Can. J. Math., Vol. XXXII, No. 5, 1980, pp. 1244–1249

A NOTE ON BUCHSBAUM RINGS AND LOCALIZATIONS OF GRADED DOMAINS

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Let $R = \bigoplus_{i \geqq 0} R_i$ be a graded integral domain, and let $p \in \text{Proj}(R)$ be a homogeneous, relevant prime ideal. Let $R_{(p)} = \{r/t | r \in R_i, t \in R_i \backslash p\}$ be the geometric local ring at p and let $R_p = \{r/t | r \in R, t \in R \backslash p\}$ be the arithmetic local ring at p. Under the mild restriction that there exists an element $r_1 \in R_1 \backslash p$, W. E. Kuan [**2**], Theorem 2, showed that r_1 is transcendental over $R_{(p)}$ and

$$R_p \cong S^{-1}(R_{(p)}[r_1]),$$

where S is the multiplicative system $R \backslash p$. It is also demonstrated in [**2**] that $R_{(p)}$ is normal (regular) if and only if R_p is normal (regular). By looking more closely at the relationship between R_p and $R_{(p)}$, we extend this result to Cohen-Macaulay (abbreviated C.M.) and Gorenstein rings.

Also, suppose (A, m) is a local (Noetherian with identity) ring of Krull dimension d. A generalization of the Cohen-Macaulay property is the requirement that every system of parameters $x_1, \ldots, x_d$ for A form a weak A-sequence: $m[(x_1, \ldots, x_{i-1}) : x_i] \subseteq (x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, d$. Equivalently, the difference $l(A/q) - e_0(q, A)$ is independent of q, as q varies over the set of parameter ideals of A. Such rings are known as Buchsbaum rings, or B-rings for short. A locally Noetherian scheme X is said to be a *Buchsbaum scheme* if the local ring at each point of X is a B-ring. See the papers of Stückrad and Vogel, [**10**] and [**11**], for further information. We show that $R_{(p)}$ is a B-ring if and only if R_p is a B-ring. Finally a method for producing varieties with non-Buchsbaum singularities is given.

First the simple

LEMMA. *Let R be an integral domain. If $S \subseteq R$ is a multiplicative system such that $S^{-1}(R)$ is quasi-local with maximal ideal m, then $S^{-1}(R) = R_p$, where $p \in$ Spec(R) is the (unique) prime ideal extending to m.*

Proof. Clearly $S^{-1}(R) \subseteq R_p$. Now let $r/s \in R_p$ and suppose $s/1 \in m$. Then $s \in p = m \cap R$, contrary to the definition of R_p. Therefore $s/1$ is a unit in $S^{-1}(R)$, and so $r/s \in S^{-1}(R)$.

Let (A, m) be a quasi-local ring and let x be transcendental over A.

Received November 3, 1978.

The extended ideal $m[x]$ is then prime in $A[x]$. Define $A^* = A[x]_{m[x]}$. Notice that $A \to A^*$ is a flat, local homomorphism.

What then is the prime in $R_{(p)}[r_1]$ which yields R_p upon localization?

THEOREM 1. *Let R be a graded domain and let $p \in \text{Proj}(R)$ with $r_1 \in R_1 \backslash p$. Then $R_p \cong R_{(p)}^* = (R_{(p)}[r_1])_{m[r_1]}$, where m is the maximal ideal of $R_{(p)}$.*

Proof. Let $S = R\backslash p$. $R_p \cong S^{-1}(R_{(p)}[r_1])$ is quasi-local, so by the Lemma, $R_p \cong (R_{(p)}[r_1])_q$ for some $q \in \text{Spec}(R_{(p)}[r_1])$. This q is the (unique) prime maximal with respect to the condition $q \cap R \subseteq p$. So first, it must be shown that $m[r_1] \cap R \subseteq p$. Suppose $x \in m[r_1]$. Then

$$x = \left(\frac{a_n}{s_n}\right){r_1}^n + \left(\frac{a_{n-1}}{s_{n-1}}\right){r_1}^{n-1} + \ldots + \frac{a_0}{s_0},$$

where $a_i/s_i \in m$. That is, $a_i \in p$, $s_i \notin p$ with a_i and s_i homogeneous of the same degree. If $x \in R$ also, then

$$x = \left(\frac{b_m}{{r_1}^m}\right){r_1}^m + \left(\frac{b_{m-1}}{{r_1}^{m-1}}\right){r_1}^{m-1} + \ldots + \frac{b_0}{1},$$

where $b_i \in R$, $\deg b_i = i$. Since r_1 is transcendental, the representation for x as a polynomial is unique. Thus, $n = m$, and for all $i = 0, \ldots, n$, $a_i/s_i = b_i/{r_1}^i$. Therefore, $b_is_i = a_i{r_1}^i \in p$ for $i = 0, \ldots, n$, so that $b_i \in p$ for $i = 0, \ldots, n$ and so x itself is in p.

Next suppose $q \in \text{Spec}(R_{(p)}[r_1])$ properly contains $m[r_1]$ with $q \cap R \subseteq p$. Pick $f \in q\backslash m[r_1]$. It is sufficient to assume that

$$f = \left(\frac{a_n}{t_n}\right){r_1}^n + \left(\frac{a_{n-1}}{t_{n-1}}\right){r_1}^{n-1} + \ldots + \frac{a_0}{t_0}$$

with $a_n/t_n \neq 0$, and for all $i = 0, \ldots, n$, either $a_i/t_i = 0$, or $a_i/t_i \notin m$, so that no a_i is a non-zero element of p. Let

$$t = \prod_{i=0}^{n} t_i$$

(if $a_i/t_i = 0$ put $t_i = 1$) and let $k = \deg t$. Then

$$tf = {r_1}^k\left(\frac{t}{{r_1}^k}\right)f \in q.$$

But

$${r_1}^k\left(\frac{t}{{r_1}^k}\right)f = \left(\frac{t}{t_n}\right)a_n{r_1}^n + \left(\frac{t}{t_{n-1}}\right)a_{n-1}{r_1}^{n-1} + \ldots + \left(\frac{t}{t_0}\right)a_0$$

is an element of $q \cap R \subseteq p$, since each t_i divides t. Moreover,

$$\deg((t/t_i)a_i{r_1}^i) = k + i \text{ for } a_i/t_i \neq 0,$$

and since p is graded, each $(t/t_i)a_i{r_1}^i$ is in p. This contradicts the hypo-

https://doi.org/10.4153/CJM-1980-092-6 Published online by Cambridge University Press

thesis that p is prime and the fact that r_1 and all t/t_i, a_i lie outside p by choice.

In the case of (A, m) local, $\mathrm{ht}(m)$ equals $\mathrm{ht}(m[x])$, see [**6**], so $\dim(A) = \dim(A^*)$. As an immediate consequence,

COROLLARY. *If R is Noetherian, R and p as in Theorem 1 then* $\dim(R_p) = \dim(R_{(p)})$.

Many other properties are invariant under the passage from A to A^*. For instance,

THEOREM 2. *Suppose (A, m) is a local ring. Then*
(a) *A is* C.M. *if and only if A^* is* C.M.
(b) *A is Gorenstein if and only if A^* is Gorenstein.*

Proof. (a) A C.M. implies $A[x]_{m[x]} \cong A^*$ C.M. Conversely, the extension

$$A \rightarrow A[x] \rightarrow A[x]_{m[x]} = A^*$$

is flat and local. Hence if A^* is C.M., then so is A. See e.g. [**5**].

(b) The proof is the same. Needed facts (A Gorenstein implies $A[x]$ locally Gorenstein and the result on flat, local extensions) can be found in [**13**].

Again, we get an immediate

COROLLARY. *If R and p are as in Theorem 1 then $R_{(p)}$ is* C.M. (*Gorenstein*) *if and only if R_p is* C.M. (*Gorenstein*).

Remarks. The fact that $R_{(p)} \rightarrow R_p$ is local was already noted in [**2**]. Combining this with flatness yields the Corollary without having to resort to Theorem 1. Also, for the special case of projective varieties over an algebraically closed field, a statement equivalent to part (a) of Theorem 2 appears as Corollaire 1.5, p. 379 of [**8**].

Notice that if (A, m) is a B-ring, $A[x]$ need not be locally a B-ring. That is Spec$(A[x])$ need not be a Buchsbaum scheme. Take a B-ring (A, m) and suppose this implication were valid. As (m, x) is maximal in $A[x]$, $A[x]_{(m,x)}$ would be a B-ring. The subsequent localization

$$(A[x]_{(m,x)})_{(m[x]A[x]_{(m,x)})} \cong A[x]_{m[x]} = A^*$$

is C.M. [**11**], Remark p. 439. From Theorem 2, A itself is C.M., a contradiction, since of course, not all B-rings are C.M. [**11**], p. 524.

However, the following is true:

THEOREM 3. *Suppose (A, m) is a local ring. Then A is a* B-*ring if and only if A^* is a* B-*ring.*

The proof uses the following cohomological characterization of Buchsbaum rings. For (A, m) local with $t = \dim_{A/m}(m/m^2)$, let

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$H^i(m, A) = H_{t-i}(m, A)$ be the i-th cohomology of the Koszul complex of A. The main result of [**9**] is quoted here from [**12**].

LEMMA. *Let (A, m) be a local ring of dimension $d > 0$. Then A is a Buchsbaum ring if and only if the canonical maps $\varphi_A{}^i$: $H^i(m, A) \to H_m{}^i(A)$ are surjective for all $i \neq d$.*

Proof of Theorem 3. Let $d = \dim(A) = \dim(A^*)$. The case $d = 0$ is trivial, since then both A and A^* are Cohen-Macaulay.

Now suppose $d > 0$. Note first that A and A^* have the same embedding dimension, say t. In fact, much more is true. See [**3**], Lemma 2, p. 75. Let $\{x_1, \ldots, x_t\}$ be a minimal generating set for m in A. Then $x_1A^* + \ldots + x_tA^* = m^*$, the maximal ideal of A^*. Given $n > 0$, write $\mathbf{x}^n$ for $x_1{}^n, \ldots, x_t{}^n$. The Koszul complex $K(\mathbf{x}^n, A^*)$ generated over A^* by $x_1{}^n, \ldots, x_t{}^n$ is isomorphic to $K(\mathbf{x}^n, A) \otimes_A A^*$. Moreover, since each module in the complex $K(\mathbf{x}^n, A)$ is finite and free over the Noetherian ring A,

$$\mathrm{Hom}_{A^*}(K(\mathbf{x}^n, A^*), A^*) \cong \mathrm{Hom}_A(K(\mathbf{x}^n, A), A) \otimes_A A^*.$$

Now for $n' > n > 0$, the A-linear map $\mathbf{x}^n \to \mathbf{x}^{n'}$ from A^t to itself induces maps of the cocomplexes $\mathrm{Hom}(K(\mathbf{x}^n, A), A) \to \mathrm{Hom}(K(\mathbf{x}^{n'}, A), A)$. Let $\varphi_{n,n'}$ be the corresponding map

$$\begin{array}{ccc} H^i(\mathbf{x}^n, A) & \longrightarrow & H^i(\mathbf{x}^{n'}, A) \\ \| & & \| \\ H^i(\mathrm{Hom}_A(K(\mathbf{x}^n, A), A) & & H^i(\mathrm{Hom}_A(K(\mathbf{x}^{n'}, A), A) \end{array}$$

Consider the diagram

$$\begin{array}{ccc} H^i(\mathbf{x}^n, A) \otimes_A A^* & \xrightarrow{\varphi_{n,n'} \otimes 1} & H^i(\mathbf{x}^{n'}, A) \otimes_A A^* \\ \| & & \downarrow \cong \\ H^i(\mathrm{Hom}_A(K(\mathbf{x}^n, A)) \otimes_A A^* & \longrightarrow & H^i(\mathrm{Hom}_A(K(\mathbf{x}^{n'}, A)) \otimes_A A^*) \\ \downarrow \cong & & \downarrow \cong \\ H^i(\mathbf{x}^n, A^*) & \xrightarrow{\varphi^*{}_{n,n'}} & H^i(\mathbf{x}^{n'}, A^*) \end{array}$$

The vertical maps are isomorphisms and the upper square commutes by Theorem 1, p. 93 of [**7**]. It is easy to check that the lower square also commutes. The maps

$$H^i(\mathbf{x}^n, A) \otimes_A A^* \overset{\cong}{\rightrightarrows} H^i(\mathbf{x}^n, A^*)$$

induce an isomorphism

$$\varphi: \varinjlim_n [H^i(\mathbf{x}^n, A) \otimes_A A^*] \to \varinjlim_n H^i(\mathbf{x}^n, A^*)$$

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which renders the lower part of the following diagram commutative.

$$\begin{array}{ccc}
H^i(\mathbf{x}, A) \otimes_A A^* & \xrightarrow{\varphi_A{}^i \otimes 1} & [\lim\limits_{\overrightarrow{n}} H^i(\mathbf{x}^n, A)] \otimes_A A^* \cong H_m{}^i(A) \otimes_A A^* \\
\Big\downarrow \cong & \searrow & \Big\downarrow \cong \\
 & & \lim\limits_{\overrightarrow{n}} [H^i(\mathbf{x}^n, A) \otimes_A A^*] \\
 & & \varphi \Big\downarrow \cong \\
H^i(\mathbf{x}, A^*) & \xrightarrow{\varphi_{A^*}{}^i} & \lim\limits_{\overrightarrow{n}} H^i(\mathbf{x}^n, A^*) \cong H_{m^*}{}^i(A^*)
\end{array}$$

Since tensor products commute with direct limits, the upper triangle commutes as well. The isomorphisms on the right are established in [**1**], Theorem 2.3.

Now assume A is a B-ring. Then by Stückrad's lemma,

$$\varphi_A{}^i : H^i(\mathbf{x}, A) \rightarrow \lim_{\overrightarrow{n}} H^i(\mathbf{x}^n, A)$$

is surjective for $i \neq d = \dim(A)$. Since tensor products are right exact, $\varphi_A{}^i \otimes 1$ is surjective for $i \neq d$. Chasing the above diagram yields that $\varphi_{A^*}{}^i$ is also surjective for $i \neq d = \dim(A^*)$.

Conversely, if A^* is a B-ring, $\varphi_{A^*}{}^i$ surjective implies $\varphi_A{}^i \otimes 1$ is surjective. Then by the faithful flatness of $A \rightarrow A^*$, $\varphi_A{}^i$ is also surjective. Hence A is a B-ring.

Remark. The sufficiency can be proven without homological methods. Using a theorem of D. Rees, [**4**], p. 277, (even though the theorem is stated there for 1-dimensional rings it is correct for arbitrary dimensions) it is possible to show that $l(A/q) - e_0(q, A)$ is independent of the choice of the parameter ideal q, since the same condition holds in A^* if A^* is a B-ring.

Combining Theorems 1 and 3 gives:

COROLLARY. *Let R be a Noetherian graded domain with $p \in \mathrm{Proj}(R)$ such that $R_1 \backslash p \neq \emptyset$. Then $R_{(p)}$ is a Buchsbaum ring if and only if R_p is a Buchsbaum ring.*

In conclusion we deduce

THEOREM 4. *Let $X \subseteq \mathbf{P}_k{}^n$ be an irreducible projective variety over an algebraically closed field. If the vertex of the associated cone $C(X) \subseteq \mathbf{A}_k{}^{n+1}$ is a Bauchsbaum singularity, then X is geometrically Cohen-Macaulay. That is, $R_{(p)}$ is C.M. for all $p \in \mathrm{Proj}(R)$, where R is the homogeneous coordinate domain of X.*

Proof. If $R_{(x_0, \ldots, x_n)}$ is a B-ring, then

$$(R_{(x_0, \ldots, x_n)})_p \cdot R_{(x_0, \ldots, x_n)} \cong R_p$$

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is Cohen-Macaulay for all $p \in \text{Proj}(R)$ [**11**], Remark p. 439. Thus by the Corollary to Theorem 2, $R_{(p)}$ is C.M. for all $p \in \text{Proj}(R)$.

In order then to produce varieties with non-Buchsbaum points, consider the associated cone, $C(X)$, of any irreducible variety $X \subseteq \mathbf{P}_k^n$ which contains a (geometrically) non-Cohen-Macaulay point. By Theorem 4, the vertex of $C(X)$ cannot be a Buchsbaum singularity.

Addendum. Theorem 2 and its Corollary hold for complete intersections: *A (resp. $R_{(p)}$) is a* C.I. *if and only if A^* (resp. R_p) is a* C.I. See L. L. Avramov, *Homology of local flat extensions and complete intersection defects*, Math. Ann. *228* (1977), 27–37 for the necessary result on flat, local extensions.

Also, the proof of Theorem 3 can be used verbatim to show that a local ring is a B-ring if and only if its completion is. This same result with a different proof appeared as Lemma 4.7 in P. Schenzel, N.V. Trung and N. T. Cuong, *Verallgemeinerte Cohen-Macaulay Moduln*, Math. Nachr. *85* (1978), 57–73.

References

1. A. Grothendieck, *Local cohomology*, Springer Lecture Notes *41* (Berlin, Heidelberg, 1967).

2. W. E. Kuan, *Some results on normality of a graded ring*, Pacific J. Math. *64* (1976), 455–463.

3. C. Lech, *Inequalities related to certain couples of local rings*, Acta. Math. *112* (1964), 69–89.

4. E. Matlis, *The multiplicity and reduction number of a one dimensional local ring*, Proc. London Math. Soc. (3) *26* (1973), 273–288.

5. H. Matsumura, *Commutative algebra* (W. A. Benjamin, New York, 1970).

6. D. G. Northcott, *Ideal theory* (Cambridge Univ. Press, 1953).

7. ——— *An introduction to homological algebra* (Cambridge Univ. Press, 1962).

8. C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie local*, IHES Publ. Math. *42*.

9. J. Stückrad, *Über die kohomologische Charakterisierung von Buchsbaum-Moduln*, Math. Nachr. (to appear).

10. J. Stückrad and W. Vogel, *Eine Verallgemeinerung der Cohen-Macaulay Ringe und Anwendungen auf ein Problem der Multiplizitätstheorie*, J. Math. Kyoto Univ. *13*-3 (1973), 513–528.

11. ——— *Über das Amsterdamer Programm von W. Gröbner*, Monatshefte für Math. *78* (1974), 433–445.

12. W. Vogel, *A non-zero divisor characterization of Buchsbaum modules*, Preprint.

13. K. Watanabe, T. Ishikawa, S. Tachibana, and K. Otsuka, *On tensor products of Gorenstein rings*, J. Math. Kyoto Univ. *9*-3 (1969), 413–423.

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