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Corrigendum

‘Well-posedness of some initial-boundary-value problems for dynamo-generated poloidal magnetic fields’

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In theorem 3.3 of our paper we gave an incomplete characterization of the spaces $D_{k/2}$ associated with the operators $\mathcal{A}^{k/2}$, $k \in \mathbb{N}$, and, as a consequence, we missed compatibility conditions in the subsequent theorem 4.5 and corollary 4.6 therein. In this corrigendum we give corrected versions of these results.

We start with an additional lemma which improves the regularity result (2.9*b*) in [3] and provides an estimate needed in the subsequent theorem.

(Henceforth, equation numbers of the form (*a*.*b*) refer to [3].)

Lemma 1. *Let $G \subset \mathbb{R}^d$, $d \geqslant 3$ be a bounded domain with C^{k+2}-boundary ∂G and $f \in H^k(G)$, $k \in \mathbb{N}_0$. Furthermore, let $u \in \mathcal{H}_0$ be a weak solution of problem (2.1). Then $u \in H^{k+2}(G)$ and we have the bound*

$$\|u\|_{H^{k+2}(G)} \leqslant C\|f\|_{H^k(G)} = C\|\Delta u\|_{H^k(G)} \tag{1}$$

with a constant C depending on G, k and d.

Proof. The case $k = 0$ is already implied by the interior regularity result (2.9*a*). In fact, $u \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ means (see [1, p. 309])

$$\|u\|_{H^2(G)} \leqslant \hat{C}(\|\hat{f}\|_{L^2(K)} + \|u\|_{L^2(K)}), \tag{2}$$

where $\hat{f}$ denotes the trivial extension of f onto $\mathbb{R}^d$ and K denotes some bounded domain such that $G \Subset K$. Combining (2.6) with the boundedness of the Green operator $\tilde{\mathcal{G}}$, we obtain

$$\|u\|_{L^2(K)} \leqslant C_K\|u\|_{\mathcal{H}} \leqslant C_K C_G\|f\|_{L^2(G)}, \tag{3}$$

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and thus (2) takes the form

$$\|u\|_{H^2(G)} \leqslant C\|f\|_{L^2(G)}. \tag{4}$$

No boundary regularity is required for this result.

The case $k > 0$ needs separate considerations of tangential and normal derivatives at ∂G. We refer in the following to the situation where ∂G has already been flattened, as explained in the paragraph before lemma 2.1 in [3], and we use the notation introduced there. So, given $g \in L^2(W^-)$, we assume $v \in H^1(W)$ to be a (weak) solution of

$$\sum_{i,j=1}^{d} \int_W a_{ij}\partial_{y_j} v \partial_{y_i} w \,\mathrm{d}y = \int_{W^-} gw \,\mathrm{d}y \tag{5}$$

for any $w \in H_0^1(W)$. Let $\hat{g}$ be again the trivial extension of g onto W. Now we assume higher tangential regularity of g, i.e. $D^\alpha g \in L^2(W^-)$ for $|\alpha| \leqslant k$, $\alpha_d = 0$, which implies $D^\alpha \hat{g} \in L^2(W)$. From interior regularity for weak solutions it follows that $D^\beta v \in L^2(V)$ for $|\beta| \leqslant k+2$, $\beta_d \leqslant 2$ and any $V \Subset W$, together with the estimate

$$\sum_{\substack{\beta_d \leqslant 2 \\ |\beta| \leqslant k+2}} \int_{V^-} |D^\beta v|^2 \,\mathrm{d}y \leqslant C\biggl(\sum_{\substack{\alpha_d = 0 \\ |\alpha| \leqslant k}} \int_{W^-} |D^\alpha g|^2 \,\mathrm{d}y + \|v\|^2_{L^2(W)}\biggr). \tag{6}$$

As to normal derivatives, note that higher interior regularity implies

$$-D^\alpha \sum_{i,j=1}^{d} \partial_{y_i}(a_{ij}\partial_{y_j} v) = D^\alpha g \tag{7}$$

to hold almost everywhere (a.e.) in W^-. Writing (7) with $\alpha = (0, \dots, 0, 1)$ in the form

$$a_{y_d y_d}\partial^3_{y_d} v = -\sum_{\substack{i+j<2d \\ i,j=1}}^{d} \partial_{y_d}\partial_{y_i}(a_{ij}\partial_{y_j} v) - 2\partial_{y_d} a_{y_d y_d}\partial^2_{y_d} v - \partial^2_{y_d} a_{y_d y_d}\partial_{y_d} v - \partial_{y_d} g, \tag{8}$$

we find, by uniform ellipticity, $\partial^3_{y_d} v$ to be bounded in W^- by the right-hand side of (8), which is at most of second order in $\partial_{y_d} v$. So, (6) may be applied, and we arrive at

$$\int_{V^-} |\partial^3_{y_d} v|^2 \,\mathrm{d}y \leqslant \tilde{C}\biggl(\sum_{\substack{\alpha_d = 0 \\ |\alpha| \leqslant k}} \int_{W^-} |D^\alpha g|^2 \,\mathrm{d}y + \|v\|^2_{L^2(W)}\biggr). \tag{9}$$

The case of arbitrary higher derivatives is now easily proved by induction. So, we find, finally, that (6) holds without restriction on α_d and β_d.

To complete the proof, one has, as usual, to cancel the change of variables, to cover G by local patches, to sum up the corresponding local estimates and to use once more (3). □

THEOREM 2 (corrected version of [3, theorem 3.3]). *Let $G \subset \mathbb{R}^d$, $d \geqslant 3$, be a bounded domain with C^∞-boundary ∂G and let $\{v_n : n \in \mathbb{N}\}$ be the complete orthonormal*

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system defined by the eigenvalue problem (3.1). Furthermore, let $\mathcal{A}$ and D_α be as defined in definition 3.2, and $\mathcal{G}$ be the Green operator associated to the Poisson problem (2.1). Then

$$\begin{gathered} D_0 = H^0(G) = L^2(G), \\ D_{1/2} = \{v|_G \colon v \in \mathcal{H}_0 \text{ and } v|_{\hat{G}} \text{ is harmonic}\} = H^1(G), \end{gathered} \tag{10}$$

i.e. in particular, any $v \in D_{1/2}$ has a unique harmonic extension $\tilde{v} \in \mathcal{H}_0$, and

$$D_1 = \mathcal{G}(L^2(G)) = \{v \in H^2(G) \colon \tilde{v} \in H^2_{\mathrm{loc}}(\mathbb{R}^d)\}. \tag{11}$$

Higher-order spaces are characterized by

$$D_{k/2} = \{v \in H^k(G) \colon \widetilde{\Delta^{i-1} v} \in H^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ for } i = 1, \dots, [k/2]\}, \quad k \in \mathbb{N} \setminus \{1\}, \tag{12}$$

where $\tilde{w} \in \mathcal{H}_0$ denotes the harmonic extension of a function $w \in D_{1/2} = H^1(G)$ and $[r] := \max\{j \in \mathbb{N} \colon j \leqslant r\}$ is the integer part of r. On $D_{k/2}$ we have the equivalence of norms:

$$\| \cdot \|_{k/2} \sim \| \cdot \|_{H^k}, \quad k \in \mathbb{N}_0. \tag{13}$$

Proof. (Concerning notation, observe that the symbols L^2 and H^k without specified domains always mean $L^2(G)$ and $H^k(G)$, respectively.) The case $k = 0$ is trivial and the case $k = 1$, i.e. (10) and $(13)_{k=1}$, is proved as in theorem 3.3 in [3]. As to $k = 2$, the proof of the first equality in (11) remains likewise untouched. The proof of the second equality in (11) (and all the rest of the proof), however, now differs from the proof in [3] in the following way.

The inclusion $\mathcal{G}(L^2(G)) \subset \{v \in H^2(G) \colon \tilde{v} \in H^2_{\mathrm{loc}}(\mathbb{R}^d)\}$ is an immediate consequence of the H^2-regularity of weak solutions. To prove the opposite inclusion, let $w \in H^2(G)$ with harmonic extension $\tilde{w} \in \mathcal{H}_0 \cap H^2_{\mathrm{loc}}(\mathbb{R}^d)$. Defining $f := -\Delta w \in L^2$, the Poisson problem (2.1) yields a solution $\tilde{u} \in \mathcal{H}_0 \cap H^2_{\mathrm{loc}}(\mathbb{R}^d)$. So, we have pointwise a.e. $\Delta(\tilde{w} - \tilde{u}) = 0$ in $\mathbb{R}^d$ for $\tilde{w} - \tilde{u} \in \mathcal{H}_0 \cap H^2_{\mathrm{loc}}(\mathbb{R}^d)$, which means $\tilde{w} - \tilde{u}$ is harmonic in $\mathbb{R}^d$ (by Weyl's lemma), and, moreover, $\tilde{w} - \tilde{u} = 0$ (by Liouville's theorem). Thus, $\tilde{w} = \tilde{u}$ and, in particular, $w = u = \mathcal{G}(f)$.

To estimate the 1-norm of $v \in D(\mathcal{A})$ observe that $\tilde{v} \in \mathcal{H}_0 \cap H^2_{\mathrm{loc}}(\mathbb{R}^d)$ for its harmonic extension, and $v_n \in C^1(\mathbb{R}^d)$ for the eigenfunctions. So, by (3.3) we can calculate

$$-(\lambda_n v_n, v)_{L^2(G)} = -\int_{\mathbb{R}^d} \nabla \tilde{v}_n \cdot \nabla \tilde{v} \, \mathrm{d}x = \int_{\mathbb{R}^d} \tilde{v}_n \Delta \tilde{v} \, \mathrm{d}x = (v_n, \Delta v)_{L^2(G)} \tag{14}$$

and therefore obtain

$$\|v\|_1^2 = \|\mathcal{A}v\|_{L^2}^2 = \sum_{n=1}^{\infty} \lambda_n^2 |(v_n, v)_{L^2}|^2 = \sum_{n=1}^{\infty} |(v_n, \Delta v)_{L^2}|^2 = \|\Delta v\|_{L^2}^2, \tag{15}$$

which implies $\|v\|_1 \leqslant C\|v\|_{H^2(G)}$ with a constant C depending only on d. To prove the opposite inequality we combine (15) with $(1)_{k=0}$:

$$\|v\|_1 = \|\Delta v\| \geqslant \frac{1}{C} \|v\|_{H^2(G)}.$$

This proves $(13)_{k=2}$.

The case $k > 2$ is proved by induction. Let $v \in D_{k/2+1}$, $k \in \mathbb{N}$, i.e. $\mathcal{A}v \in D_{k/2}$. By assumption we have $\mathcal{A}v \in H^k(G)$ and

$$\widetilde{\Delta^{i-1}\mathcal{A}}v \in H^2_{\text{loc}}(\mathbb{R}^d) \quad \text{for } i = 1, \dots, [k/2]. \tag{16}$$

(Note that for $v \in D_{3/2}$ condition (16) does not yet make sense and can be omitted.) By (15) the condition $\mathcal{A}v \in H^k(G)$ means $\Delta v \in H^k(G)$, and lemma 1 implies $v \in H^{k+2}(G)$. Moreover, we have $\tilde{v} \in H^2_{\text{loc}}(\mathbb{R}^d)$, which complements condition (16). So, we conclude

$$v \in \{v \in H^{k+2}(G)\colon \widetilde{\Delta^{i-1}v} \in H^2_{\text{loc}}(\mathbb{R}^d) \text{ for } i = 1, \dots, [k/2] + 1\}. \tag{17}$$

To prove the opposite inclusion, let v be as in (17). We set $w := \Delta v$ and have, by assumption,

$$w \in \{v \in H^k(G)\colon \widetilde{\Delta^{i-1}v} \in H^2_{\text{loc}}(\mathbb{R}^d) \text{ for } i = 1, \dots, [k/2]\} = D_{k/2}.$$

Computing the $k/2 + 1$-norm of v, we find, with (14),

$$\|v\|^2_{k/2+1} = \sum_{n=1}^{\infty} \lambda_n^k |\lambda_n (v_n, v)_{L^2}|^2 = \sum_{n=1}^{\infty} \lambda_n^k |(v_n, w)_{L^2}|^2 = \|w\|^2_{k/2} < \infty, \tag{18}$$

and thus, $v \in D_{k/2+1}$. This completes the proof of (12).

As to the equivalence (13) we proceed likewise by induction. Assuming $v \in D_{k/2+1}$, $k \in \mathbb{N}$, we find, by (18) and by assumption,

$$\|v\|_{k/2+1} = \|\Delta v\|_{k/2} \leqslant C\|\Delta v\|_{H^k} \leqslant \tilde{C}\|v\|_{H^{k+2}},$$

whereas the opposite inequality follows by (1):

$$\|v\|_{H^{k+2}} \leqslant C\|\Delta v\|_{H_k} \leqslant \tilde{C}\|\Delta v\|_{k/2} = \tilde{C}\|v\|_{k/2+1}.$$

This completes the proof. □

Remark 3. Iterating (14), one finds on D_α for integer values α the following alternative formulation of the α-norm:

$$\|v\|_k = \|\Delta^k v\|_{L^2(G)}, \qquad v \in D_k,\ k \in \mathbb{N},$$

and for half-integer values:

$$\|v\|_{k+1/2} = \|\nabla \Delta^k v\|_{L^2(G)}, \quad v \in D_k,\ k \in \mathbb{N}.$$

In view of theorem 2 we must in the following discriminate between H^k and $D_{k/2}$. So, a corrected version of [3, theorem 4.5] on higher regularity now reads as follows.

Theorem 4 (corrected version of [3, theorem 4.5]). *Let $T > 0$, $k \in \mathbb{N} \setminus \{1\}$ and $a, b, c \in C^k_1(\bar{G} \times [0, T])$. Furthermore, let $v_0 \in D_{(k+1)/2}$, $-a(\cdot, 0)\mathcal{A}v_0 + \mathcal{B}|_{t=0}v_0 + f(0) \in D_{(k-1)/2}$ and $f \in C^1([0, T], H^k(G))$. Then the weak solution v of problem (4.2) in [3] satisfies*

$$v \in L^2((0, T), D_{k/2+1}), \qquad \dot{v} \in L^2((0, T), D_{k/2}), \qquad \ddot{v} \in L^2((0, T), D_{k/2-1}).$$

Theorem 4 differs from theorem 4.5 in [3] (in addition to the fact that $f \in C^1([0,T], D_{k/2})$ has been replaced by $f \in C^1([0,T], H^k(G))$) by the additional condition $-a(\cdot,0)\mathcal{A}v_0 + \mathcal{B}|_{t=0}v_0 + f(0) \in D_{(k-1)/2}$, which is now no longer implied by the conditions on v_0 and the coefficients, and must, therefore, be stated explicitly. The condition is needed in the proof of $\ddot{v} \in L^2((0,T), D_{k/2-1})$. In fact, differentiating $(4.4)_1$ with respect to t yields an equation of type (4.17) and applying $\mathcal{A}^{k/2-1}$ results in an evolution equation for $\mathcal{A}^{k/2-1}\dot{v} =: \omega$. Together with the initial value $\omega_0 := \mathcal{A}^{k/2-1}(-a(\cdot,0)\mathcal{A}v_0 + \mathcal{B}|_{t=0}v_0 + f(0)) \in D_{1/2}$, we have then an initial-value problem to which theorem 4.3 in [3] applies with the result (among others) $\dot{\omega} \in L^2((0,T), L^2(G))$ and hence $\ddot{v} \in L^2((0,T), D_{k/2-1})$. Otherwise the proof of theorem 4.5 in [3] remains unchanged. Of course, higher temporal regularity would require further compatibility conditions. For classical solutions, however, the regularity stated in theorem 4 is enough.

As to corollary 4.6 in [3], observe that $u \in C^2(\bar{G})$ and $\tilde{u} \in C^1(\mathbb{R}^d)$ imply $\tilde{u} \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$. Thus, the corollary now takes the following supplemented form, where again we aim at sufficient (and not necessarily sharp) conditions in terms of classical derivatives for existence of classical solutions.

Corollary 5 (corrected version of [3, corollary 4.6]). *Let $G \subset \mathbb{R}^d$, $d \geqslant 3$, be a bounded domain with $C^{k+3/2}$-boundary, $k > 1 + d/2$, $u_0 \in C^{k+1}(\bar{G})$ and $a, b, c \in C^k_1(\bar{G} \times [0,T])$, $u_\infty \in C^2([0,T])$ for any $T > 0$. Furthermore, let $u_0 - u_\infty(0)$, $\Delta^i u_0$, and $\Delta^{i-1}(a_0\Delta u_0 + b_0 \cdot \nabla u_0 + c_0 u_0 - \dot{u}_\infty(0))$, $i = 1, \dots, [(k-1)/2]$, where $a_0 = a(\cdot, 0)$, etc., all C^1-match to their harmonic extensions. Then problem (1.1) has a unique classical solution u, i.e. $u \in C^2_1(G \times \mathbb{R}_+) \cap C^2(\hat{G} \times \mathbb{R}_+)$ satisfies pointwise equations (1.1).*

Remark 6. In $d = 3$ we may choose $k = 3$, and the compatibility conditions amount to

$$(u_0 - u_\infty(0))^{\sim}, \quad \widetilde{\Delta u_0} \in C^1(\mathbb{R}^3) \tag{19}$$

and

$$(a_0\Delta u_0 + b_0 \cdot \nabla u_0 + c_0 u_0 - \dot{u}_\infty(0))^{\sim} \in C^1(\mathbb{R}^3). \tag{20}$$

So, in the case $u_\infty = 0$ admissible initial values u_0 are, for instance, $C^4(\bar{G})$-functions with $\partial_n^i u_0|_{\partial G} = 0$, $i = 0, \dots, 3$, where ∂_n denotes the normal derivative at ∂G. In the case when $u_0 = u_\infty = \text{const.} > 0$, which was interesting in applications [2], condition (20) requires the coefficient c_0 to have a C^1-smooth harmonic extension.

Remark 7. Appendix E in [3], which provides simpler proofs in the case of a time-independent principal coefficient, now loses some of its significance. The idea was to absorb the principal coefficient a into the definition of the operator $\mathcal{A} =: \mathcal{A}_a$. In that case, the sequence $(w^{(n)})$ of Galerkin approximations can be shown to converge in $C([0,T], D_{1/2})$ to some limit function w, and $\dot{w} \in C^1([0,T], D_{-1/2})$ follows then by the evolution equation $(\mathrm{E}2)_1$, since $w^{(n)} \in C([0,T], D_{1/2})$ implies $-\mathcal{A}_a w^{(n)} + Q^{(n)}(\mathcal{B}w^{(n)} + f) \in C([0,T], D_{-1/2})$. The latter conclusion, however, no longer works if $D_{1/2}$ is replaced by $D_{k/2}$ and $D_{-1/2}$ is replaced by $D_{k/2-1}$, with $k > 3$, since the lower-order terms do not preserve the boundary behaviour that is now required for elements of $D_{k/2}$, $k > 1$. So, theorem E.2 now holds only in the case

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of vanishing lower-order coefficients, i.e. $b = c = 0$, while a compatibility condition involving the principal coefficient a arises from the condition $w_0 \in D_{(k+1)/2}$ as $\mathcal{A}_a$ corresponds to $-a\Delta$ on H^2-functions.

We take the opportunity to correct another blunder in the proof of theorem E.2: of course, $Q^{(n)} - Q^{(m)}$ is always a projection operator with norm 1 as long as $n > m$. Nevertheless, with $f = \sum_{n=1}^{\infty} c_n w_n \in C([0,T], L^2(G))$, the norm

$$\max_{[0,T]} \|(Q^{(n)} - Q^{(m)})f\|_{L^2_a} = \max_{[0,T]} \left(\sum_{\nu=m+1}^{n} |c_\nu(t)|^2 \right)^{1/2}$$

clearly vanishes in the limit $n, m \to \infty$. The same argument applies to the projected initial value $(Q^{(n)} - Q^{(m)})w_0$, whereas the lower-order term $Q^{(n)}\mathcal{B}w^{(n)} - Q^{(m)}\mathcal{B}w^{(m)}$ is no longer present.

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