Bohr radius

$$r_B = \frac{4\pi\hbar^2}{m_e e^2} = \alpha^{-1}\frac{\hbar}{m_e c} \tag{13.78}$$

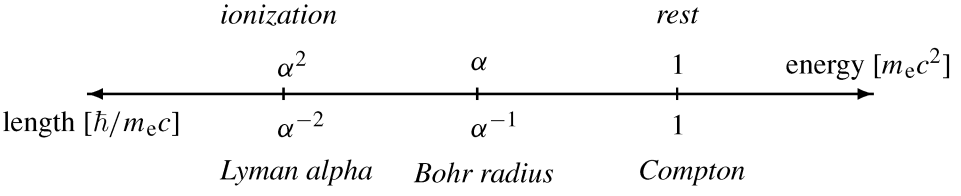
and the energy scale is set by the ionization energy

$$\frac{e^2}{4\pi r_B} = \frac{e^4 m_e}{(4\pi)^2\hbar^2} = \alpha^2(m_e c^2) \tag{13.79}$$

with

$$\alpha = \frac{e^2}{4\pi\hbar c}\,, \tag{13.80}$$

the Sommerfeld fine-structure constant written in Heaviside–Lorentz units. The ionization energy corresponds to the length $\alpha^{-1} r_B$ which approximately equals the wavelength of the Lyman alpha line. The scales compare as



Since $\alpha \simeq 1/137$ in nature, the scales are well separated.

These scales necessarily reappear in the Pauli–Fierz Hamiltonian with the crucial difference that the physical mass m_e of the electron is replaced by its bare mass m. To have a concrete example let us discuss the hydrogen atom as the simplest two-particle case. We assume that the nucleus is infinitely heavy. Then the Pauli–Fierz Hamiltonian reads

$$H = \frac{1}{2m}\big(\sigma\cdot(p + c^{-1}eA_\varphi(x))\big)^2 + H_f - e^2 V_{\varphi\mathrm{coul}}(x)\,, \tag{13.81}$$

where $-e$ is the charge of the electron and for the purpose of this subsection only we set

$$V_{\varphi\mathrm{coul}}(x) = \int \mathrm{d}^3k\,|\widehat{\varphi}(k)|^2|k|^{-2}\mathrm{e}^{-\mathrm{i}k\cdot x}\,. \tag{13.82}$$

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We transform to dimensionless form, such that the energy unit is $\alpha^2(mc^2)$, the length scale for the electron is $r_B = \hbar/\alpha mc$, and that for the photons is $\alpha^{-1}r_B$. This is achieved through the canonical transformation U defined through

$$U^*a(k,\lambda)U = (\alpha^{-2}\lambda_c)^{3/2}a(\alpha^{-2}\lambda_c k,\lambda)\,,$$
$$U^*xU = \alpha^{-1}\lambda_c x\,,\quad U^*pU = \alpha\lambda_c^{-1}p\,, \tag{13.83}$$

where now the Compton wavelength

$$\lambda_c = \hbar/mc \tag{13.84}$$

depends on the bare mass rather than the physical mass as in (13.77). Then

$$U^*HU = \alpha^2 mc^2\Big(\frac{1}{2}\big(\sigma\cdot(-\mathrm{i}\nabla_x - \sqrt{4\pi\alpha}\alpha A_{\tilde\varphi}(\alpha x))\big)^2$$
$$+\sum_{\lambda=1,2}\int \mathrm{d}^3k|k|a^*(k,\lambda)a(k,\lambda) - 4\pi V_{\varphi_\alpha\mathrm{coul}}(x)\Big) \tag{13.85}$$

with

$$A_{\tilde\varphi}(x) = \sum_{\lambda=1,2}\int \mathrm{d}^3k\widehat{\varphi}(\alpha^2\lambda_c^{-1}k)\frac{1}{\sqrt{2|k|}}e_\lambda(k)\big(\mathrm{e}^{\mathrm{i}k\cdot x}a(k,\lambda) + \mathrm{e}^{-\mathrm{i}k\cdot x}a^*(k,\lambda)\big) \tag{13.86}$$

and $\widehat{\varphi}_\alpha(k) = \widehat{\varphi}(\alpha k/\lambda_c)$. We infer from (13.85) that the Maxwell field is weakly coupled to the electron. Thus, *cum grano salis*, perturbation theory around $\alpha = 0$ should provide a qualitatively correct picture. In particular, spectral lines should be rather sharp. In addition, since $A_{\tilde\varphi}$ varies only on the scale $\alpha^{-1}r_B$, the dipole approximation $A_{\tilde\varphi}(\alpha x) \cong A_{\tilde\varphi}(0)$ will suffice as long as the electron remains bound to the nucleus.

The dimensionless form (13.85) teaches us also how to choose the wave number cutoff $\widehat{\varphi}$. Thus, if $\widehat{\varphi} = (2\pi)^{-3/2}$ for $|k| < \Lambda$, $\widehat{\varphi} = 0$ for $|k| \geq \Lambda$, then $\Lambda \gg 1/r_B$ to have a negligible smearing of the Coulomb potential. On the other hand, at the scale of the rest energy of the electron, the Pauli–Fierz model cannot be expected to describe the physics correctly. Thus the cutoff should satisfy

$$1 \ll \Lambda r_B \ll \alpha^{-1}\,. \tag{13.87}$$

It is instructive to compare the atomic units with relativistic units. In the latter case the scale transformation U reads

$$U^*a(k,\lambda)U = \lambda_c^{3/2}a(\lambda_c k,\lambda)\,,\quad U^*xU = \lambda_c x\,,\quad U^*pU = \lambda_c^{-1}p\,. \tag{13.88}$$

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Then

$$U^*HU = mc^2\Big(\frac{1}{2}\big(\sigma\cdot(-\mathrm{i}\nabla_x - \sqrt{4\pi\alpha}A_{\varphi_{\lambda_c}}(x))\big)^2 + \sum_{\lambda=1,2}\int \mathrm{d}^3k|k|a^*(k,\lambda)a(k,\lambda) - 4\pi\alpha V_{\varphi_{\lambda_c}\mathrm{coul}}(x)\Big) \quad (13.89)$$

with the form factor in units of the Compton wavelength, $\widehat{\varphi}_{\lambda_c} = \widehat{\varphi}(k/\lambda_c)$. Note that the cutoff depends through the Compton wavelength on the bare electron mass.

13.5 Conservation laws

The Pauli–Fierz Hamiltonian (13.42) is invariant under translations and rotations. Therefore the total momentum and the total angular momentum will be conserved. One only has to identify the generators of these symmetries. The generator for the translations of the j-th particle is its momentum p_j, which means

$$\mathrm{e}^{\mathrm{i}a\cdot p_j/\hbar}q_j\mathrm{e}^{-\mathrm{i}a\cdot p_j/\hbar} = q_j + a\,. \quad (13.90)$$

Similarly the field translations are generated by the momentum of the Maxwell field

$$P_\mathrm{f} = \sum_{\lambda=1,2}\int \mathrm{d}^3k\hbar k a^*(k,\lambda)a(k,\lambda) \quad (13.91)$$

with the property that

$$\mathrm{e}^{\mathrm{i}a\cdot P_\mathrm{f}/\hbar}a(k,\lambda)\mathrm{e}^{-\mathrm{i}a\cdot P_\mathrm{f}/\hbar} = \mathrm{e}^{-\mathrm{i}a\cdot k}a(k,\lambda)\,. \quad (13.92)$$

Thus the total momentum

$$P = \sum_{j=1}^{N} p_j + P_\mathrm{f} \quad (13.93)$$

must be conserved and indeed

$$[H, P] = 0\,. \quad (13.94)$$

Next we consider a rotation R by an angle θ relative to the axis of rotation $\widehat{n}$ through the origin. For position and momentum we have

$$\mathrm{e}^{\mathrm{i}\theta\widehat{n}\cdot(q_j\times p_j)/\hbar}q_j\mathrm{e}^{-\mathrm{i}\theta\widehat{n}\cdot(q_j\times p_j)/\hbar} = Rq_j\,,\quad \mathrm{e}^{\mathrm{i}\theta\widehat{n}\cdot(q_j\times p_j)/\hbar}p_j\mathrm{e}^{-\mathrm{i}\theta\widehat{n}\cdot(q_j\times p_j)/\hbar} = Rp_j\,. \quad (13.95)$$

For the Maxwell field we define the angular momentum relative to the origin

$$J_\mathrm{f} = -\sum_{\lambda=1,2}\int \mathrm{d}^3k a^*(k,\lambda)(k\times\mathrm{i}\hbar\nabla_k)a(k,\lambda) \quad (13.96)$$

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and the helicity

$$S_{\rm f} = c^{-1}\int {\rm d}^3x\, E(x)\times A(x) = {\rm i}\hbar\int {\rm d}^3k\widehat{k}\big(a^*(k,2)a(k,1) - a^*(k,1)a(k,2)\big) \tag{13.97}$$

with $\widehat{k} = k/|k|$. Their sum rotates the vector potential as

$${\rm e}^{{\rm i}\theta\widehat{n}\cdot(J_{\rm f}+S_{\rm f})/\hbar}A(x){\rm e}^{-{\rm i}\theta\widehat{n}\cdot(J_{\rm f}+S_{\rm f})/\hbar} = RA(R^{-1}x) \tag{13.98}$$

and correspondingly for the transverse electric field $E_\perp(x)$. We conclude that the total angular momentum

$$J = \sum_{j=1}^{N}(q_j\times p_j) + J_{\rm f} + S_{\rm f} \tag{13.99}$$

is conserved and indeed

$$[H, J] = 0\,. \tag{13.100}$$

If the j-th particle carries spin σ_j, then

$$J = \sum_{j=1}^{N}(q_j\times p_j) + \sum_{j=1}^{N}\frac{1}{2}\hbar\sigma_j + J_{\rm f} + S_{\rm f} \tag{13.101}$$

is the conserved total angular momentum.

The helicity $S_{\rm f}$ is diagonalized through transforming to circularly polarized photons. We define the left-circularly and right-circularly polarized annihilation operators

$$a_+(k) = \frac{1}{\sqrt{2}}\big(a(k,1) - {\rm i}a(k,2)\big)\,,\quad a_-(k) = \frac{1}{\sqrt{2}}\big(a(k,1) + {\rm i}a(k,2)\big)\,. \tag{13.102}$$

Then

$$S_{\rm f} = \int {\rm d}^3k\widehat{k}\big(a_+^*(k)a_+(k) - a_-^*(k)a_-(k)\big)\,, \tag{13.103}$$

which establishes that the photon has spin 1. However, only two helicity states are admissible, $+1$ for left and -1 for right polarization. The corresponding one-photon states are

$$e_\pm(k)(2\pi)^{-3/2}{\rm e}^{{\rm i}(k\cdot x\mp\omega t)},\quad e_\pm(k) = \frac{1}{\sqrt{2}}\big(e_1(k)\pm{\rm i}e_2(k)\big)\,. \tag{13.104}$$

For the $+$ index the photon state represents a plane wave whose polarization vector rotates in a right-handed sense about k and thus appears to an observer facing the incoming wave as left polarized.

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13.6 Boundary conditions and the Casimir effect

So far we took for granted that the Maxwell field lives in infinite space. In many applications one has a macroscopically finite geometry, like a cavity or a wave guide, and it is necessary to include it as a boundary condition into the Hamiltonian. For concreteness, let us assume then some bounded region Λ whose surface $\partial\Lambda$ is defined through a perfect, grounded conductor. Momentarily there are no charges inside Λ. Then the Maxwell equations are

$$c^{-1}\partial_t B = -\nabla \times E\,, \quad c^{-1}\partial_t E = \nabla \times B\,, \quad \nabla \cdot E = 0\,, \quad \nabla \cdot B = 0\,. \tag{13.105}$$

If $\widehat{n}(x)$ denotes the outward normal at $x \in \partial\Lambda$, the boundary conditions for a perfect conductor are

$$\widehat{n} \cdot B(x) = 0\,, \quad \widehat{n} \times E(x) = 0 \quad \text{at} \quad x \in \partial\Lambda\,. \tag{13.106}$$

The rules of canonical quantization apply as before, only the final expressions are less explicit, since (13.105) together with the boundary conditions (13.106) cannot be solved through simple Fourier transformation. Let $L^2(\Lambda, \mathbb{R}^3)$ be the space of (complex valued) vector fields on Λ. $A \in L^2$ is divergence free if $\nabla \cdot A = 0$ and we denote by $Q_\Lambda^\perp$ the projection onto all such fields. The quantum mechanical Fock space is built up from $Q_\Lambda^\perp L^2$ as one-particle Hilbert space. Notationally it is slightly more convenient to start from $L^2(\Lambda, \mathbb{R}^3)$ and incorporate the projection into the definition of the quantized fields. We introduce then the three-component Bose field $a(x), a^*(x)$ satisfying

$$[a_\alpha(x), a^*_{\alpha'}(x')] = \delta_{\alpha\alpha'}\delta(x - x') \tag{13.107}$$

with all other commutators vanishing. The quantized Maxwell field will depend only on $Q_\Lambda^\perp a$ and $Q_\Lambda^\perp a^*$.

As before the vector potential A satisfies the Coulomb gauge, which implies

$$c^{-2}\partial_t^2 A = \Delta A\,, \quad \nabla \cdot A = 0 \tag{13.108}$$

with boundary conditions

$$\widehat{n} \cdot (\nabla \times A) = 0\,, \quad \widehat{n} \times A = 0 \quad \text{at} \quad x \in \partial\Lambda\,. \tag{13.109}$$

Since $E_\perp = -c^{-1}\partial_t A$, one can write the solution to (13.108), (13.109) on $Q_\Lambda^\perp L^2$ as in (13.108)

$$\begin{pmatrix} A(t) \\ E_\perp(t) \end{pmatrix} = \begin{pmatrix} \cos \Omega t & -c\Omega^{-1} \sin \Omega t \\ c^{-1}\Omega \sin \Omega t & \cos \Omega t \end{pmatrix} \begin{pmatrix} A \\ E_\perp \end{pmatrix}\,, \tag{13.110}$$

where, as a linear operator, $\Omega = c(-\Delta \otimes \mathbb{1})^{1/2}$ restricted to $Q_\Lambda^\perp L^2$ and with the mixed Dirichlet–Neumann boundary condition (13.109). Ω is a positive

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self-adjoint operator. In analogy to (13.35), (13.36) the canonically quantized fields are obtained as

$$A(x) = c\sqrt{\hbar/2}\,\Omega^{-1/2} Q_\Lambda^\perp \big(a(x) + a^*(x)\big)\,, \tag{13.111}$$

$$E_\perp(x) = \sqrt{\hbar/2}\,\Omega^{1/2} Q_\Lambda^\perp \mathrm{i}\big(a(x) - a^*(x)\big)\,. \tag{13.112}$$

Clearly their commutation relations are

$$[A_\alpha(x), -c^{-1} E_{\perp\alpha'}(x')] = \mathrm{i}\hbar (Q_\Lambda^\perp)_{\alpha\alpha'}(x, x') \tag{13.113}$$

with the right-hand side denoting the integral kernel of $Q_\Lambda^\perp$ in $L^2(\Lambda, \mathbb{R}^3)$. The field energy is a sum over the energy in each mode, which in position space becomes

$$H_\mathrm{f} = \hbar \int_\Lambda \mathrm{d}^3x a^*(x) \cdot \Omega Q_\Lambda^\perp a(x)\,. \tag{13.114}$$

In case there are charges enclosed in the cavity, their mutual Coulomb interaction has to respect the perfect conductor boundary condition (13.106). For example, since $E_\parallel$ is not quantized, for a single charge at q the potential ϕ_Λ satisfies the Poisson equation

$$\Delta\phi_\Lambda(x) = e\varphi(x - q)\,, \quad \phi_\Lambda(x) = 0 \quad \text{for} \quad x \in \partial\Lambda \tag{13.115}$$

and the potential acting on the particle is given by

$$e\phi_{\Lambda\varphi}(q) = e \int \mathrm{d}^3x \varphi(q - x)\phi_\Lambda(x)\,. \tag{13.116}$$

Close to the surface $\phi_\Lambda(x)$ is determined by the image charge and looks like an attractive Coulomb potential. Thus we have to add phenomenologically to the Hamiltonian a surface potential V_sur which keeps the particle confined to the cavity Λ. Altogether the Pauli–Fierz Hamiltonian for a single charge enclosed in a cavity is

$$H = \frac{1}{2m}\big(p - c^{-1}eA_\varphi(q)\big)^2 + e\phi_{\Lambda\varphi}(q) + V_\mathrm{sur}(q) + H_\mathrm{f}\,. \tag{13.117}$$

To return to the charge-free situation, according to (13.114) we calibrated the ground state energy of the cavity at zero, which is an acceptable choice for a closed cavity. If, however, the cavity is open, as for example two plane parallel, grounded metal plates, then the natural zero of energy refers to the energy of the field vacuum in infinite space. In the presence of the plates this vacuum energy is lowered by an amount which depends on the separation of the plates. Therefore there is an effective attractive force between the plates – the famous Casimir effect. Together with the spectrum of the black-body radiation it provides the most direct evidence for the quantum nature of the Maxwell field.

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If one adopts the boundary conditions as in (13.109), the energy difference – with and without plates – diverges because of high-frequency modes, which reflects the fact that the metal plates cannot be perfect conductors up to arbitrarily high frequencies. We therefore choose a cutoff function g with $g(\omega) = 1$ for small ω and rapidly decreasing at infinity. The plates are parallel to each other, have a distance d, and an area ℓ^2 which is taken to be very large. Then the energy difference per unit area is given by

$$\frac{1}{\ell^2}\Delta E(d) = \frac{\pi^2\hbar c}{4d^3}\Big(\frac{1}{2}G(0) + \sum_{n=1}^{\infty} G(n) - \int_0^\infty \mathrm{d}\kappa\, G(\kappa)\Big)\,, \tag{13.118}$$

where

$$G(\kappa) = 2\int_\kappa^\infty \mathrm{d}u u^2 g(\pi u/d)\,. \tag{13.119}$$

For analytic g one can use in (13.118) the Euler–MacLaurin summation formula,

$$\frac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n) - \int_0^\infty \mathrm{d}\kappa\, F(\kappa) = -\frac{1}{12}F'(0) + \frac{1}{720}F'''(0) + \text{higher derivatives}, \tag{13.120}$$

and note that $F'(0) = 0$, $F'''(0) = -4$, since $g(0) = 1$, whereas every extra derivative carries a factor $1/d$. Thus to leading order

$$\frac{1}{\ell^2}\Delta E(d) = -\frac{\pi^2\hbar c}{720d^3} + \mathcal{O}(d^{-4})\,, \tag{13.121}$$

independently of the choice of the cutoff function g, and the force per unit area between the conducting plates is given by

$$\frac{1}{\ell^2}F(d) = -\frac{\pi^2\hbar c}{240d^4} + \mathcal{O}(d^{-5})\,. \tag{13.122}$$

13.7 Dipole and single-photon approximation

Even for a single charge the Pauli–Fierz Hamiltonian resists exact diagonalization and one has to rely on approximations. As suggested by (13.85), since the coupling to the photon field is weak, an obvious strategy is to expand in α. Such a perturbative treatment is covered extensively in standard texts and there is no need to repeat it here. Since one of our aims is to explain why perturbation theory works so well, we will make contact with the conventional results later on. Another strategy is to truncate the Hamiltonian to taste, so as not to throw out the physics. In essence there are only two such schemes, the dipole approximation and the single-photon approximation.

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(i) Dipole approximation

We consider a single charge confined by an external potential $e\phi_{\rm ex}$, centered at the origin. Since the potential inhibits large excursions, one loses little by evaluating the vector potential at the origin instead of at q, the true position of the charged particle. This leads to the dipole Hamiltonian

$$H = \frac{1}{2m}\big(p - c^{-1}eA_\varphi(0)\big)^2 + e\phi_{\rm ex}(q) + H_{\rm f}\,. \tag{13.123}$$

The interaction $p \cdot A_\varphi(0)$ couples p to the fluctuating vector potential at the origin. We can transform it to a fluctuating electric field coupled to the position q through the unitary operator

$$U = \exp[{\rm i}c^{-1}eq \cdot A_\varphi(0)/\hbar]\,. \tag{13.124}$$

Then

$$\begin{aligned} &U^* pU = p + c^{-1}eA_\varphi(0)\,, \quad U^* qU = q\,,\\ &U^* a(k,\lambda)U = a(k,\lambda) + {\rm i}q \cdot e_\lambda(k)e\widehat{\varphi}(k)\sqrt{1/2\hbar\omega}\,, \end{aligned} \tag{13.125}$$

which imply

$$U^* HU = \frac{1}{2m}p^2 + e\phi_{\rm ex}(q) + H_{\rm f} - eq \cdot E_{\perp\varphi}(0) + \frac{1}{2}\Big(\frac{2}{3}\int {\rm d}^3k e^2|\widehat{\varphi}(k)|^2\Big)q^2\,. \tag{13.126}$$

The extra harmonic potential balances $q \cdot E_{\perp\varphi}$ so as to make the sum of the last three terms positive.

Even in the form (13.123), respectively (13.126), H is not tractable and in a second approximation one assumes the external potential to be harmonic. Then the dipole Hamiltonian reads

$$H = \frac{1}{2m}\big(p - c^{-1}eA_\varphi(0)\big)^2 + \frac{1}{2}m\omega_0^2 q^2 + H_{\rm f}\,. \tag{13.127}$$

Clearly, the Hamiltonian is quadratic in the dynamical variables and consequently the Heisenberg equations of motion are linear,

$$\begin{aligned} &\dot{q}(t) = \frac{1}{m}\big(p(t) - c^{-1}eA_\varphi(0,t)\big)\,, \quad \dot{p}(t) = -m\omega_0^2 q(t)\,,\\ &c^{-2}\partial_t^2 A(x,t) = \Delta A(x,t) + (e/c)\delta_\varphi^\perp(x)\dot{q}(t) \end{aligned} \tag{13.128}$$

with $\delta^\perp$ the transverse δ-function of (13.38). As before the index φ denotes convolution with the form factor φ. At this point (13.128) can be solved as classical equations of motion. One obtains the exact line shape, the Lamb shift, and the Rayleigh scattering of light from a bound charge. It should be noted that, since the

energy levels of the harmonic oscillator are equidistant, several emitted photons will interfere, which makes the emission spectrum distinct from, say, the hydrogen atom; compare with section 17.4.

Even though the equations of motion (13.128) are linear, their solution is not a back of the envelope computation and one often resorts to yet another approximation, the rotating wave approximation. One starts from (13.126) with the harmonic potential $\frac{1}{2}m\omega_0^2 q^2$, which already includes the last summand in (13.126), and rewrites the harmonic oscillator in terms of its creation and annihilation operator b, b^*. Then

$$H = \hbar\omega_0 b^* b - \mathrm{i}\sqrt{\hbar/2m\omega_0}(b - b^*) \cdot eE_{\perp\varphi}(0) + H_\mathrm{f}\,. \tag{13.129}$$

In the coupling, one ignores the counter-rotating terms ba and b^*a^*, which results in

$$H_\mathrm{rw} = (b, a) \cdot \mathsf{h}(b, a)^t\,. \tag{13.130}$$

Our notation emphasizes that the rotating wave Hamiltonian H_rw is quadratic in (b, a) and should be regarded as the second quantization of the one-particle Hamiltonian h. The one-particle space is $\mathcal{K} = \mathbb{C}^3 \oplus (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)$, the $\mathbb{C}^3$ subspace corresponding to b, b^*. A wave function in $\mathcal{K}$ is of the form $(\chi, \psi(k, \lambda))$, χ the one-particle amplitude for the oscillator and $\psi(k, \lambda)$ the one-particle photon amplitude. h acting on a pair (χ, ψ) is defined by

$$\mathsf{h}\begin{pmatrix}\chi\\ \psi(k,\lambda)\end{pmatrix} = \begin{pmatrix}\hbar\omega_0\chi - \frac{1}{2}\sum_{\lambda=1,2}\int \mathrm{d}^3k e\widehat{\varphi}^*\hbar\sqrt{\omega/m\omega_0}e_\lambda(k)\psi(k,\lambda)\\ -\frac{1}{2}e\widehat{\varphi}\hbar\sqrt{\omega/m\omega_0}e_\lambda(k)\cdot\chi + \hbar\omega\psi(k,\lambda)\end{pmatrix}, \tag{13.131}$$

$\widehat{\varphi} = \widehat{\varphi}(k)$, $\omega = \omega(k)$. h will reappear as the Friedrichs–Lee Hamiltonian. For $e = 0$, the eigenvalue $\hbar\omega_0$ is embedded in the continuous spectrum $[0, \infty)$. The coupling turns this eigenvalue into a resonance; compare with section 17.3.

A further popular variant is to set $\omega_0 = 0$ in (13.127) and to regard the Hamiltonian as describing a freely propagating charge. One finds that the mass of the particle is increased due to the coupling with the field. However, quantitatively such a result cannot be trusted, since the dipole approximation is based on the assumption that the electron remains close to the origin. There is no such mechanism for a free particle.

(ii) Single-photon approximation

We restrict the Fock space to $\mathbb{C} \oplus \mathfrak{h}$. Then the wave functions ψ are pairs $(\psi_0(x), \psi_1(x, k, \lambda))$. $\psi_0(x)$ is the wave function for an electron and no photon

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present, while $\psi_1(x, k, \lambda)$ is the wave function for the electron plus one photon with momentum $\hbar k$ and helicity λ. The correspondingly restricted Pauli–Fierz Hamiltonian is denoted by H_1. From (13.39), setting $N = 1$, $\hbar = 1 = c$, one infers

$$(H_1\psi)_0(x) = \frac{1}{2m}p^2\psi_0(x) + \sum_{\lambda=1,2}\int \mathrm{d}^3k e\widehat{\varphi}\frac{1}{\sqrt{2\omega}}\mathrm{e}^{\mathrm{i}k\cdot x}\frac{1}{m}e_\lambda \cdot p\psi_1(x, k, \lambda)\,,$$
$$(H_1\psi)_1(x, k, \lambda) = \Big(\frac{1}{2m}p^2 + \omega\Big)\psi_1(x, k, \lambda) + e\widehat{\varphi}\frac{1}{\sqrt{2\omega}}\mathrm{e}^{-\mathrm{i}k\cdot x}\frac{1}{m}e_\lambda \cdot p\psi_0(x)\,, \tag{13.132}$$

where the A_φ^2 contribution has been neglected. H_1 is a two-particle problem with a translation-invariant interaction. The electron has kinetic energy $\frac{1}{2m}p^2$. The photon can be either "dead" (ψ_0) or "alive" (ψ_1). The kinetic energy is zero in the dead state and $\hbar\omega$ in the alive state. Through the interaction a photon is either created or annihilated, which corresponds to a transition between dead and alive. Because of the $1/\sqrt{\omega}$-factor, this interaction has a long range and decays only as $r^{-3/2}$ in the relative distance between the electron and the photon.

Notes and references

Section 13.1

The Hamiltonian form of the Abraham model in the Coulomb gauge is standard and explained in Cohen-Tannoudji *et al.* (1989) and Sakurai (1986), for example.

Section 13.2

The name "Pauli–Fierz" is not accurate historically. The Hamiltonian (13.42) appears at the beginning of paragraph two of Pauli and Fierz (1938) as a matter of fact, without citation. Pauli and Fierz study the generation of infrared photons in Compton scattering. Cohen-Tannoudij *et al.* (1989) call (13.42) "of basic importance" and Milonni (1994) refers to (13.42) simply as "the Hamiltonian". Thus despite its fundamental nature the Hamiltonian (13.42) carries no specific name in the literature. Lately, "nonrelativistic quantum electrodynamics" and "Pauli–Fierz" have become common usage in some quarters. We stick to the latter convention, which is certainly better than to be speechless.

The quantization of the electromagnetic field as a system of harmonic oscillators was common knowledge right after the advent of quantum mechanics through the work of Dirac (1927), Landau (1927), Jordan and Pauli (1928), Fermi (1930), and Landau and Peierls (1930), and was immediately applied to atomic radiation

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by many quantum theorists. The systematic derivation of the Hamiltonian (13.42) is not so well documented and was presumably regarded as more or less obvious, although the advantage of the Coulomb gauge was only slowly realized. The review articles by Breit (1932) and by Fermi (1932) and the research monograph by Heitler (1936, 1958) explain the quantization in its modern form, in essence. Since "one cannot comb the hair on a sphere", the polarization vectors $e_\lambda(k)$ are necessarily discontinuous in k, which causes poor decay in their Fourier transform. We refer to Lieb and Loss (2004) for a formulation using only the transverse projection.

The size of atoms as based exclusively on the Coulomb Hamiltonian is a long-standing open problem. We refer to Lieb (1990, 2001).

Textbooks on nonrelativistic quantum electrodynamics are listed in Notes and References to section 3.2.

Section 13.3

Criteria for self-adjointness are given in Reed and Simon (1980, 1975). In our context the Kato–Rellich theorem has been applied by Nelson (1964b) and Fröhlich (1974), amongst others. Self-adjointness without restriction on the magnitude of the charge is proved by Hiroshima (2000b, 2002). A review is Hiroshima (2001).

Section 13.5

A more detailed treatment of conservation laws is Huang (1998).

Section 13.6

Casimir (1948) discovered the attraction of two conducting plates through vacuum fluctuations. Casimir and Polder (1948) compute the attractive force between two atoms, the retarded van der Waals force, and the force between an atom and a wall. The forces are minute and direct experimental evidence had to wait for a while. We refer to Sparnaay (1958) and Lamoreaux (1997). On the theoretical side a complete coverage is Milloni (1994), Huang (1998), with the finite-temperature corrections discussed by Schwinger *et al.* (1978), Bordag *et al.* (2000), and Feinberg *et al.* (2000).

Section 13.7

Apparently the first systematic study of the dipole approximation with a harmonic external potential is Kramers (1948) and van Kampen (1951). Various aspects are covered by Senitzky (1960), Schwabl and Thirring (1964), Ford, Kac and Mazur

(1965), Ullersma (1966), Ford, Lewis and O'Connell (1988a, 1988b), Grabert, Schramm and Ingold (1988), Unruh and Zurek (1989). A mathematical study is the series by Arai (1981, 1983a, 1983b, 1990, 1991). Since the dipole approximation provides a reasonable description of radiation processes, one might regard the harmonic potential as the lowest-order approximation and expand in the anharmonicity. This program has been carried through in Maassen (1984), Spohn (1997), Maassen, Gută and Botvich (1999), and Fidaleo and Liverani (1999). If the anharmonicity is small, in fact so small that the external potential remains convex and grows as $\frac{1}{2}m\omega_0^2 q^2$ for large q, then the convergence of the time-dependent Dyson series can be controlled uniformly in t. With such a strong estimate one can show that qualitatively the properties of the damped harmonic oscillator persist into the nonlinear regime.

The dipole approximation is not restricted to a single particle. For example one may consider two harmonically bound charges with their center of charge at r_1 and r_2. Then the kinetic energies are approximated by $(p_j - c^{-1}e_j A_\varphi(r_j))^2/2m_j$, $j = 1, 2$. Denoting $R = |r_1 - r_2|$, one is interested in the ground state energy, $E(R)$, as a function of the separation. Because of retardation $E(R) \cong -R^{-7}$ for large R and $E(R) \cong -R^{-6}$ in an intermediate regime.

If $\phi_{\text{ex}} = 0$, then the Hamiltonian (13.123) can be unitarily transformed to $H' = (p^2/2m_{\text{eff}}) + H_{\text{f}}$. m_{eff} agrees with the effective mass of the Abraham model to lowest order in $|v|/c$; compare with section 4.1.

The single-photon approximation was already used in disguise by Dirac (1927) and Weisskopf and Wigner (1930). It is instructive to extend this approximation by cutting Fock space at N photons (Hübner and Spohn, unpublished manuscript; Skibsted 1998). If one artificially adds to the space of single-photon wave functions a one-dimensional subspace for a "dead" photon, then the theory has a structure very similar to an $(N+1)$-particle Schrödinger equation. The photons interact only indirectly through the atom. The cluster decomposition consists of n free photons and $N - n$ photons bound by the atom, $n = 0, 1, \ldots, N$.