A DOUBLE-INFINITY CONFIGURATION

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The double-six configuration in classical 3-dimensional projective geometry has been discussed by a number of authors.[1] It consists of two sets $a_1, \cdots, a_6$ and $b_1, \cdots, b_6$ of six lines such that no two lines of the same set intersect, and a_i meets b_j if and only if $i \neq j$. The existence of a double-six in the 3-dimensional projective geometry over a field F has been proved by Hirschfeld in [2] for all fields F except those of 2, 3 and 5 elements. For an arbitrary 3-dimensional projective geometry in which the number of points on a line is at least 5 but is not 6, the existence of a double-six follows from the fact that the geometry is a geometry over a division ring D with a subfield F satisfying the conditions of Hirschfeld's theorem.

In the classical theory, it is proved that no line a_7 meeting $b_1, \cdots b_6$ can be added to the system. This proof depends on the commutativity of the coordinate system. We show that it is possible for a non-Pappusian geometry to contain a double configuration with infinitely many lines in each set.

Let D be a division ring. We denote by $\Gamma(D)$ the 3-dimensional projective geometry over D. A system consisting of two sets $\{a_i | i \in I\}$ and $\{b_i \, | i \in I\}$ of lines of $\Gamma(D)$, each indexed by the index set I of cardinal c, is called a double-c configuration if

(i) no two members of the same set intersect, and
(ii) a_i meets b_j if and only if $i \neq j$.

THEOREM. *Let c be any cardinal number. Then there exists a 3-dimensional projective geometry in which there is a double-c configuration.*

PROOF. Let V be a 4-dimensional left vector space over a division ring D. Then the points, lines and planes of $\Gamma(D)$ are the 1-, 2- and 3-dimensional subspaces of V. We denote by $\langle v_1, \cdots, v_r \rangle$ the subspace of V spanned by the elements $v_1, \cdots, v_r \in V$. Let e_1, e_2, e_3, e_4 be a basis of V. For $\alpha \in D$, let m_α, n_α be the lines $\langle e_1 + \alpha e_2, e_3 + \alpha e_4 \rangle$, $\langle e_1 + \alpha e_3, e_2 + \alpha e_4 \rangle$ respectively. Then $M = \{m_\alpha | \alpha \in D\}$ and $N = \{n_\alpha | \alpha \in D\}$ are families of

[1] For an account of the classical theory, see Baker [1] pp. 159—164.

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skew lines. The line m_α meets n_β if and only if $\alpha\beta = \beta\alpha$. If $\{\alpha_i | i \in I\}$ and $\{\beta_i | i \in I\}$ are subsets of D satisfying the condition

$$(*) \qquad \alpha_i\beta_j = \beta_j\alpha_i \text{ if and only if } i \neq j,$$

then the sets of lines $\{m_{\alpha_i} | i \in I\}$ and $\{n_{\beta_i} | i \in I\}$ form a double-c configuration, where c is the cardinal of I.

It remains to show that, for given c, there exists a division ring D with two subsets $\{\alpha_i | i \in I\}$ and $\{\beta_i | i \in I\}$, each indexed by a set I of cardinal c, satisfying the condition $(*)$. The following construction was suggested by B. H. Neumann. We take any set I of cardinal c. For each $i \in I$, let F_i be the free group on the two generators α_i, β_i. Let G be the restricted direct product of the F_i. Then the subsets $\{\alpha_i | i \in I\}$ and $\{\beta_i \in I\}$ of G satisfy $(*)$. We embed G in the multiplicative group of a division ring.

By Neumann [4] Corollary 3.3, each F_i can be ordered. By [4] Theorem 3.6, G can be ordered. By Neumann [3] Theorem 5.9, this implies that G can be embedded in the multiplicative group of a division ring D. This division ring D clearly has the required properties.

References

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[4] B. H. Neumann, 'On ordered groups,' *Amer. J. Math.* 71 (1949) 1—18.

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