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RESEARCH ARTICLE

The Atiyah class on algebraic stacks

Nikolas Kuhn

University of Oslo, Moltke Moes vei 35, 0851 Oslo, Norway; E-mail: ntkuhn@posteo.net
Current affiliation: Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG; E-mail: kuhnn@maths.ox.ac.uk.

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Abstract
We generalize Illusie's definition of the Atiyah class to complexes with quasi-coherent cohomology on arbitrary algebraic stacks. We show that this gives a global obstruction theory for moduli stacks of complexes in algebraic geometry without derived methods. We give a similar generalization of the reduced Atiyah class, and we show various useful properties for working with Atiyah classes, such as compatibilities between the reduced and ordinary Atiyah class, and compatibility with tensor products and determinants.

It is a classical fact that the deformation-obstruction theory of a coherent sheaf E on a projective variety X is governed by the groups $\mathrm{Ext}^i_X(E, E)$ for $i = 1, 2$ [5, Section 7]. The Atiyah class, as defined in [7] in the algebraic setting, globalizes and generalizes this correspondence – roughly speaking, it measures how much a family of sheaves varies over a base. When working over a moduli space, these self-Ext groups often give an important extra structure in the form of an *obstruction theory* (often with additional properties), which is the foundational ingredient in enumerative sheaf theories. Examples are the famous Donaldson–Thomas theory [23], PT-theory [18], [6], moduli spaces of stable sheaves on surfaces [14] and, more recently, CY4 theory [15]. In each of these cases, the technical tool used to obtain the obstruction theory is the Atiyah class.

In many cases, particularly for wall-crossing arguments as in [14], [8], [9], it is necessary to consider moduli stacks that include properly semi-stable objects which may have positive-dimensional stabilizers. To work with these, one would like an obstruction theory *on the stack*. This is constructed in [14] for moduli stacks of objects with a two-term resolution, and in [8] by using derived moduli stacks of perfect complexes.

The main result of this paper is to generalize Illusie's construction of the Atiyah class to algebraic stacks, using Olsson's definition of the cotangent complex [16]. We also treat some variants, such as Gillam's reduced Atiyah class [3], and a version of the Atiyah class for exact sequences. We then show various compatibility properties for the Atiyah class and its variants.

As a second main result, we show that the Atiyah class indeed gives an obstruction theory for moduli stacks of perfect complexes (Theorem 1.4). The main new part here is that it captures the infinitesimal automorphisms. For simplicity, we only treat the absolute case of a proper scheme over a field, although the relative case follows along the same lines.

Roughly, our construction of the Atiyah class proceeds by presenting a given algebraic stack $\mathcal{X}$ as a groupoid $W \rightrightarrows X$ in algebraic spaces.

Then, up to descent, the Atiyah class of a sheaf E on $\mathcal{X}$ should morally be obtained by taking mapping cones of a commutative square involving the Atiyah classes of the pullback of E to X and W, respectively. Since taking mapping cones in the derived category is not functorial, some work is needed to make this into a definition. For this purpose, in §2.3, we introduce a topos $W_{||}$ associated to the groupoid $W \rightrightarrows X$, whose objects are certain diagrams involving sheaves of W and X. Then the desired mapping cone operation can be encoded as a functor that takes a complex of sheaves of modules on $W_{||}$ and yields a complex on W. With this technical tool, the strategy to define the Atiyah class goes through. (From a higher categorical viewpoint, the use of $W_{||}$ allows us to keep track of the necessary coherence data used in the pushout operation.)

Relation to existing work

Throughout, we build on the constructions and results of [6], [7] and [3], which we generalize to algebraic stacks. Our construction also recovers the G-equivariant Atiyah class considered by Ricolfi [20]. Throughout, we use Olsson's definition of the quasi-coherent derived category and the cotangent complex for algebraic stacks developed in [16] and [10]; see also the excellent discussion in [4, §1].

Generalizations of the Atiyah class to the theory of algebraic stacks have appeared before in different contexts: In the setting of derived algebraic geometry of Schürg, Toën and Vezzosi [21], a perfect complex E on a derived geometry stack $\mathcal{Y}$ with a perfect cotangent complex gives rise to a map from $\mathcal{Y}$ to the (derived) moduli stack of perfect complexes. In ([21], Appendix A), they define the Atiyah class as the induced pullback map on cotangent complexes and argue that it recovers Illusie's definition in the case of schemes. Moreover, Lurie has constructed the Atiyah class in the context of spectral algebraic geometry [12, §19.2.2]. In the purely classical setting, our definition goes beyond these, as we do not require the perfectness assumptions made in [21], and since the current version of [12] presently only deals with Deligne–Mumford stacks.[1]

Finally, the paper [1] shows how to construct virtual cycles from perfect obstruction theories on algebraic stacks and thus fits in neatly with the viewpoint taken here, that it is often advantageous to consider fundamental constructions directly on the moduli stack.

Notations and conventions

For a Grothendieck topos T and a ring R in T, we let $\mathrm{Mod}(R)$ denote the category of R-modules and $D(R)$ its derived category. We identify $\mathrm{Mod}(R)$ with the subcategory of $D(R)$ generated by complexes concentrated in degree zero.

For any abelian category $\mathcal{A}$, write $C(\mathcal{A})$ for the category of complexes of objects in $\mathcal{A}$. Write $C^{\leq 0}(\mathcal{A})$ and $C^{[-1,0]}(\mathcal{A})$ for the full sub-category of complexes bounded in degrees ≤ 0 and in degrees $-1, 0$, respectively. When $\mathcal{A} = \mathrm{Mod}(R)$, we simply write $C(R)$ and $C^{\leq 0}(R)$, $C^{[-1,0]}(R)$, respectively.

We will use the following convention regarding shift functors: For any complex E of R-modules, we have a natural isomorphism

$$E[1] = \mathbb{Z}[1] \otimes_{\mathbb{Z}} E.$$

If E, F are complexes, then by $E \otimes F[1]$ we mean $(E \otimes F)[1]$ rather than $E \otimes (F[1])$.

All tensor products, pullbacks and duals of objects in a derived category be in the derived sense. For modules and complexes, we will consider the underived tensor products and pullbacks but will point out when these do not necessarily compute the derived operation.

We use the definition of algebraic stacks as in the Stacks project [22, Tag 026N].

1. Statements of results

In this section, we introduce the Atiyah class on an algebraic stack and several variants, and we state their basic properties and mutual relations. The proofs will be given in the following sections.

[1]We thank the referee for explaining these results in derived geometry.

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1.1. The Atiyah class

The Atiyah class of a vector bundle E on a smooth scheme X is a linear map that turns a vector field v on X into a class $\alpha_v \in \mathrm{Ext}^1(E, E)$ that measures how E varies in the directions of v and is compatible with any pullbacks of schemes. Via the cup product, it also gives rise to a map $H^1(T_X) \to \mathrm{Ext}^2(E, E)$ – here, the source is naturally identified with the collection of infinitesimal deformations of X via the Kodaira–Spencer map. This provides an *obstruction class* to extending E over infinitesimal deformations X' of X: One can extend E to a vector bundle on X' if and only if the class in $\mathrm{Ext}^2(E, E)$ obtained from X' is zero. An appropriate way to write the Atiyah class that generalizes beyond the case of smooth schemes is as a map

$$E \to L_X \otimes E[1]$$

in the derived category of X, where L_X denotes the *cotangent complex* of X as defined by Illusie [7].[2] If one further replaces X by an algebraic stack $\mathcal{X}$, one would like that the Atiyah class also captures how the (infinitesimal) stabilizer groups of $\mathcal{X}$ act on E. This is where we pick up.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and let $E \in D^-_{qcoh}(\mathcal{X})$. In §4.1, we define the *Atiyah class of E over $\mathcal{Y}$*, which is a natural map

$$\mathrm{at}_E : E \to L_{\mathcal{X}/\mathcal{Y}} \otimes E[1].$$

Here, $L_{\mathcal{X}/\mathcal{Y}}$ denotes Olsson's generalization of the relative cotangent complex to algebraic stacks [16], which we review in §2.1. We also use the notation $\mathrm{at}_{E,\mathcal{X}/\mathcal{Y}}$ when we want to emphasize the dependence on f. If E is dualizable in the derived category (equivalently, a perfect complex; see [4, Lemma 4.3]), then the data of at_E is equivalent to that of a map

$$\mathrm{at}'_E : E \otimes E^\vee[-1] \to L_{\mathcal{X}/\mathcal{Y}},$$

which we also call the Atiyah class.

We now list a series of fundamental properties of the Atiyah class. The proofs will be given in §5: Let F, E be objects of $D^-_{qcoh}(\mathcal{X})$.

Functoriality
Given a map $F \to E$ in $D^-_{qcoh}(\mathcal{X})$, the induced diagram

$$\begin{tikzcd} F \arrow[r, "\mathrm{at}_F"] \arrow[d] & L_{\mathcal{X}/\mathcal{Y}} \otimes F[1] \arrow[d] \\ E \arrow[r, "\mathrm{at}_E"] & L_{\mathcal{X}/\mathcal{Y}} \otimes E[1] \end{tikzcd}$$

commutes.

Pullback
Given another morphism $f' : \mathcal{X}' \to \mathcal{Y}'$ together with maps $A : \mathcal{X}' \to \mathcal{X}$ and $B : \mathcal{Y}' \to \mathcal{Y}$, and a 2-isomorphism $B \circ f' \Rightarrow f \circ A$, the induced diagram

$$\begin{tikzcd} A^*E \arrow[r, "A^*\mathrm{at}_E"] \arrow[d] & A^*L_{\mathcal{X}/\mathcal{Y}} \otimes A^*E[1] \arrow[d] \\ A^*E \arrow[r, "\mathrm{at}_{A^*E}"] & L_{\mathcal{X}'/\mathcal{Y}'} \otimes A^*E[1]. \end{tikzcd}$$

[2]If X is smooth, this is just the dual of the tangent bundle (i.e., the sheaf of differentials on X). In general, it is a somewhat complicated object in the derived category of X that retains the good properties of differentials from the smooth case (e.g., the exact sequence of relative differentials).

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commutes. If E is perfect, then equivalently, the diagram

$$\begin{tikzcd} A^*E \otimes A^*E^\vee[-1] \arrow[r, "A^*\,\mathrm{at}'_E"] \arrow[dr, "\mathrm{at}'_{A^*E}"'] & A^*L_{\mathcal{X}/\mathcal{Y}} \arrow[d] \\ & L_{\mathcal{X}'/\mathcal{Y}'} \end{tikzcd}$$

commutes.

Tensor products

Identify $E \otimes L_{\mathcal{X}/\mathcal{Y}}[1] \otimes F \simeq L_{\mathcal{X}/\mathcal{Y}}[1] \otimes E \otimes F$ using the standard symmetry isomorphism of the derived tensor product. Then, up to this identification, we have an equality

$$\mathrm{at}_{E\otimes F} = \mathrm{at}_E \otimes F + E \otimes \mathrm{at}_F .$$

As a special case of this, if E and F are perfect and at_F is trivial (e.g., if F is pulled back from $\mathcal{Y}$), then the following diagram commutes:

$$\begin{tikzcd} E \otimes E^\vee[-1] \arrow[r, "\mathrm{at}'_E"] \arrow[d] & L_{\mathcal{X}/\mathcal{Y}} \arrow[d, equal] \\ E \otimes F \otimes E^\vee \otimes F^\vee[-1] \arrow[r, "\mathrm{at}'_{E\otimes F}"] & L_{\mathcal{X}/\mathcal{Y}}. \end{tikzcd}$$

Here, the left vertical map is induced by the diagonal map $\mathcal{O}_X \to F \otimes F^\vee$ and the symmetry isomorphisms of the tensor product.

Determinants

Suppose that E is perfect and consider the natural trace map $\mathrm{tr} : \mathrm{Hom}(E, L_{\mathcal{X}/\mathcal{Y}}[1] \otimes E) \to \mathrm{Hom}(\mathcal{O}_{\mathcal{X}}, L_{\mathcal{X}/\mathcal{Y}}[1])$. Then we have $\mathrm{at}_{\det(E)} = \mathrm{tr}(\mathrm{at}_E) \otimes \det(E)$ as morphisms $\det E \to L_{\mathcal{X}/\mathcal{Y}} \otimes \det E[1]$, at least when E has a global finite length resolution by locally free sheaves. In particular, if the latter condition holds, the following diagram commutes

$$\begin{tikzcd} \mathcal{O}_X[-1] \arrow[d] \arrow[dr, "\mathrm{at}'_{\det E}"] & \\ E \otimes E^\vee[-1] \arrow[r, "\mathrm{at}'_E"] & L_{\mathcal{X}/\mathcal{Y}}, \end{tikzcd}$$

where the left vertical map is induced by the natural diagonal map $\mathcal{O}_X \to E \otimes E^\vee$.

Pushforward

Consider a cartesian diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "p"] & \mathcal{Y}' \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

and suppose that $\mathcal{X}' \to \mathcal{X}$ is concentrated [4, Definition 2.4]. Let $E \in D^-_{qcoh}(\mathcal{X}')$. Then we have a commutative diagram

$$\begin{tikzcd} Rp_*E \arrow[r, "\mathrm{at}_{Rp_*E}"] \arrow[d] & L_{\mathcal{X}/\mathcal{Y}}[1] \otimes Rp_*E \arrow[d] \\ Rp_*E \arrow[r, "Rp_*(\mathrm{at}_E)"] & Rp_*(L_{\mathcal{X}'/\mathcal{Y}'}[1] \otimes E). \end{tikzcd} \tag{1.1}$$

Suppose that, moreover, the diagram is Cartesian and that the morphisms $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{X}$ are Tor-independent, and that E is a perfect complex. Then the right vertical morphism in (1.1) is an isomorphism, and this gives a natural identification $Rp_*(\mathrm{at}_E) = \mathrm{at}_{Rp_*E}$ as morphisms $Rp_*E \to L_{\mathcal{X}/\mathcal{Y}}[1] \otimes Rp_*E$. If, moreover, Rp_*E is perfect, this can be restated as commutativity of the following diagram:

$$\begin{tikzcd} Rp_*E \otimes (Rp_*E)^\vee[-1] \arrow[d] \arrow[dr] & \\ (Rp_*(E \otimes E^\vee))^\vee[-1] \arrow[r] & L_{\mathcal{X}/\mathcal{Y}}. \end{tikzcd}$$

1.2. The reduced Atiyah class

Let $f : \mathcal{X} \to \mathcal{Y}$ be a map of algebraic stacks and let $E \in D^-_{qcoh}(\mathcal{Y})$. Let

$$F \to f^*E \to G \xrightarrow{+1}$$

be an exact triangle in $D^-_{qcoh}(\mathcal{X})$ such that $R\operatorname{Hom}^{-1}(F, G) = 0$. For example, this applies if E is a sheaf and G is a quotient of the ordinary pullback of E as a quasi-coherent sheaf.

Then the *reduced Atiyah class* associated to this data is a natural map

$$\overline{\mathrm{at}}_{E,\mathcal{X}/\mathcal{Y},G} : F \to L_{\mathcal{X}/\mathcal{Y}} \otimes G.$$

If G is dualizable, this corresponds to a map

$$\overline{\mathrm{at}}'_{E,\mathcal{X}/\mathcal{Y},G} : F \otimes G^\vee \to L_{\mathcal{X}/\mathcal{Y}}.$$

We also write $\overline{\mathrm{at}}_E$ and $\overline{\mathrm{at}}'_E$ if the rest of the data is understood.

Proposition 1.1. *Assume that E, F and G are dualizable. We have the following compatibility between the reduced Atiyah class and the ordinary Atiyah class of f^*E: The diagram*

$$\begin{tikzcd} F \otimes G^\vee \arrow[r] \arrow[d] & f^*E \otimes f^*E^\vee \arrow[d] \\ L_{X/Y} \arrow[r] & f^*L_Y[1] \end{tikzcd}$$

anti-commutes.

1.3. Atiyah class of an exact sequence

Let $\mathcal{X} \to \mathcal{Y}$ be a map of algebraic stacks and let

$$\underline{E} := [0 \to F \to E \to G \to 0]$$

be an exact sequence of bounded above complexes of $\mathcal{O}_\mathcal{X}$ modules with quasi-coherent cohomology sheaves. Assume that the images of F, E, G in $D^-_{qcoh}(\mathcal{X})$ are perfect complexes and that their duals

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lie again in $D^-_{qcoh}(\mathcal{X})$ (the latter is automatic if, for example, $\mathcal{X}$ is quasi-compact). Then there is a canonical way to complete the natural map $F \otimes G^\vee \to E \otimes E^\vee$ in $D^-_{qcoh}(\mathcal{X})$ to a triangle

$$F \otimes G^\vee \to E \otimes E^\vee \to E \otimes E^\vee / F \otimes G^\vee \xrightarrow{+1} . \tag{1.2}$$

Moreover, there exists a natural mophism

$$\mathrm{at}_{\underline{E}} : \frac{E \otimes E^\vee}{F \otimes G^\vee}[-1] \to L_{\mathcal{X}/\mathcal{Y}},$$

which we call the *Atiyah class of the exact sequence* $\underline{E}$.

Proposition 1.2. *We have a natural commutative diagram*

$$\begin{tikzcd} F \otimes F^\vee[-1] \arrow[r] \arrow[dr, "\mathrm{at}_F"'] & \frac{E\otimes E^\vee}{F\otimes G^\vee}[-1] \arrow[d, "\mathrm{at}_{\underline{E}}"] \\ & L_{\mathcal{X}/\mathcal{Y}} \end{tikzcd}$$

where the horizontal map is the shift of the morphism

$$F \otimes F^\vee \simeq \frac{F \otimes E^\vee}{F \otimes G^\vee} \hookrightarrow \frac{E \otimes E^\vee}{F \otimes G^\vee}.$$

Proposition 1.3. *Let* $\mathcal{X} \xrightarrow{f} \mathcal{Y} \to \mathcal{Z}$ *be maps of algebraic stacks with f flat and let E be a bounded above complex of* $\mathcal{O}_{\mathcal{Y}}$*-modules with quasi-coherent cohomology. Let* $E_{\mathcal{X}} := f^*E$ *and suppose we are given an exact sequence* $\underline{E}_{\mathcal{X}}$ *of the form*

$$0 \to F \to E_X \to G \to 0.$$

Then we have a natural morphism of distinguished triangles

$$\begin{tikzcd} E_X \otimes E_X^\vee[-1] \arrow[r] \arrow[d] & \frac{E_X\otimes E_X^\vee}{F\otimes G^\vee}[-1] \arrow[r] \arrow[d] & F \otimes G^\vee \arrow[r, "+1"] \arrow[d] & {} \\ f^*L_{Y/Z} \arrow[r] & L_{X/Z} \arrow[r] & L_{X/Y} \arrow[r, "+1"] & {} \end{tikzcd}$$

Here, the morphisms in the upper row are shifts of the ones in (1.2), except for the last one which is minus the map $F \otimes G^\vee \to E_X \otimes E_X^\vee$ *provided there.* [3]

1.4. Deformation theoretic properties

We present two important examples of how the Atiyah class can be used to construct obstruction theories. For simplicity, we work over a base field k and let X be a smooth and proper scheme of dimension d over k.

Obstruction theory on moduli spaces of sheaves

Let $\mathcal{M}$ be a stack over Spec k and let $E \in D^-_{qcoh}(X \times \mathcal{M})$ be perfect. Consider the Atiyah class map of E relative to X:

[3] In the usual conventions (e.g., [22, Tag 0145]), the upper row is obtained by rotating (1.2) to the right twice and then *flipping all signs*.

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$$\mathrm{at}_E : E \to L_{X\times\mathcal{M}/X} \otimes E[1] \simeq \pi_{\mathcal{M}}^* L_{\mathcal{M}} \otimes E[1].$$

Since E is dualizable, and by the projection formula, this data is equivalent to a map

$$\mathcal{O}_{\mathcal{M}} \to R\pi_*(\pi_{\mathcal{M}}^* L_{\mathcal{M}} \otimes (E \otimes E^\vee))[1] \simeq L_{\mathcal{M}} \otimes R\pi_*(E \otimes E^\vee)[1].$$

Using dualizability again, we obtain a morphism

$$\mathrm{At}_E : R\pi_{\mathcal{M}*}(E \otimes E^\vee)^\vee[-1] \to L_{\mathcal{M}}. \tag{1.3}$$

Theorem 1.4. *Suppose that $\mathcal{M}$ is an open substack of the moduli stack of coherent sheaves on X. Then At_E is an obstruction theory. More generally, this holds when $\mathcal{M}$ is an open substack of a moduli space of universally gluable perfect complexes on X.*

This is proven in §§6.3–6.5 for moduli of sheaves. The statement for complexes is addressed in Remark 6.16. Recall that an obstruction theory on an algebraic stack $\mathcal{Y}$ consists of a map $g : \mathbb{E} \to L_{\mathcal{Y}}$ in $D_{qcoh}(\mathcal{Y})$, so that $h^i(g)$ is an isomorphism for $i \geq 0$, and so that $h^{-1}(g)$ is surjective [14, Definition 2.4.1].

Obstruction theory on Quot-schemes

Let $\mathcal{Y}$ be an algebraic stack and let E be a $\mathcal{Y}$-flat coherent sheaf on $X \times \mathcal{Y}$. Let $f : \mathcal{Q} \to \mathcal{Y}$ be an open substack of the relative Quot-scheme of E over $\mathcal{Y}$ and let

$$0 \to F \to E_{\mathcal{Q}} \to G \to 0$$

be the universal exact sequence on $X \times \mathcal{Q}$. We consider the associated reduced Atiyah class $\overline{\mathrm{at}}_E := \overline{\mathrm{at}}_{E,V\times\mathcal{X}/V\times\mathcal{Y},G}$ as a map $\overline{\mathrm{at}}_E : F \to \pi_{\mathcal{X}}^* L_{\mathcal{X}/\mathcal{Y}} \otimes G$ in the derived category. As before, this data is equivalent to a morphism

$$\overline{\mathrm{At}}_E : Rf_*(G \otimes F^\vee)^\vee \to L_{\mathcal{Q}/\mathcal{Y}}.$$

Proposition 1.5. *The map $\overline{\mathrm{At}}_E$ is a relative obstruction theory for $f : \mathcal{Q} \to \mathcal{Y}$.*

2. Preliminaries

2.1. Derived category and cotangent complex of an algebraic stack

Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{O}_{\mathcal{X}}$ denote its structure sheaf in the lisse-étale topos on $\mathcal{X}$. Given a smooth cover $X \to \mathcal{X}$, where X is an algebraic space, one can form the strictly simplicial algebraic space $X_\bullet = X_{\bullet,\mathrm{et}}$, which we consider as a strictly simplicial topos with respect to the étale topology on every component. We write $X_{\bullet,\text{lis-et}}$ for the strictly simplicial topos obtained by taking the corresponding lisse-étale topos in place of each X_n. It is shown in [16, 4.6] that we have flat morphisms of topoi

$$X_{\bullet,\mathrm{et}} \xleftarrow{\epsilon} X_{\bullet,\text{lis-et}} \xrightarrow{\pi} \mathcal{X}_{\text{lis-et}}.$$

Moreover, the functor ϵ_* is exact and preserves flatness. We define η_X^* to be the composition $\epsilon_* \circ \pi^* : \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Mod}(\mathcal{O}_{X_\bullet})$. It defines a functor on the categories of chain complexes and due to exactness also on the derived categories, both of which we also denote by η_X^*. Let $\eta_{X_*} := R\pi_* \circ \epsilon^* : D(\mathcal{O}_{X_\bullet}) \to D(\mathcal{O}_{\mathcal{X}})$. By ([10], Example 2.2.5), the functors η_X^* and η_{X_*} restrict to mutually inverse equivalences on the derived categories with quasi-coherent cohomology sheaves

$$\eta_X^* : D_{qcoh}(\mathcal{O}_{\mathcal{X}}) \to D_{qcoh}(\mathcal{O}_{X_\bullet}), \tag{2.1}$$

$$\eta_{X*} : D_{qcoh}(\mathcal{O}_{X_\bullet}) \to D_{qcoh}(\mathcal{O}_{\mathcal{X}}). \tag{2.2}$$

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Given a smooth surjective map of algebraic spaces $g : W \to X$ over $\mathcal{X}$, let $W_\bullet$ be the induced hypercover for the map $W \to \mathcal{X}$ and let $g_\bullet : W_\bullet \to X_\bullet$ be the map induced by g. Then there is a canonical natural isomorphism between the functors $g^*\eta_X^*$ and η_W^* on the levels of sheaves, which induces isomorphisms between the induced functors on complexes and derived categories, respectively. We recall the notion of the cotangent complex of algebraic stacks as given in [16, §8]: Given a morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$, choose a 2-commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathcal{Y}, \end{array} \tag{2.3}$$

where X, Y are algebraic spaces and where the maps $Y \to \mathcal{Y}$ and $X \to \mathcal{X}_Y := Y \times_{\mathcal{Y}} \mathcal{X}$ are smooth and surjective. Let $X_\bullet$ and $Y_\bullet$ be the strictly simplicial algebraic spaces associated to $X \to \mathcal{X}$ and $Y \to \mathcal{Y}$, respectively. One defines a complex $L_{\mathcal{X}/\mathcal{Y},X/Y}$ on $X_\bullet$ whose restriction to X_n is given by the complex

$$L_{X_n/Y_n} \to \Omega_{X_n/\mathcal{X}_{Y_n}},$$

where $\Omega_{X_n/\mathcal{X}_{Y_n}}$ is placed in degree one, and the map is induced from the natural map of differentials $h^0(L_{X_n/Y_n}) \simeq \Omega_{X_n/Y_n} \to \Omega_{X_n/\mathcal{X}_{Y_n}}$. It is shown in [16] that this defines an element of $D^{\leq 1}_{qcoh}(X_\bullet)$ and that the element $\eta_* L_{\mathcal{X}/\mathcal{Y},X/Y} \in D^{\leq 1}_{qcoh}(\mathcal{X})$ is independent of the choice of diagram (2.3) up to canonical isomorphisms. This is used to define $L_{\mathcal{X}/\mathcal{Y}}$, so that one has a canonical isomorphism $\eta^* L_{\mathcal{X}/\mathcal{Y}} \simeq L_{\mathcal{X}/\mathcal{Y},X/Y}$ for any choice of diagram (2.3).

Remark 2.1. Define $\Omega_{X_\bullet/\mathcal{X}_{Y_\bullet}}$ to be the $\mathcal{O}_{X_\bullet}$-module which on X_n is given by $\Omega_{X_n/\mathcal{X}_{Y_n}}$ with the obvious pullback maps. Then we may restate

$$L_{\mathcal{X}/\mathcal{Y},X/Y} = \mathrm{Cone}\Big(L_{X_\bullet/Y_\bullet} \xrightarrow{-} \Omega_{X_\bullet/\mathcal{X}_{Y_\bullet}}\Big)[-1].$$

Here, $L_{X_\bullet/Y_\bullet}$ is the usual cotangent complex for the map of topoi $X_\bullet \to Y_\bullet$, and the map indicated by '$-$' is minus the natural map whose restriction to the n-th simplicial degree is given by the composition $L_{X_n/Y_n} \xrightarrow{\tau_{\geq 0}} \Omega_{X_n/Y_n} \to \Omega_{X_n/\mathcal{X}_{Y_n}}$.

2.2. *Simplicial methods*

We recall some notation and basic facts about simplicial rings and simplicial sheaves of modules in general topoi. For a general reference, see Illusie's book [7]. Throughout this subsection, let T denote a topos.

Simplicial modules

For a simplicial ring A in T, we denote by $A-\mathrm{Mods}$ the category of A-modules. When A is an ordinary, we regard it as a constant simplicial ring, so that $A-\mathrm{Mods}$ denotes the category of *simplicial* A-modules. In either case, we denote by $D^\Delta(A)$ the derived category obtained by localizing $A-\mathrm{Mods}$ at the class of quasi-isomorphisms.

Dold–Kan correspondence

Let A be an ordinary ring in T. The *normalized chain functor* induces an equivalence of abelian categories $N : A-\mathrm{Mods} \to C^{\leq 0}(A)$; see [7, I 1.3]. It sends homotopic maps to homotopic maps and there are natural identifications $\pi_i(M) \simeq h^{-i}(NM)$ for a simplicial A-module M. In particular, N preserves quasi-isomorphisms and induces an equivalence $N : D^\Delta(A) \to D^{\leq 0}(A)$.

Cones and distinguished triangles

As in [7, I 3.2.1], let σ denote the simplicial $\mathbb{Z}$-module satisfying $N\sigma = \mathbb{Z}[1]$ and let γ be the simplicial $\mathbb{Z}$-module such that $N\gamma$ is the complex $\mathbb{Z} \to \mathbb{Z}$ concentrated in degrees $[-1, 0]$. Let A be a simplicial ring in the topos T and let E be an A-module. We write $\sigma E := \sigma \otimes_{\mathbb{Z}} E$ and $\gamma E := \gamma \otimes_{\mathbb{Z}} E$. One has canonical isomorphisms $\pi_i(E) = \pi_{i+1}(\sigma E)$ for any $i \geq 0$. We have a natural exact sequence of A-modules

$$0 \to E \xrightarrow{\iota} \gamma E \xrightarrow{q} \sigma E \to 0.$$

For a map $\alpha : E \to F$ of A-modules, we define

$$\mathrm{Cone}^{\Delta}(\alpha) := \mathrm{Coker}(E \xrightarrow{(\iota,\alpha)} \gamma E \oplus F).$$

We have the sequence of natural maps

$$E \xrightarrow{\alpha} F \to \mathrm{Cone}^{\Delta}(\alpha) \to \sigma E, \tag{2.4}$$

and the induced maps on homotopy groups fit into a long exact sequence. One declares a sequence $E \to F \to G \to \sigma E$ in $D^{\Delta}(A)$ to be a *distinguished triangle* if it is isomorphic in $D^{\Delta}(A)$ to a sequence of the form (2.4); see [7, I 3.2.2]. If A is an ordinary ring in T, and E a simplicial A-module, then in $D^{\leq 0}(A)$, we have natural isomorphisms $N\sigma E \simeq (NE)[1]$, and the Dold–Kan correspondence preserves the notions of distinguished triangle.

Derived tensor product

Let A be a simplicial ring in T. The derived tensor product defines a functor $D^{\Delta}(A) \times D^{\Delta}(A) \to D^{\Delta}(A), (E, F) \mapsto E \otimes_A F$, which can be computed as follows: For any quasi-isomorphism $L \to E$ where L is a flat (i.e., degreewise flat) A-module, the derived tensor product is computed by the (usual) tensor product $L \otimes_A F$ of A-modules (taken degree-wise). The analogous statement holds with a flat replacement of F. For fixed E, the functor $E \otimes - : D^{\Delta}(A) \to D^{\Delta}(A)$ is naturally triangulated, and similarly for $- \otimes E$. If A is an ordinary ring and E, F are simplicial A-modules, then we have canonical natural isomorphisms $N(E \otimes^{\ell} F) \simeq NE \otimes^{L} NF$ in $D^{\leq 0}(A)$, which are compatible with the symmetry isomorphism of the tensor product.

Now let $P \to B$ be a morphism of A-algebras. Then, the derived tensor product $B \otimes_P -$ induces a triangulated functor $D^{\Delta}(P) \to D^{\Delta}(B)$, which is left-adjoint to the functor $D^{\Delta}(B) \to D^{\Delta}(P), N \mapsto N_P$ given by restriction of scalars.

We have the following:

Lemma 2.2. *Suppose that $P \to B$ is a quasi-isomorphism of A-algebras. Then the derived tensor product and restriction of scalars are mutually inverse equivalences of categories. In other words, the natural adjunction maps $M \to B \otimes_P M$ for M in $D^{\Delta}(P)$ and $B \otimes_P N_P \to N_P$ for N in $D^{\Delta}(B)$ are isomorphisms.*

Proof. This is [7, I Corollaire 3.3.4.6]. □

Simplicial resolutions

Let $A \to B$ be a map of ordinary rings in a topos T. We denote by

$$P_A(B)$$

the *standard simplicial resolution of B over A* [7, I 1.5]. It is a simplicial A-algebra and flat over A in each degree. There is a natural quasi-isomorphism $P_A(B) \to B$, where we regard B as a constant simplicial A-algebra in T.

We will use the following result.

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Lemma 2.3. *Let*

$$\begin{tikzcd} W_2 \arrow[r, "a"] \arrow[d, "h"] & W_1 \arrow[d, "g"] \\ Y_2 \arrow[r, "b"] & Y_1 \end{tikzcd}$$

be a commutative diagram of locally ringed topoi with enough points. Assume that a and b are flat. Then the natural map $a^{-1}P_{g^{-1}\mathcal{O}_{Y_1}}(\mathcal{O}_{W_1}) \to P_{h^{-1}\mathcal{O}_{Y_2}}(\mathcal{O}_{W_2})$ of simplicial sheaves of rings on W_2 is flat in each degree.

Proof. This can be checked on stalks of W_2. Since taking the standard simplicial resolution commutes with pullback of topoi and with filtered direct limits, we are reduced to the following setting: We have a diagram in the category of local rings

$$\begin{tikzcd} B_2 & B_1 \arrow[l] \\ A_2 \arrow[u] & A_1 \arrow[l] \arrow[u] \end{tikzcd}$$

with B_2 flat over B_1 and A_2 flat over A_1, and we need to show that the natural map $P_{A_1}(B_1) \to P_{A_2}(B_2)$ is degreewise flat. Denote this map by $F : P \to R$ with n-th part $F_n : P_n \to R_n$ for $n \geq 0$. We also define $F_{-1} : B_1 \to B_2$. We show by induction on $n \geq -1$ that F_n is flat and injective. The base case follows from the fact that a flat morphism of local rings is faithfully flat and therefore injective. By the construction of the standard simplicial resolution, we have that P_{n+1} is the free polynomial algebra $A_1[P_n]$ over the set of elements of P_n. The analogue is true for R_{n+1}, and the map F_{n+1} is the map $P_{n+1} = A_1[P_n] \to A_2[R_n] = R_{n+1}$ obtained by functoriality of this construction. The induction step then follows from Lemma 2.4 (note that $A_1 \to A_2$ is injective since it is a flat map of local rings). □

Lemma 2.4. *Consider a commutative diagram of rings*

$$\begin{tikzcd} B_1 \arrow[r] \arrow[d] & B_2 \arrow[d] \\ A_1 \arrow[r] & A_2 \end{tikzcd}$$

where the map $A_1 \to A_2$ is flat and injective and $B_1 \to B_2$ is injective. Then the induced map between polynomial algebras $A_1[B_1] \to A_2[B_2]$ is flat and injective.

Proof. The injectivity is clear. The map $A_1[B_1] \to A_2[B_1] = A_2 \otimes_{A_1} A_1[B_1]$ is a base change of a flat map, hence flat, and $A_2[B_1] \to A_2[B_2]$ is a free algebra by injectivity, hence also flat. Since a composition of flat morphisms is flat, the result follows. □

Module of principal parts

Let $A \to B$ be a map of rings in T. The *(first) module of principal parts* for the ring map $A \to B$ is given by $P^1_{B/A} := (B \otimes_A B)/I_\Delta^2$, where I_Δ is the kernel of the multiplication map $B \otimes_A B \to B$. (Here, the tensor product is in general not a derived one). The A-module $P^1_{B/A}$ is naturally a (B, B)-bimodule. Recall that $\Omega^1_{B/A} = I_\Delta/I_\Delta^2$, so that we have an exact sequence of (B, B)-bimodules called the *exact sequence of principal parts*

$$0 \to \Omega^1_{B/A} \to P^1_{B/A} \to B \to 0.$$

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Here, for each of the outer terms, the two B-module structures agree. We will denote this sequence by $\underline{P}^1_{B/A}$. The map $b \mapsto b \otimes 1$ gives a splitting of this sequence for the left B-module structures, and $b \to 1 \otimes b$ gives a splitting for the right B-module structures. Now let E be a B-module. Then we set $P^1_{B/A}(E) := P^1_{B/A} \otimes_B E$, where we take the tensor product with respect to the right B-module structure on $P^1_{B/A}$. Equivalently, $P^1_{B/A}(E) = B \otimes_A E/(I^2_\Delta B \otimes_A E)$. Then the *sequence of principal parts for* E is $\underline{P}^1_{B/A}(E) := \underline{P}^1_{B/A} \otimes_B E$, or explicitly,

$$0 \to \Omega_{B/A} \otimes_B E \to P^1_{B/A}(E) \to E \to 0.$$

We usually regard this as a sequence of B-modules with respect to the *left* B-module structure. Note that it is in general not split.

Illusie's Atiyah class

With these ingredients, we recall Illusie's definition of Atiyah class: Let $f : X \to Y$ be a morphism of ringed topoi and let $E \in D^{\leq 0}(X)$. Let $P := P_{f^{-1}\mathcal{O}_Y}(\mathcal{O}_X)$ be the standard simplicial resolution. By the Dold–Kan correspondence, we may regard E as an object of $D^\Delta(\mathcal{O}_X)$ and thus of $D^\Delta(P)$ by restriction of scalars. We have the exact sequence of principal parts associated to E_P, which is an exact sequence of P-modules:

$$\underline{P}^1_{P/f^{-1}\mathcal{O}_Y}(E) : \quad 0 \to \Omega^1_{P/f^{-1}\mathcal{O}_Y} \otimes_P E_P \to P^1_{P/f^{-1}\mathcal{O}_Y}(E_P) \to E_P \to 0.$$

Note that the leftmost term here computes the derived tensor product since $\Omega_{P/f^{-1}\mathcal{O}_Y}$ is flat over P. Moreover, it is canonically quasi-isomorphic to the restriction of scalars of $L_{X/Y} \otimes_{\mathcal{O}_X} E$. From the sequence $\underline{P}_{P/f^{-1}\mathcal{O}_Y}(E_P)$, we obtain a morphism $E_P \to \sigma(L_{X/Y} \otimes_{\mathcal{O}_X} E)_P$ in $D^\Delta(P)$. Extending scalars to $\mathcal{O}_X$, this defines a canonical morphism $E \to \sigma L_{X/Y} \otimes_{\mathcal{O}_X} E$ in $D^\Delta(\mathcal{O}_X)$. The Atiyah class is the corresponding morphism

$$E \to L_{X/Y}[1] \otimes E$$

in $D^{\leq 0}(\mathcal{O}_X)$ obtained via the Dold–Kan correspondence.

2.3. The parallel arrow category.

Let

$$W \underset{t}{\overset{s}{\rightrightarrows}} X$$

be a diagram of topoi. We obtain an induced topos $W_\parallel$ whose objects are tuples $(A_X, A_W, s^\sharp, t^\sharp)$, where A_X and A_W are objects of X and W, respectively, and $s^\sharp : s^{-1}A_X \to A_W$ and $t^\sharp : t^{-1}A_X \to A_W$ are morphisms in W. Giving a ring $R = (R_X, R_W, s^\sharp, t^\sharp)$ in $W_\parallel$ is equivalent to giving rings on X and W, and giving s, t the structure of morphism of ringed topoi. Given such an R, we use the following notation: For an R_X-module M_X, we write $s^*_R M_X := s^{-1}M_X \otimes_{s^{-1}R_X} R_W$ and $t^*_R M_X := t^{-1}M_X \otimes_{t^{-1}R_X} R_W$ for the respective (in general un-derived) base change of M_X. Then an R-module is given by a tuple $M = (M_X, M_W, s^*, t^*)$, where M_X and M_W are R_X and R_W-modules, respectively, and where $s^* : s^*_R M_X \to M_W$ and $t^* : t^*_R M_X \to M_W$ are morphisms of R_W-modules.

We define a functor $\mathrm{Cone}_R : C(R) \to C(R_W)$ by

$$\mathrm{Cone}_R : (M_X, M_W, s^*, t^*) \mapsto \mathrm{Cone}(s^*_R M_X \oplus t^*_R M_X \xrightarrow{-s^* \oplus t^*} M_W).$$

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If R is a *simplicial* ring in $W_{\|}$, the analogous discussion holds for R-modules, and we get a functor

$$\begin{aligned} \mathrm{Cone}_R^{\Delta} : R - \mathrm{Mods} &\to R_W - \mathrm{Mods} \\ (M_X, M_W, s^*, t^*) &\mapsto \mathrm{Cone}^{\Delta}(s_R^* M_X \oplus t_R^* M_X \xrightarrow{-s^* \oplus t^*} M_W). \end{aligned}$$

Under some natural flatness assumptions, we have induced functors on the derived categories:

Lemma 2.5.

i) *Let R be a ring on $W_{\|}$ with $s^\sharp : s^{-1}R_X \to R_W$ and $t^\sharp : t^{-1}R_X \to R_W$ flat. Then* Cone_R *descends to a triangulated functor of derived categories $D(R) \to D(R_W)$ (also denoted* Cone_R*).*
ii) *Let R be a simplicial ring on $W_{\|}$ with $s^\sharp : s^{-1}R_X \to R_W$ and $t^\sharp : t^{-1}R_X \to R_W$ flat. Then* Cone_R^{Δ} *descends to a triangulated functor $D^{\Delta}(R) \to D^{\Delta}(R_W)$ (also denoted* Cone_R^{Δ}*).*
iii) *Let R be an ordinary ring $W_{\|}$, viewed as a constant simplicial ring, and assume that $s^\sharp$ and $t^\sharp$ are flat. Then the two constructions in i) and ii) are compatible with the Dold–Kan correspondence, in the sense that the two functors obtained by traversing the outer edges of the diagram*

$$\begin{tikzcd} D^{\Delta}(R) \arrow[r, "\mathrm{Cone}_R^{\Delta}"] \arrow[d] & D^{\Delta}(R_W) \arrow[d] \\ D^{\leq 0}(R) \arrow[r, "\mathrm{Cone}_R"] & D^{\leq 0}(R_W) \end{tikzcd}$$

are related by a canonical natural isomorphism.

Proof. We prove 2.5. For any ring S, the abelian category of complexes $C^{[-1,0]}(S)$ concentrated in degrees -1 and 0 is canonically identified with the category of maps of S-modules.

The functor $\mathrm{Cone}_R : C(R) \to C(R_W)$ factors as $C(R) \to C(C^{[-1,0]}(R_W)) \to C(R_W)$, where the first map is induced from the functor

$$\begin{aligned} \Gamma : \mathrm{Mod}(R) &\to C^{[-1,0]}(R_W) \\ (M_X, M_W, s^*, t^*) &\mapsto \left[s_R^* M_X \oplus t_R^* M_X \xrightarrow{-s^* \oplus t^*} M_W \right], \end{aligned}$$

and the second map is taking the mapping cone. By the flatness assumption, Γ is exact and therefore induces a triangulated functor $D(R) \to D(C^{[-1,0]}(R_W))$.

Therefore, it is enough to show that for any ring S, the mapping cone functor $C(C^{[-1,0]}(S)) \to C(S)$ descends to a triangulated functor of derived categories. We may regard $C(C^{[-1,0]}(S))$ as a category of double complexes $A^{i,j}$ with nonzero entries only for $i \in \{-1, 0\}$. Then the mapping cone is exactly given by taking the associated double complex with the sign conventions of [22, Remark 0G6A]. By [22, Remark 0G6A and Remark 0G6D], taking the total complex defines a triangulated functor of the *homotopy categories* $K(C^{[-1,0]}(S)) \to K(S)$. To see that this preserves quasi-isomorphisms, one can use that the spectral sequence for the double complex converges; see [22, Lemma 0132]. Hence, it descends to a functor $D(C^{[-1,0]}(S)) \to D(S)$. This finishes the proof of 2.5.

Part ii) follows from an analogous argument with simplicial modules. Here, one uses additionally that base change along a degree-wise flat map of simplicial rings preserves quasi-isomorphism of modules. For S any simplicial ring, and $[A_{-1} \xrightarrow{\alpha} A_0] \in C^{[-1,0]}(S)$, we describe the triangulated structure on the mapping cone functor: We have

$$\begin{aligned} \mathrm{Cone}^{\Delta}(\sigma\alpha) &= \mathrm{Coker}(\sigma A_{-1} \to \gamma\sigma A_{-1} \oplus \sigma A_0), \\ \sigma\, \mathrm{Cone}^{\Delta}(\alpha) &= \mathrm{Coker}(\sigma A_{-1} \to \sigma\gamma A_{-1} \oplus \sigma A_0). \end{aligned}$$

We take the canonical isomorphism between them induced by the symmetry isomorphism of the tensor product $\gamma\sigma A_{-1} \simeq \sigma\gamma A_{-1}$.

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The compatibility (iii) follows from the constructions by using the basic compatibilities of the Dold–Kan correspondence, in particular, that it is compatible with the symmetry isomorphisms of tensor products on the level of derived categories. □

Lemma 2.6. *Let R be a ring in $W_{\|}$. For any complex of R-modules E, we have a natural morphism $c_E : \mathrm{Cone}_R(E) \to E_W[1]$. If the pullback maps $s^* : s^*_R E_X \to E_W$ and $t^* : t^*_R E_X \to E_W$ are quasi-isomorphisms, then so is c_E. The same picture holds in the category of simplicial $\mathcal{O}_{\mathcal{X}}$ modules with the obvious modifications, and the two situations are compatible via the Dold–Kan correspondence.*

Proof. We define c_E as the composition

$$\mathrm{Cone}_R(E) \to s^*_R E_X[1] \oplus t^*_R E_X[1] \xrightarrow{s^* \oplus 0} E_W[1]$$

(the alternative choice of second map $(0 \oplus t^*)$ gives the same map up to chain-homotopy). One checks directly that this is a quasi-isomorphism when s^* and t^* are quasi-isomorphisms. The proofs of the remaining statements are left to the reader. □

We have the following result regarding tensor products:

Lemma 2.7. i) *Let R be a simplicial ring on $W_{\|}$ with flat pullback maps $s^\sharp, t^\sharp$ and let L, E be R-modules. Then there is a natural map $\mathrm{Cone}^\Delta_R(L \otimes_R E) \to \mathrm{Cone}^\Delta_R(L) \otimes_{R_W} E_W$. If either of L and E are flat and if $s^* : s^*_R E_X \to E_W$ and $t^* : t^*_R E_X \to E_W$ are quasi-isomorphisms, then $\mathrm{Cone}^\Delta_R(L \otimes_R E) \to \mathrm{Cone}^\Delta_R(L) \otimes_{R_W} E_W$ is a quasi-isomorphism. In particular, for any $E \in D^\Delta(R)$, we have a canonical 2-morphism*

$$\begin{tikzcd} D^\Delta(R) \arrow[r, "\mathrm{Cone}^\Delta_R"] \arrow[d, "- \otimes_R E"'] & D^\Delta(R_W) \arrow[d, "- \otimes_{R_W} E_W"] \\ D^\Delta(R) \arrow[r, "\mathrm{Cone}^\Delta_R"'] \arrow[ur, Rightarrow] & D^\Delta(R_W), \end{tikzcd}$$

which is an isomorphism if the pullback maps s^*, t^* of E are isomorphisms in $D^\Delta(R_W)$.

ii) *The analogous statement holds if R is an ordinary ring and L, E are bounded above complexes of R-modules.*

iii) *The natural isomorphisms in the derived category in i) and ii) are compatible via the Dold–Kan correspondence.*

Proof. We only address i). Part ii) is analogous, and iii) can be seen by tracing through the argument and using the compatibilites of the Dold–Kan correspondence. Note that for any R-module F, we have

$$\mathrm{Cone}^\Delta_R(F) = \mathrm{Cone}^\Delta(s^*_R F_X \to \mathrm{Cone}^\Delta(t^*_R F_X \xrightarrow{t^*} F_Z)),$$

where the map in the outer cone is the composition of $-s^*$ with the inclusion $F_Z \to \mathrm{Cone}^\Delta(t^*)$. For any commutative triangle

$$\begin{tikzcd} K \arrow[d, "u"] \arrow[dr, "vu"] & \\ M \arrow[r, "v"'] & N \end{tikzcd}$$

in R_W − Mods, we obtain induced maps

$$\mathrm{Cone}^\Delta(u) \to \mathrm{Cone}^\Delta(vu) \xrightarrow{\alpha} \mathrm{Cone}^\Delta(v),$$

which form the first three terms of an exact triangle. In particular, if u is a quasi-isomorphism, then $\mathrm{Cone}^{\Delta}(u)$ is acyclic, so α is also a quasi-isomorphism. Applying this to the triangle

$$\begin{tikzcd} t_R^* L_X \otimes_{R_W} t_R^* E_X \arrow[d, "1\otimes t^*"] \arrow[dr, "t^*\otimes t^*"] & \\ t_R^* L_X \otimes_{R_W} E_W \arrow[r, "t^*\otimes 1"] & L_W \otimes_{R_W} E_W, \end{tikzcd}$$

we get a natural map,

$$b : \mathrm{Cone}^{\Delta}(t^* \otimes t^*) \to \mathrm{Cone}^{\Delta}(t^* \otimes 1) = \mathrm{Cone}^{\Delta}(t_R^* L_X \to L_W) \otimes E_W,$$

which is a quasi-isomorphism if $t^* : t_R^* E_X \to E_W$ is one and if additionally either one of L or E is flat.

Similarly, we have the triangle

$$\begin{tikzcd} s_R^* L_X \otimes_{R_W} s_R^* E_X \arrow[d, "-1\otimes s^*"] \arrow[dr] & \\ s_R^* L_X \otimes_{R_W} E_W \arrow[r] & \mathrm{Cone}^{\Delta}(t_R^* L_X \xrightarrow{t^*} L_W) \otimes_{R_W} E_W, \end{tikzcd}$$

where the horizontal map is obtained by applying $- \otimes_{R_W} E_W$ to the composition

$$s_R^* L_X \xrightarrow{-s^*} L_W \to \mathrm{Cone}^{\Delta}(t_R^* L_X \xrightarrow{t^*} L_W).$$

We get an induced morphism

$$\begin{aligned} c : \mathrm{Cone}^{\Delta}\Big(s_R^* L_X \otimes_{R_W} s_R^* E_X \to \mathrm{Cone}^{\Delta}(t_R^* L_X \to L_W) \otimes_{R_W} E_W\Big) \\ \to \mathrm{Cone}_R^{\Delta}(L) \otimes_{R_W} E_W, \end{aligned}$$

which again is a quasi-isomorphism if $s^* : s_R^* E_X \to E_W$ is and if one of L or E is flat. Putting together b and c, we get a natural morphism

$$\mathrm{Cone}_R^{\Delta}(L \otimes_R E) \to \mathrm{Cone}_R^{\Delta}(L) \otimes_{R_W} E_W,$$

which is a quasi-isomorphism if L (resp. E) is flat and that both pullback maps of E are quasi-isomorphisms. □

Variant 2.8. Let $W_\wedge$ denote the topos associated to the diagram

$$X \xleftarrow{s} W \xrightarrow{t} X.$$

Its objects are tuples $(A'_X, A''_X, A_W, s^\sharp, t^\sharp)$, where A'_X, A''_X are objects of X, where A_W is an object of W and where $s^\sharp : s^{-1}A'_X \to A_W$ and $t^\sharp : t^{-1}A''_X \to A_W$ are morphisms in W. Then everything in the preceding subsection goes through with $W_\wedge$ in place of $W_\|$ and straightforward modifications. Note also that we have an exact pullback functor $\mathrm{Sh}(W_\|) \to \mathrm{Sh}(W_\wedge)$ given on objects by

$$(A_X, A_W, s^\sharp, t^\sharp) \mapsto (A_X, A_X, A_W, s^\sharp, t^\sharp).$$

If $R_{\|}$ is a ring on $W_{\|}$ and $R_{\wedge}$ its restriction to $W_{\wedge}$, then $\mathrm{Cone}_{R_{\|}}$ factors as

$$D(R_{\|}) \to D(R_{\wedge}) \xrightarrow{\mathrm{Cone}_{R_{\wedge}}} D(R_W).$$

The analogous picture holds for the simplicial versions.

Application to algebraic stacks

We apply the preceding discussion to the derived categories of algebraic stacks and the cotangent complex. This lays the groundwork for our definition of the Atiyah class on a stack using the modules of principal parts.

Situation 2.9. Consider the diagram (2.3) with the associated strictly simplicial algebraic spaces $X_\bullet$ and $Y_\bullet$. Let $W := X \times_{\mathcal{X}_Y} X$, with associated projections $s, t : W \to X$ on the first and second factor, respectively, and let $h : W \to Y$ be the induced map. Let $W_\bullet$ be the strictly simplicial algebraic space associated to the covering $W \xrightarrow{s} X \to \mathcal{X}$. One has canonical isomorphisms $W_n \simeq X_n \times_{\mathcal{X}_{Y_n}} X_n$ and maps $h_n : W_n \to Y_n$. We denote by $W_{\|}$ the topos associated to the diagram

$$W_\bullet \underset{t_\bullet}{\overset{s_\bullet}{\rightrightarrows}} X_\bullet.$$

It has a natural structure of ringed topos with flat pullback maps $s_\bullet^\sharp, t_\bullet^\sharp$, and we write $\mathcal{O}_{W_{\|}}$ for the structure sheaf.

We denote by $(Y_{\|}, \mathcal{O}_{Y_{\|}})$ the ringed topos associated to the diagram

$$Y_\bullet \rightrightarrows Y_\bullet$$

with both arrows the identity. There is a natural map of topoi $h_{\|} : W_{\|} \to Y_{\|}$.

For E a complex of $\mathcal{O}_{\mathcal{X}}$-modules, we write $E_{X_\bullet} := \eta_X^* E$ and $E_{W_\bullet} := \eta_W^* E$. We have natural morphisms $s_\bullet^* E_{X_\bullet} \to E_{W_\bullet}$ and $t_\bullet^* E_{X_\bullet} \to E_{W_\bullet}$. Thus, we get naturally a complex of $\mathcal{O}_{W_{\|}}$-modules, which we denote $E_{W_{\|}}$.

In Situation 2.9, we have the following properties:

Lemma 2.10. *For any a complex of $\mathcal{O}_{\mathcal{X}}$-modules E, we have a natural morphism* $\mathrm{Cone}_{\mathcal{O}_{W_{\|}}}(E_{W_{\|}}) \to E_{W_\bullet}[1]$. *If E has quasi-coherent (or more generally, Cartesian) cohomology sheaves, this map is a quasi-isomorphism. The same picture holds in the category of simplicial $\mathcal{O}_{\mathcal{X}}$ modules with the obvious modifications, and the two situations are compatible via the Dold–Kan correspondence.*

Proof. This is a restatement of Lemma 2.6, using that the pullback maps s^* and t^* are quasi-isomorphisms whenever E has quasi-coherent cohomology sheaves. □

Lemma 2.11. *In $D_{qcoh}(W_\bullet)$, the object $\eta_W^* L_{\mathcal{X}/\mathcal{Y}}$ is naturally isomorphic to*

$$\mathrm{Cone}_{\mathcal{O}_{W_{\|}}}(L_{W_{\|}/Y_{\|}})[-1] = \mathrm{Cone}\Big(s_\bullet^* L_{X_\bullet/Y_\bullet} \oplus t_\bullet^* L_{X_\bullet/Y_\bullet} \xrightarrow{-s_\bullet^*+t_\bullet^*} L_{W_\bullet/Y_\bullet}\Big)[-1].$$

This isomorphism is functorial in the diagram (2.3).

Proof. The displayed equality is immediate from the definition of $W_{\|}$ and the definition of cotangent complex for a morphism of topoi. By the construction of the cotangent complex and Remark 2.1, we have a canonical isomorphism

$$\eta_X^* L_{\mathcal{X}/\mathcal{Y}} \simeq L_{\mathcal{X}/\mathcal{Y}, X/Y} = \mathrm{Cone}\Big(L_{X_\bullet/Y_\bullet} \xrightarrow{-} \Omega_{X_\bullet/\mathcal{X}_{Y_\bullet}}\Big)[-1]. \tag{2.5}$$

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We pull this isomorphism back via $s_\bullet$. Due to the diagram

$$\begin{tikzcd} W_\bullet \arrow[r, "t_\bullet"] \arrow[d, "s_\bullet"] & X_\bullet \arrow[d] & \\ X_\bullet \arrow[r] & \mathcal{X}_{Y_\bullet} \arrow[r] & Y_\bullet, \end{tikzcd}$$

we have natural quasi-isomorphisms of complexes

$$s_\bullet^*\Omega_{X_\bullet/\mathcal{X}_{Y_\bullet}} \simeq \Omega_{W_\bullet/X_\bullet,t_\bullet} \leftarrow \mathrm{Cone}(t_\bullet^* L_{X_\bullet/Y_\bullet} \xrightarrow{t^*} L_{W_\bullet/Y_\bullet}).$$

We claim that the following diagram commutes:

$$\begin{tikzcd} s_\bullet^* L_{X_\bullet/Y_\bullet} \arrow[r, "a"] \arrow[d] & \mathrm{Cone}(t_\bullet^* L_{X_\bullet/Y_\bullet} \xrightarrow{t^*} L_{W_\bullet/Y_\bullet}) \arrow[d] \\ s_\bullet^*\Omega_{X_\bullet/\mathcal{X}_{Y_\bullet}} \arrow[r, "\sim"] & \Omega_{W_\bullet/X_\bullet,t_\bullet}, \end{tikzcd}$$

where the upper horizontal map is given by the pullback $s^* : s_\bullet^* L_{X_\bullet/Y_\bullet} \to L_{W_\bullet/Y_\bullet}$ followed by the inclusion of $L_{W_\bullet/Y_\bullet}$ into the cone. Indeed, this follows from the observation that the map $s_\bullet^* L_{X_\bullet/Y_\bullet} \to \Omega_{W_\bullet/X_\bullet,t_\bullet}$ factors through $L_{W_\bullet/Y_\bullet}$. It follows from this that, after pulling back the right-hand side of (2.5) along $s_\bullet$, the result is naturally quasi-isomorphic to

$$\mathrm{Cone}\left(s_\bullet^* L_{X_\bullet/Y_\bullet} \xrightarrow{-a} \mathrm{Cone}\left(t_\bullet^* L_{X_\bullet/Y_\bullet} \xrightarrow{t^*} L_{W_\bullet/Y_\bullet}\right)\right)[-1].$$

An easy calculation shows that this iterated cone is identical to

$$\mathrm{Cone}\left(s_\bullet^* L_{X_\bullet/Y_\bullet} \oplus t_\bullet^* L_{X_\bullet/Y_\bullet} \xrightarrow{-s_\bullet^*+t_\bullet^*} L_{W_\bullet/Y_\bullet}\right)[-1],$$

as desired. The compatibility with pullback follows from the compatibility of (2.5) and the usual pullback compatibilities of the cotangent complex for algebraic spaces. □

Remark 2.12. In Situation 2.9, suppose that R is a simplicial ring on $W_\|$ with a given map $R \to \mathcal{O}_{W_\|}$. Then we have the following diagram of functors, which commutes up to canonical natural isomorphisms:

$$\begin{tikzcd} D^\Delta(R) \arrow[r, "\mathcal{O}_{W_\|}\otimes^\ell_R -"] \arrow[d, "\mathrm{Cone}_R^\Delta"] & D^\Delta(\mathcal{O}_{W_\|}) \arrow[r, "N"] \arrow[d, "\mathrm{Cone}^\Delta_{\mathcal{O}_{W_\|}}"] & D^{\leq 0}(\mathcal{O}_{W_\|}) \arrow[d, "\mathrm{Cone}_{\mathcal{O}_{W_\|}}"] & \\ D^\Delta(R_W) \arrow[r, "\mathcal{O}_{W_\bullet}\otimes^\ell_{R_W} -"] & D^\Delta(\mathcal{O}_{W_\bullet}) \arrow[r, "N"] & D^{\leq 0}(\mathcal{O}_{W_\bullet}) \arrow[r, "\eta_{W*}"] & D^{\leq 0}(\mathcal{X}). \end{tikzcd} \tag{2.6}$$

For the following lemma, let $W_\wedge$ be the ringed topos associated to the diagram $X_\bullet \xleftarrow{s_\bullet} W_\bullet \xrightarrow{t_\bullet} X_\bullet$. Recall that we have a natural restriction functor $D(W_\|) \to D(W_\wedge)$. The following will be used in the proof of Lemma 4.4.

Lemma 2.13. *Let $E_{W_\|} = (E_{X_\bullet}, E_{W_\bullet}, s^*, t^*)$ be a complex of $\mathcal{O}_{W_\|}$-modules such that $E_{X_\bullet}$ has quasi-coherent cohomology sheaves and such that the pullback maps s^* and t^* are quasi-isomorphisms. Let $F := \eta_{W*}E_{W_\bullet} \in D_{qcoh}(\mathcal{X})$, and let $F_{W_\|} = (\eta_X^* F, \eta_W^* F)$ denote the induced object of $D(W_\|)$. Let $E_{W_\wedge}$ and $F_{W_\wedge}$ denote the images of $E_{W_\|}$ and $F_{W_\|}$ in $D(W_\wedge)$, respectively. Then there is a natural isomorphism $F_{W_\wedge} \to E_{W_\wedge}$ in $D(W_\wedge)$.*

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Proof. Let $W_{\wedge,lis-et}$ be the ringed topos associated to the diagram

$$X_{\bullet,lis-et} \xleftarrow{s_\bullet} W_{\bullet,lis-et} \xrightarrow{t_\bullet} X_{\bullet,lis-et}.$$

From the diagram of ringed topoi

$$\begin{tikzcd}
X_\bullet & X_{\bullet,lis-et} \arrow[l, "\epsilon"'] \arrow[drr, "\pi_X"] & & \\
W_\bullet \arrow[u, "s_\bullet"] \arrow[d, "t_\bullet"'] & W_{\bullet,lis-et} \arrow[l, "\epsilon"'] \arrow[u, "s_\bullet"] \arrow[d, "t_\bullet"'] \arrow[rr, "\pi_W"] & & \mathcal{X} \\
X_\bullet & X_{\bullet,lis-et} \arrow[l, "\epsilon"'] \arrow[urr, "\pi_X"'] & &
\end{tikzcd}$$

we see that there is an induced morphism $\epsilon_\wedge : W_{\wedge,lis-et} \to W_\wedge$. Let $F_{\wedge,lis-et} = (\pi_X^* F, \pi_W^* F, \pi_X^* F)$, with pullback maps induced by the identifications $\pi_W^* = s^*\pi_X^*$ and $\pi_W^* = t^*\pi_X^*$, respectively. Since the functors ϵ_* are exact, we have $F_\wedge = \epsilon_{\wedge *} F_{\wedge,lis-et}$. In particular, since $\epsilon_*\epsilon^*$ is naturally isomorphic to the identity, it is enough to show that there is a natural quasi-isomorphism $\epsilon_\wedge^* E_{W_\wedge} \to F_{\wedge,lis-et}$.

We are reduced to the following situation: Let $G = (G_X, G_W, G_X, s^*, t^*)$ be a complex of $\mathcal{O}_{W_{\wedge,lis-et}}$-modules such that G_X has quasicoherent cohomology sheaves, and such that s^*,t^* are quasi-isomorphisms, and suppose that $F = R(\pi_W)_* G_W$. Then we need to show that G is naturally isomorphic to $F_{\wedge,lis-et}$. Without loss of generality, we may assume that G is K-injective. Then G_W and G_X are themselves K-injective. We can therefore assume that $F = \pi_{W*} G_W$. Let also $F' := \pi_{X*} G_X$. The morphism $s^* : s^* G_X \to G_W$ corresponds to a morphism $G_X \to s_* G_W$, and by applying π_{W*} we obtain a map $\sigma : F' \to F$. Similarly, we obtain $\tau : F' \to F$ from $t^* : t^* G_X \to G_W$. The maps σ and τ are quasi-isomorphisms, as one can check after applying π_W^*, since F' and F have quasi-coherent comohology.

Then we have the following commutative diagram of complexes of $\mathcal{O}_{W_{\bullet,lis-et}}$-modules:

$$\begin{tikzcd}
s^*\pi_X^* F \arrow[d, "s^*"'] & s^*\pi_X^* F' \arrow[l, "s^*\pi_X^*\sigma"'] \arrow[r] \arrow[d, "\pi_W^*\sigma"'] & s^* G_X \arrow[d] \\
\pi_W^* F & \pi_W^* F \arrow[l] \arrow[r] & G_W \\
t^*\pi_X^* F \arrow[u, "t^*"] & t^*\pi_X^* F' \arrow[l, "t^*\pi_X^*\tau"] \arrow[r] \arrow[u, "\pi_W^*\tau"] & t^* G_X. \arrow[u]
\end{tikzcd}$$

The horizontal maps going to the right are the quasi-isomorphisms coming from the push-pull adjunctions associated to π_X and π_W, respectively. Setting $\widetilde{F}_{\wedge,lis-et} = (\pi_X^* F', \pi_W^* F, \pi_X^* F', s^*\pi_X^*\sigma, t^*\pi_X^*\tau)$, it follows that we have quasi-isomorphisms

$$F_{\wedge,lis-et} \leftarrow \widetilde{F}_{\wedge,lis-et} \to G,$$

as desired. □

2.4. *Tensor triangulated categories and additivity of traces*

Traces in a closed symmetric monoidal category

Let $\mathcal{C}$ be a symmetric closed monoidal category with product $- \otimes -$, unit $\mathcal{O}$, and internal Hom-functor $\mathcal{H}om(-,-)$. Let τ denote the symmetry morphism of the tensor product. We let E, F, G denote arbitrary elements of $\mathcal{C}$ and write $E^\vee := \mathcal{H}om(E, \mathcal{O})$. Recall that we have an adjunction between $- \otimes E$ and $\mathcal{H}om(E, -)$. We have various natural maps in $\mathcal{C}$:

(1) *Evaluation.* We have a natural map $ev : \mathcal{H}om(E,F) \otimes E \to F$, corresponding to the identity on $\mathcal{H}om(E,F)$ under adjunction.
(2) As a special case, we get the evaluation map $ev : E^\vee \otimes E \to \mathcal{O}$.
(3) *Composition.* We have a natural map $comp : \mathcal{H}om(F,G) \otimes \mathcal{H}om(E,F) \to \mathcal{H}om(E,G)$. This map is adjoint to the map $\mathcal{H}om(F,G) \otimes \mathcal{H}om(E,F) \otimes E \to G$ which is obtained from composing two evaluation maps.
(4) As a special case, we get the composition map $comp : F \otimes E^\vee \to \mathcal{H}om(E,F)$, where we use the canonical isomorphism $F \simeq \mathcal{H}om(\mathcal{O},F)$.
(5) *Diagonal.* We have a diagonal map $s : \mathcal{O} \to \mathcal{H}om(E,E)$ which corresponds to id_E under adjunction.

We recall the notion of dualizable object:

Definition 2.14. The object E is called *dualizable* if the composition map $comp : E \otimes E^\vee \to \mathcal{H}om(E,E)$ is an isomorphism. See the discussion preceding Definition 2.2 in [19].

For dualizable E and arbitrary F, the canonical map $comp : F \otimes E^\vee \to \mathcal{H}om(E,F)$ is an isomorphism, and we will often use this isomorphism to identify the source and target.

Let E be a dualizable object of $\mathcal{C}$. Then we have further natural maps

(1) *Trace I.* We define the *trace map* $\mathrm{tr} : \mathcal{H}om(E,E) \to \mathcal{O}$ as the composition $\mathcal{H}om(E,E) \simeq E \otimes E^\vee \xrightarrow{\tau} E^\vee \otimes E \xrightarrow{ev} \mathcal{O}$.
(2) *Trace II.* We have a map $\mathrm{tr} : \mathcal{H}om(E, F \otimes E) \to F$ given by the composition $\mathcal{H}om(E, F \otimes E) \simeq F \otimes E \otimes E^\vee \xrightarrow{F \otimes tr} F$. By abuse of language, we call this also the *trace map*.
(3) *Diagonal II.* We have the composition $\eta := comp^{-1} \circ s : \mathcal{O} \to E \otimes E^\vee$, which we will also call *diagonal*.

A morphism $f : E \to F \otimes E$ corresponds by adjunction to a map $\mathcal{O} \to \mathcal{H}om(E, F \otimes E)$. When E is dualizable, we define tr_f as the composition $\mathcal{O} \to \mathcal{H}om(E, F \otimes E) \xrightarrow{\mathrm{tr}} F$.

As a consequence of the definition and the naturality of adjunction, we have the following:

Lemma 2.15. *For $f : E \to F \otimes E$, the map tr_f is equal to the composition*

$$\mathcal{O} \xrightarrow{\eta} E \otimes E^\vee \xrightarrow{f \otimes \mathrm{id}_{E^\vee}} F \otimes E \otimes E^\vee \xrightarrow{\mathrm{id}_F \otimes \tau} F \otimes E^\vee \otimes E \xrightarrow{\mathrm{id}_F \otimes ev} F,$$

where τ denotes the symmetry isomorphism of the tensor product.

Proof. The composition of the final two maps is by definition equal to $\mathrm{id}_F \otimes \mathrm{tr}$. Now consider the following diagram:

$$\begin{tikzcd}
 & E \otimes E^\vee \arrow[r, "f \otimes \mathrm{id}_{E^\vee}"] \arrow[d, "comp"] & F \otimes E \otimes E^\vee \arrow[d, "comp"] \arrow[dr, "\mathrm{id}_F \otimes \mathrm{tr}"] & \\
\mathcal{O} \arrow[ur, "\eta"] \arrow[r, "s"] & \mathcal{H}om(E,E) \arrow[r, "\mathcal{H}om(E,f)"] & \mathcal{H}om(E, F \otimes E) \arrow[r, "\mathrm{tr}"] & F.
\end{tikzcd}$$

The left and right triangles commute by definition, while the square commutes by naturality of the composition map. By naturality of adjunction, the first two arrows in the lower row compose to the

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map adjoint to f. Thus, by definition, the lower row composes to tr_f, and the same must be true for the composition along the top. $\square$

We will need the following compatibility of traces with tensor products

Lemma 2.16. *Let $f : E \to F \otimes E$ be a morphism in $\mathcal{C}$ with E dualizable and let V be another dualizable object of $\mathcal{C}$. Then*

$$\mathrm{tr}(f \otimes \mathrm{id}_V) = \mathrm{tr}(f) \circ \mathrm{tr}(\mathrm{id}_V).$$

Proof. This follows by applying Corollary 5.9 of [19]. $\square$

Criterion for additivity of traces

We now assume that, say, $\mathcal{C} = D(R)$ is the derived category of a category of sheaves of modules over a ring R in a topos T. In particular, it is closed monoidal with a compatible triangulated structure. Let F, E, G be dualizable objects of $D(R)$ that are part of an exact triangle

$$F \to E \to G \to F[1]. \tag{2.7}$$

Suppose that we have a commutative diagram for some element $L \in D(R)$:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "g"] & E \arrow[r] \arrow[d, "f"] & G \arrow[d, "h"] \\ L \otimes F \arrow[r] & L \otimes E \arrow[r] & L \otimes G, \end{tikzcd}$$

where the lower row is obtained from the upper by tensoring with L. We want a criterion that allows us to conclude that $tr(f) = tr(g) + tr(h)$. We follow the strategy in [13, §8].

We will make the following

Assumption 2.17. The objects F, E, G are elements of the bounded above derived category $D^-(R)$.

Situation 2.18. Suppose Assumption 2.17 holds. (This is likely unnecessary if one works with K-flat complexes in what follows.) Choose a representation of (2.7) by a short exact sequence $0 \to F \to E \to G \to 0$ of bounded above complexes of flat R-modules (or suppose one is given). Further, choose an injective resolution J of R, so that the derived dual of a complex is explicitly realized by $(-)^\vee := \mathcal{H}om_R(-, J)$. Then we have again an exact sequence $0 \to G^\vee \to E^\vee \to F^\vee \to 0$. We define complexes W and $\overline{W}$ via

$$W := E \otimes E^\vee / F \otimes G^\vee$$
$$\overline{W} := E^\vee \otimes E / G^\vee \otimes F.$$

The tensor product here is the ordinary tensor product of complexes, which represents the derived tensor product due to the choice of F and E. We have a natural isomorphism $W \to \overline{W}$ coming from the symmetry isomorphism of the tensor product. Moreover, we have natural inclusions of complexes

$$F \quad \otimes F^\vee \hookrightarrow W \hookleftarrow G \quad \otimes G^\vee,$$

and

$$F^\vee \otimes F \quad \hookrightarrow \overline{W} \hookleftarrow G^\vee \otimes G.$$

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The inclusions into W are given by

$$F \otimes F^\vee \simeq F \otimes E^\vee / F \otimes G^\vee \subset W$$

and

$$G \otimes G^\vee \simeq E \otimes G^\vee / F \otimes G^\vee \subset W,$$

respectively, and analogously for the inclusions into $\overline{W}$.

Lemma 2.19. *Let the notation be as in Situation 2.18.*

(i) *Let $\overline{\eta}$ be the composition $\mathcal{O} \xrightarrow{\eta} E \otimes E^\vee \to W$ in $D(R)$. Then the following diagram commutes:*

$$\begin{tikzcd} & \mathcal{O} \arrow[dl, "\eta"'] \arrow[d, "\overline{\eta}"] \arrow[dr, "(\eta,\eta)"] & \\ E \otimes E^\vee \arrow[r] & W & F \otimes F^\vee \oplus G \otimes G^\vee. \arrow[l] \end{tikzcd}$$

(ii) *There exists a natural map of complexes $\overline{\mathrm{ev}} : \overline{W} \to J$ whose image in $D(R)$ makes the following diagram commute:*

$$\begin{tikzcd} E^\vee \otimes E \arrow[r] \arrow[dr, "\mathrm{ev}"'] & \overline{W} \arrow[d, "\overline{\mathrm{ev}}"] & (F^\vee \otimes F) \oplus (G^\vee \otimes G) \arrow[l] \arrow[dl, "\mathrm{ev} \oplus \mathrm{ev}"] \\ & \mathcal{O}. & \end{tikzcd}$$

Proof. The map in (ii) is the natural quotient map of $ev : E^\vee \otimes E \to J$, which vanishes on the subcomplex $G^\vee \otimes F$. The commutativity of the right triangle follows by unwinding the definitions.

(i) We let $V := (F \otimes E^\vee \oplus E \otimes G^\vee)/(F \otimes G^\vee)$, where the quotient is with respect to the antidiagonal inclusion. Then we have a natural exact sequence of complexes

$$0 \to V \xrightarrow{j_1} E \otimes E^\vee \to G \otimes F^\vee \to 0.$$

Moreover, we have natural maps $j_2 : V \to F \otimes F^\vee$ and $j_3 : V \to G \otimes G^\vee$ obtained by composing projection onto a factor with a respective quotient map. We claim that the following diagram of complexes commutes:

$$\begin{tikzcd} & V \arrow[dl, "j_1"'] \arrow[dr, "(j_2,j_3)"] & \\ E \otimes E^\vee \arrow[r] & W & (F \otimes F^\vee) \oplus (G \otimes G^\vee). \arrow[l] \end{tikzcd}$$

By precomposing with the surjection $F \otimes E^\vee \oplus E \otimes G^\vee \to V$, this reduces to the commutativity of the following two diagrams,

$$\begin{tikzcd} & F \otimes E^\vee \arrow[dl] \arrow[dr] & \\ E \otimes E^\vee \arrow[r] & W & F \otimes F^\vee \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} & E \otimes G^\vee \arrow[dl] \arrow[dr] & \\ E \otimes E^\vee \arrow[r] & W & G \otimes G^\vee, \arrow[l] \end{tikzcd}$$

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which one can see directly. Now the problem is reduced to finding a map $\eta_V : \mathcal{O} \to V$ in the derived category such that

$$\begin{tikzcd}
 & \mathcal{O} \arrow[dl, "\eta"'] \arrow[d, "\eta_V"] \arrow[dr, "(\eta,\eta)"] & \\
E \otimes E^\vee & V \arrow[l] & F \otimes F^\vee \oplus G \otimes G^\vee \arrow[l]
\end{tikzcd}$$

commutes. After dualizing in the derived category, this follows from (ii). $\square$

Now we have the following result:

Proposition 2.20 [13, §8]. *Suppose there exists a dotted arrow making its two adjacent squares in the following diagram commute. Then $tr(f) = tr(g) + tr(h)$.*

$$\begin{tikzcd}
 & \mathcal{O} \arrow[dl, "(\eta,\eta)"'] \arrow[d, "\overline{\eta}"] \arrow[dr, "\eta"] & \\
E \otimes E^\vee \arrow[r] \arrow[d] & W \arrow[d, dashed] & F \otimes F^\vee \oplus G \otimes G^\vee \arrow[l] \arrow[d] \\
L \otimes E \otimes E^\vee \arrow[r] \arrow[d] & L \otimes W \arrow[d] & (L \otimes F \otimes F^\vee) \oplus (L \otimes G \otimes G^\vee) \arrow[l] \arrow[d] \\
L \otimes E^\vee \otimes E \arrow[r] \arrow[dr, "\mathrm{ev}"'] & L \otimes \overline{W} \arrow[d, "\overline{\mathrm{ev}}"] & (L \otimes F^\vee \otimes F) \oplus (L \otimes G^\vee \otimes G) \arrow[l] \arrow[dl, "\mathrm{ev} \oplus \mathrm{ev}"] \\
 & L &
\end{tikzcd}$$

Proof. Everything else in the diagram commutes due to Lemma 2.19 and the definitions. By Lemma 2.15, the composition along the left side equals $\mathrm{tr}(f)$, and the composition along the right side equals $\mathrm{tr}(g) + \mathrm{tr}(h)$. $\square$

3. Constructions for topoi

3.1. Construction of the reduced Atiyah class

We review the construction of the reduced Atiyah class for a map of ringed topoi, following the ideas in [3].

A diagram of exact sequences

Let $\mathcal{A}$ be an abelian category and consider the following diagram in $\mathcal{A}$ in which the solid arrows commute:

$$\begin{tikzcd}
E \arrow[r, "e"'] \arrow[d, "q"] & E'' \arrow[l, bend right, dotted, "s"'] \arrow[d, "q''"] \\
G \arrow[r, "f"'] & G''.
\end{tikzcd} \tag{3.1}$$

Assume that s is a section of e (i.e., $e \circ s = \mathrm{id}_{E''}$). Then, the composition $q \circ s$ induces a morphism $\delta : \mathrm{Ker}(q'') \to \mathrm{Ker}(f)$.

Now suppose this diagram extends to a commutative diagram in which all solid rows and columns are exact and where the dotted arrows give a spliting of the middle exact sequence

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$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F' \arrow[r, "a"] \arrow[d, "j'"] & F \arrow[r, "d"] \arrow[d, "j"] & F'' \arrow[r] \arrow[d, "j''"] & 0 \\
0 \arrow[r] & E' \arrow[r, "b"'] \arrow[d] & E \arrow[r, "e"'] \arrow[d, "q"] \arrow[l, dotted, bend right, "r"'] & E'' \arrow[r] \arrow[d, "q''"] \arrow[l, dotted, bend right, "s"'] & 0 \\
0 \arrow[r] & G' \arrow[r, "c"'] \arrow[d] & G \arrow[r, "f"'] \arrow[d] & G'' \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Then we have the morphism $\delta : F'' \to G'$ as described above. From the composition $r \circ j : F \to E'$, we get an induced map $\mathrm{Coker}(a) \to \mathrm{Coker}(j')$ and thus another map $F'' \to G'$. By a diagram chase, one checks this to be equal to $-\delta$.

Now assume that, further, $\mathcal{A}$ is the category of modules over a simplicial ring A in a topos. Then, the above morphisms fit into the following diagrams of triangles in $D^{\Delta}(A)$:

$$\begin{tikzcd}
F' \arrow[r] \arrow[d, equal] & F' \arrow[r] \arrow[d] & F'' \arrow[r] \arrow[d, "-\delta"] & \sigma F' \arrow[d, equal] \\
F' \arrow[r] & E' \arrow[r] & G' \arrow[r] & \sigma F',
\end{tikzcd}
\qquad
\begin{tikzcd}
F'' \arrow[r] \arrow[d, "\delta"] & E'' \arrow[r] \arrow[d] & G'' \arrow[r] \arrow[d, equal] & \sigma F'' \arrow[d, "\sigma\delta"] \\
G' \arrow[r] & G \arrow[r] & G'' \arrow[r] & \sigma G'.
\end{tikzcd} \tag{3.2}$$

This shows that composing δ with the connecting map $G' \to \sigma F'$ yields *minus* the connecting map associated to the exact sequence of the F's, while composing $\sigma\delta$ with the connecting map $G'' \to \sigma F''$ yields the connecting map associated to the exact sequence of the G's.

The reduced Atiyah class

Now let $f : X \to Y$ be a morphism of ringed topoi. Let $E \in C^{\leq 0}(\mathcal{O}_Y)$ be a complex whose components are Tor-independent to f, so that, in particular, the component-wise pullback $E_X := f^*E$ equals the derived pullback. Let

$$0 \to F \to E_X \to G \to 0 \tag{3.3}$$

be an exact sequence in $C^{\leq 0}(\mathcal{O}_X)$, which we also view as a sequence of simplicial $\mathcal{O}_X$-modules via the Dold–Kan correspondence. Let $R := P_{f^{-1}\mathcal{O}_Y}(\mathcal{O}_X)$ be the standard simplicial resolution and let $E_R := f^{-1}E \otimes_{f^{-1}\mathcal{O}_Y} R$. Denote by $\cdot\,|_R$ restriction of scalars from $\mathcal{O}_X$ to R. Since R is flat, the natural morphism $E_R \to E_X|_R$ is a quasi-isomorphism, and it is termwise surjective, since $R \to \mathcal{O}_X$ is. Let $G_R := G|_R$ and let F_R be the kernel of the induced map $E_R \to G_R$. We have an induced map of exact sequences of R-modules, in which the vertical arrows are quasi-isomorphisms

$$\begin{tikzcd}
0 \arrow[r] & F_R \arrow[r] \arrow[d] & E_R \arrow[r] \arrow[d] & G_R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F|_R \arrow[r] & E_X|_R \arrow[r] & G|_R \arrow[r] & 0.
\end{tikzcd}$$

We now take the exact sequence of principal parts with respect to the upper row. This gives the following commutative diagram of solid arrows with exact rows and columns:

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$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega_{R/f^{-1}\mathcal{O}_Y} \otimes_R F_R \arrow[r] \arrow[d] & P^1_{R/f^{-1}\mathcal{O}_Y}(F_R) \arrow[r] \arrow[d] & F_R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_{R/f^{-1}\mathcal{O}_Y} \otimes_R E_R \arrow[r] \arrow[d] & P^1_{R/f^{-1}\mathcal{O}_Y}(E_R) \arrow[r] \arrow[d] \arrow[l, dotted, bend right, "r"'] & E_R \arrow[r] \arrow[d] \arrow[l, dotted, bend right, "s"'] & 0 \\
0 \arrow[r] & \Omega_{R/f^{-1}\mathcal{O}_Y} \otimes_R G_R \arrow[r] \arrow[d] & P^1_{R/f^{-1}\mathcal{O}_Y}(G_R) \arrow[r] \arrow[d] & G_R \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{3.4}
$$

Here, the arrows denoted by s and r come from a splitting of the middle exact sequence, which is defined as follows: Since $E_R = R \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}E$, we have by definition

$$
\begin{aligned}
P^1_{R/f^{-1}\mathcal{O}_Y}(E_R) &= (R \otimes_{f^{-1}\mathcal{O}_Y} R/I_\Delta^2) \otimes_R E_R \\
&= (R \otimes_{f^{-1}\mathcal{O}_Y} R \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}E)/(I_\Delta^2 \cdot R \otimes R \otimes f^{-1}E).
\end{aligned}
$$

Then s is given on local sections by $a \otimes m \mapsto a \otimes 1 \otimes m$, which one checks to be a morphism of left R-modules. We see that the lower right square of (3.4) is of the form (3.1). In particular, we get the induced morphism

$$
\delta : F_R \to \Omega_{R/f^{-1}\mathcal{O}_Y} \otimes_R G_R. \tag{3.5}
$$

By passing to derived categories, and taking extensions of scalars along $R \to \mathcal{O}_X$, this corresponds to a morphism

$$
\overline{\mathrm{at}}_{E,X/Y,G} : F \to L_{X/Y} \otimes G
$$

in $D^\Delta(\mathcal{O}_X)$, or equivalently $D^{\le 0}(\mathcal{O}_X)$, which we call the *reduced Atiyah class* of E over Y with respect to the sequence (3.3) on X. We also write $\overline{\mathrm{at}}_E$ if the remaining data is understood.

Remark 3.1. The morphism $\delta : F_R \to \Omega_{R/f^{-1}\mathcal{O}_Y} \otimes_R G_R$ obtained by using the lower right corner of (3.4) is explicitly given as follows: For a local section $f = \sum r_i \otimes f^{-1}e_i$, where r_i are sections of R and e_i are sections of E (over Y), we have

$$
\delta : f \mapsto \sum dr_i \otimes \overline{e_i}.
$$

Remark 3.2. The following triangles commute:

1)

$$
\begin{tikzcd}
F \arrow[r, "-\overline{\mathrm{at}}_E"] \arrow[dr, "\mathrm{at}_F"'] & L_{X/Y} \otimes G \arrow[d] \\
 & L_{X/Y}[1] \otimes F,
\end{tikzcd}
$$

2)

$$\begin{tikzcd} G \arrow[d] \arrow[dr, "\mathrm{at}_G"] & \\ F[1] \arrow[r, "\overline{\mathrm{at}}_E[1]"'] & L_{X/Y}[1] \otimes G. \end{tikzcd}$$

In both cases, the vertical morphisms are induced from the connecting map $G \to F[1]$ of the given exact sequence. This follows from the morphisms of triangles (3.2).

Remark 3.3. In general, the reduced Atiyah class depends on slightly more than a map $\varphi : f^*E \to G$ in the derived category $D(\mathcal{O}_X)$, analogously to how cones in the derived category are unique only up to non-unique isomorphism. However, for any given φ, by a variation of the above using cocones/cones instead of kernels/cokernels, one can define a reduced Atiyah class $F := \mathrm{cocone}(\varphi) \to L_{X/Y} \otimes G$, which is well defined *up to an element of* $\mathrm{Ext}^{-1}(F, G)$.

Lemma 3.4 (Shift invariance). *Consider the reduced Atiyah class* $\overline{\mathrm{at}}_{E[1]}$ *associated to the shift of the sequence (3.3). Then, the following diagram commutes:*

$$\begin{tikzcd} F[1] \arrow[r, "\overline{\mathrm{at}}_{E[1]}"] \arrow[dr, "\overline{\mathrm{at}}_E[1]"'] & L_{X/Y} \otimes (G[1]) \arrow[d, "\tau"] \\ & L_{X/Y} \otimes G[1], \end{tikzcd}$$

where τ *denotes the compatibility morphism making the tensor product* $L_{X/Y} \otimes -$ *into a triangulated functor.*

Proof. On the level of simplicial modules, the shift functor corresponds to taking a tensor product $\sigma \otimes_{\mathbb{Z}} -$, where σ is the simplicial $\mathbb{Z}$-module defined in §2.2. The statement then follows from the fact that for any R-module M, we have canonical isomorphisms of exact sequences, functorial in M, which are induced by the properties of the tensor product:

$$\begin{tikzcd} 0 \arrow[r] & \Omega_{R/f^{-1}\mathcal{O}_Y} \otimes_R (\sigma \otimes_{\mathbb{Z}} M) \arrow[r] \arrow[d, "\sim"] & P^1_{R/f^{-1}\mathcal{O}_Y}(\sigma \otimes_{\mathbb{Z}} M) \arrow[r] \arrow[d, "\sim"] & \sigma \otimes_{\mathbb{Z}} M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \sigma \otimes_{\mathbb{Z}} \Omega_{R/f^{-1}\mathcal{O}_Y} \otimes_R M \arrow[r] & \sigma \otimes_{\mathbb{Z}} P^1_{R/f^{-1}\mathcal{O}_Y}(M) \arrow[r] & \sigma \otimes_{\mathbb{Z}} M \arrow[r] & 0. \end{tikzcd}$$

□

We extend the definition of the reduced Atiyah class to objects in $D^-(X)$ (cf. Construction 5.6 for the analogue for the usual Atiyah class):

Construction 3.5. Let $f : X \to Y$ be a morphism of ringed topoi, let $E \in D^-(Y)$, with $E_X := f^*E$ and let

$$F \to E_X \to G \xrightarrow{+1} F[1] \tag{3.6}$$

be an exact triangle in $D^-(X)$. Assume that $\mathrm{Ext}^{-1}(F, G) = 0$. Choose $N > 0$ so that the $E[N], F[N]$ and $G[N]$ all lie in $D^{\leq 0}(X)$. By Remark 3.3, we obtain a well-defined map

$$\overline{\mathrm{at}}_{E[N]} : F[N] \to L_{X/Y} \otimes (G[N]).$$

We define

$$\overline{\mathrm{at}}_E := (\tau^{(N)} \circ \mathrm{at}_{E[N]})[-N],$$

where $\tau^{(N)} : L_{X/Y} \otimes (G[N]) \to L_{X/Y} \otimes G[N]$ is the N-fold application of the compatibility map of tensor product with the shift functor. This is independent of choice of N by Lemma 3.4.

3.2. The reduced Atiyah class via the graded cotangent complex

Let $f : X \to Y$ be a morphism of ringed topoi and let E be an $\mathcal{O}_Y$-module, Tor-independent to f. Let $E_X := f^*E$ and let

$$0 \to F \to E_X \to G \to 0$$

be an exact sequence of $\mathcal{O}_X$-modules. We will use the $\mathbb{Z}$-graded cotangent complex as defined in [7, IV 2]. For a $\mathbb{Z}$-graded ring $A = \oplus A_i$, we let k^i denote the functor that sends a graded A-module $M = \oplus M_i$ to the A_0-module given by its i-th graded piece M_i.

Let $X[E_X]$ denote the graded ringed topos whose underlying site is the étale site of X and whose sheaf of rings is given by $\mathcal{O}_X \oplus E$, where E is placed in degree 1, and similarly for $Y[E], X[G]$. Let $q : X[G] \to X$ be the morphism induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X \oplus G$ and, similarly, $r : X[E_X] \to X$.

Proposition 3.6. (i) *We have a canonical natural isomorphism*

$$k^1 q_* L^{gr}_{X[E_X]/X} \simeq E_X$$

of objects of $D(\mathcal{O}_X)$.

(ii) *We have a canonical natural isomorphism*

$$k^1 q_* L^{gr}_{X[G]/X[E_X]} \simeq F[1]$$

of objects of $D(\mathcal{O}_X)$.

(iii) *Up to the isomorphisms in 3.6 and 3.6, applying $k^1 \circ q_*$ to the connecting homomorphism of graded cotangent complexes associated to the composition $X[G] \to X[E_X] \to X[F]$ recovers the inclusion $F \to E$ up to a shift.*

Proof. Statement 3.6 is [7, IV (2.2.5)], the other parts can be deduced from that as in [3, §1.8]: Consider the distinguished triangle in $D^{gr}(X[G])$

$$L^{gr}_{X[E_X]/X}\,|_{X[G]} \to L^{gr}_{X[G]/X} \to L^{gr}_{X[E_X]/X[G]} \xrightarrow{+1} L^{gr}_{X[E_X]/X}[1].$$

Applying $k^1 \circ q_*$ and the isomorphisms of 3.6, we obtain a distinguished triangle in $D(X)$:

$$E_X \to G \to k^1 \circ q_* L^{gr}_{X[E_X]/X[G]} \to E_X[1].$$

It follows that there is a unique isomorphism $k^1 \circ q_* L^{gr}_{X[E_X]/X[G]} \simeq F$ that identifies the connecting map on cohomology sheaves with the inclusion $F \to E$. $\square$

Consider the transitivity triangle of graded cotangent complexes associated to the morphism of graded ringed topoi $X[G] \to X[E_X] \to Y[E]$ and let

$$L^{gr}_{X[G]/X[E_X]} \to L^{gr}_{X[E_X]/Y[E]}|_{X[G]}\ [1]$$

be the connecting map. Applying $k^1 \circ q_*$ and the isomorphisms of Proposition 3.6, we obtain a map

$$F[1] \to (L_{X/Y} \otimes G)[1]. \tag{3.7}$$

Proposition 3.7. *The map (3.7) agrees with the shift $\overline{\mathrm{at}}_{E,X/Y,G}[1]$ of the reduced Atiyah class.*

Proof. The definition in [3, 2.3] goes through in our setting and gives the same resulting notion of reduced Atiyah class. By Theorem 2.6 there, the map defined in this way agrees with (3.7) up to a shift. □

Corollary 3.8. *The reduced Atiyah class is preserved under Tor-independent pullback: Consider a commutative diagram of ringed topoi*

$$\begin{tikzcd} X' \arrow[r, "\alpha"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "\beta"] & Y \end{tikzcd}$$

such that β is Tor-independent to E and such that α is Tor-independent to E_X and G. Then the diagram

$$\begin{tikzcd} \alpha^*F \arrow[r, "\alpha^*\overline{\mathrm{at}}_{E,X/Y,G}"] \arrow[dr, "\overline{\mathrm{at}}_{\beta^*E,X'/Y',\alpha^*E}"'] & \alpha^*L_{X/Y} \otimes \alpha^*G \arrow[d] \\ & L_{X'/Y'} \otimes \alpha^*G \end{tikzcd}$$

commutes.

Proof. This follows directly from Proposition 3.7 and the functoriality of the transitivity triangle for graded cotangent complexes. □

Lemma 3.9. *Consider a commutative diagram of ringed topoi*

$$\begin{tikzcd} W \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[d] & \\ & Y' \arrow[r] & Y \arrow[r] & Z. \end{tikzcd}$$

We assume further that X and Y' are Tor-independent over Y and that the induced square of rings on X' obtained by pulling back the structure sheaves of Y, Y' and X is cocartesian. Then the diagram of (shifted) connecting maps

$$\begin{tikzcd} L_{W/X'} \arrow[r] \arrow[d] & L_{X'/Y'}|_W\ [1] \simeq L_{X/Y}|_W\ [1] \arrow[d] \\ L_{X'/X}|_W\ [1] \simeq L_{Y'/Y}|_W\ [1] \arrow[r] & L_{Y/Z}|_W\ [2] \end{tikzcd}$$

on W anti-commutes. Here, $\cdot\ |_W$ denotes pullback to W.

Proof. By taking suitable simplicial resolutions, we are reduced to the setting that we have a diagram

$$\begin{tikzcd} & C \arrow[r] & C' \arrow[r] & D \\ A \arrow[r] & B \arrow[r] \arrow[u] & B' \arrow[u] & \end{tikzcd}$$

of simplicial rings in a topos T, in which the square is cocartesian and in which all maps are free in each simplicial degree.

We then get a diagram of D-modules

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \Omega_{B/A} \otimes_B D \arrow[r] \arrow[d] & \Omega_{C/A} \otimes_C D \arrow[r] \arrow[d] & \substack{\Omega_{C/B} \otimes_C D \\ = \Omega_{C'/B'} \otimes_{C'} D} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega_{B'/A} \otimes_B' D \arrow[r] \arrow[d] & \Omega_{D/A} \arrow[r] \arrow[d] & \Omega_{D/B'} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \substack{\Omega_{B'/B} \otimes_{B'} D \\ = \Omega_{C'/C} \otimes_{C'} D} \arrow[r] \arrow[d] & \Omega_{D/C} \arrow[r] \arrow[d] & \Omega_{D/C'} \arrow[r] \arrow[d] & 0. \\ & 0 & 0 & 0 & \end{tikzcd}$$

It is now a basic exercise in homological algebra to show that the following induced diagram of connecting maps in $D^{\Delta}(D)$ anti-commutes:

$$\begin{tikzcd} \Omega_{D/C'} \arrow[r, "\delta"] \arrow[d, "\delta"] & \sigma\Omega_{C'/C} \arrow[d, "\sigma\delta"] \\ \sigma\Omega_{C'/B'} \otimes_{C'} D \arrow[r, "\sigma\delta"] & \sigma^2\Omega_{B/A} \otimes_B D. \end{tikzcd}$$

□

Proposition 3.10. *Let E, F, G and $f : X \to Y$ be as before and let $Y \to Z$ be a morphism of ringed topoi. Then the following diagram anti-commutes:*

$$\begin{tikzcd} F \arrow[rr, "\overline{\mathrm{at}}_{E,X/Y,G}"] \arrow[d] & & L_{X/Y} \otimes G \arrow[d] \\ E \arrow[r, "\mathrm{at}_{E,Y/Z}"] & L_{Y/Z}[1] \otimes E \arrow[r] & L_{Y/Z}[1] \otimes G, \end{tikzcd}$$

where the right vertical map is obtained from the connecting homomorphism of cotangent complexes by tensoring with G.

Proof. We apply Lemma 3.9 to the diagram of ringed topoi

$$\begin{tikzcd} X[G] \arrow[r] & X[E] \arrow[r] \arrow[d] & X \arrow[d] & \\ & Y[E] \arrow[r] & Y \arrow[r] & Z, \end{tikzcd}$$

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which gives the anti-commutative diagram

$$\begin{tikzcd} L_{X[G]/X[E]} \arrow[r] \arrow[d] & L_{X/Y}|_{X[G]}[1] \arrow[d] \\ L_{Y[E]/Y}|_{X[G]}[1] \arrow[r] & L_{Y/Z}|_{X[G]}[2]. \end{tikzcd}$$

By applying $k^1 \circ q_*$ and using the identifications of Proposition 3.6, the result follows. □

3.3. The Atiyah class for an exact sequence

Let $f : X \to Y$ be a morphism of ringed topoi and let $0 \to F \to E \to G \to 0$ be an exact sequence of bounded above complexes of $\mathcal{O}_X$-modules, such that F, E and G are dualizable and such that their duals lie again in $D^-(X)$. After taking appropriate resolutions and up to shifting, we may assume that E, F and G are concentrated in degrees ≤ 0 and that they have flat components. Let $\mathcal{O}_X \to J$ be an injective resolution and let $F^\vee, E^\vee$ and $G^\vee$ be the complexes obtained by applying $\mathcal{H}om_X(-, J)$ to F, E and G, respectively. Let N be large enough so that $E^\vee, F^\vee$ and $G^\vee$ have no nonzero cohomology in degrees $\geq N$ and set $\overline{E} := (\tau^{\leq N} E)[N]$, and similarly for $\overline{F}$, $\overline{G}$. We let $J' := \tau^{\leq N} J[N]$. Then the sequence $0 \to \overline{G} \to \overline{E} \to \overline{F} \to 0$ is exact, and we have the natural commutative diagram of complexes

$$\begin{tikzcd} \overline{G} \otimes E \arrow[r] \arrow[d] & \overline{G} \otimes G \arrow[d] \\ \overline{E} \otimes E \arrow[r] & \overline{J}. \end{tikzcd}$$

Using the Alexander–Whitney map of the Dold–Kan correspondence, we get such a diagram in the category of simplicial $\mathcal{O}_X$-modules.

Now let $R = P^1_{f^{-1}\mathcal{O}_Y}(\mathcal{O}_X)$. Then we have the following commutative diagram of R-modules in which the rows are exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \overline{G} \otimes \Omega^1_{R/f^{-1}\mathcal{O}_Y} \otimes F \arrow[r] \arrow[d, "\beta"] & \overline{G} \otimes P^1_{R/f^{-1}\mathcal{O}_Y}(F) \arrow[r] \arrow[d] & \overline{G} \otimes F \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \overline{E} \otimes \Omega^1_{R/f^{-1}\mathcal{O}_Y} \otimes E \arrow[r] \arrow[d, "\alpha"] & \overline{E} \otimes P^1_{R/f^{-1}\mathcal{O}_Y}(E) \arrow[r] \arrow[d] & \overline{E} \otimes E \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega^1_{R/f^{-1}\mathcal{O}_Y} \otimes \overline{J} \arrow[r] & \alpha_*(\overline{E} \otimes P^1_{R/f^{-1}\mathcal{O}_Y}(E)) \arrow[r] & \overline{E} \otimes E \arrow[r] & 0. \end{tikzcd}$$

Here, the last row is obtained from the second by pushout along the map α, and α is induced from the evaluation map $\overline{E} \otimes E \to J$. Abbreviate $P(E) := P^1_{R/f^{-1}\mathcal{O}_Y}$. Since the composition $\alpha \circ \beta$ is zero, we can quotient out $\overline{G} \otimes \Omega^1_{R/f^{-1}\mathcal{O}_Y} \otimes F$ to obtain a diagram:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \overline{G} \otimes F \arrow[r] \arrow[d] & \overline{G} \otimes F \arrow[r] \arrow[d, "\iota"] & 0 \\ 0 \arrow[r] & \Omega^1_{R/f^{-1}\mathcal{O}_Y} \otimes \overline{J} \arrow[r] & \alpha_*(\overline{E} \otimes P(E)) \arrow[r] & \overline{E} \otimes E \arrow[r] & 0. \end{tikzcd}$$

Here, the vertical maps are injections of R-modules, so we can take the quotient exact sequence

$$0 \to \Omega^1_{R/f^{-1}\mathcal{O}_Y} \otimes \overline{J} \to \frac{\alpha_*(\overline{E} \otimes P(E))}{\iota(\overline{G} \otimes F)} \to \frac{\overline{E} \otimes E}{\overline{G} \otimes F} \to 0.$$

The induced connecting map defines (after extending scalars to $\mathcal{O}_X$, applying the Dold–Kan correspondence, and shifting) a morphism in $D^-(\mathcal{O}_X)$

$$\frac{E^\vee \otimes E}{G^\vee \otimes F}[-1] \to L_{X/Y},$$

which we call the Atiyah class $\mathrm{at}_{\underline{E}} = \mathrm{at}_{\underline{E},X/Y}$ of the exact sequence $0 \to F \to E \to G \to 0$.

Remark 3.11. It is easy to see by standard arguments that the result of the construction is independent of the choice of J. The dependence on N is not addressed here.

We observe the following directly from the construction.

Corollary 3.12. *The morphism* $\mathrm{at}_{\underline{E}}$ *is compatible with the usual Atiyah class (i.e., the diagram*

$$\begin{tikzcd}
E^\vee \otimes E[-1] \arrow[d] \arrow[dr, "\mathrm{at}'_E"] & \\
\frac{E^\vee \otimes E}{G^\vee \otimes F}[-1] \arrow[r, "\mathrm{at}_{\underline{E}}"] & L_{X/Y}.
\end{tikzcd}$$

commutes).

Corollary 3.13. *The map* $\mathrm{at}_{\underline{E},X/Y}$ *is functorial in* Y *(i.e., given a map* $Y \to Y'$*, the composition*

$$\frac{E^\vee \otimes E}{G^\vee \otimes F}[-1] \xrightarrow{\mathrm{at}_{\underline{E},X/Y'}} L_{X/Y'} \to L_{X/Y}$$

equals $\mathrm{at}_{\underline{E},X/Y}$ *(assuming we make the same choices of* N *in each construction)).*

Lemma 3.14. *The map* $\mathrm{at}_{\underline{E},X/Y}$ *is functorial for morphisms* $a : X' \to X$*. More precisely, given such a morphism, the diagram*

$$\begin{tikzcd}
a^*\left(\frac{E^\vee \otimes E}{G^\vee \otimes F}[-1]\right) \arrow[r, "\mathrm{at}_{\underline{E}}"] \arrow[d] & a^*L_{X/Y} \arrow[d] \\
\frac{(a^*E)^\vee \otimes a^*E}{(a^*G)^\vee \otimes a^*F}[-1] \arrow[r, "\mathrm{at}_{a^*\underline{E}}"] & L_{X'/Y}
\end{tikzcd}$$

commutes (assuming we make the same choices of N *in the construction). Here, we assume that* E, F, G *are already given by bounded above complexes with flat components.*

Proof. The main point is that to compute the derived pullback of $\overline{E}$ (and similarly $\overline{G}, \overline{F}$), we may use either a flat resolution of $\overline{E}$ – denote this $La^*\overline{E}$ by abuse of notation – or repeat the construction with a^*E in place of E, which we denote $\overline{a^*E}$. The two are related by a natural map $La^*E \to \overline{a^*E}$, which is a quasi-isomorphism, as can be checked locally where it follows from the assumption that E is a perfect complex. The details are left to the reader. □

Now suppose that we have an exact sequence $0 \to F \to E_X \to G \to 0$ as in (3.3). Then the Atiyah class for the exact sequence is related to the reduced Atiyah class.

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Proposition 3.15. *We have a commutative diagram*

$$\begin{tikzcd} \frac{E^\vee \otimes E}{G^\vee \otimes F}[-1] \arrow[r, "-"] \arrow[dr, "\mathrm{at}_{\underline{E}}"'] & G^\vee \otimes F \arrow[d, "\overline{\mathrm{at}}'_E"] \\ & L_{X/Y}, \end{tikzcd}$$

where the horizontal map is minus the natural connecting homomorphism.

Proof. We may work with the sequence

$$0 \to F_R \to E_R \to G_R \to 0$$

instead of F, E, G as in the construction of the reduced Atiyah class. We need to show that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \overline{G} \otimes F_R \arrow[r] \arrow[d] & \overline{E} \otimes E_R \arrow[r] \arrow[d] & \frac{\overline{E} \otimes E_R}{\overline{G} \otimes F_R} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L_{X/Y} \otimes \overline{J} \arrow[r] & \frac{\alpha_*(\overline{E} \otimes P(E_R))}{\iota(\overline{G} \otimes F_R)} \arrow[r] & \frac{\overline{E} \otimes E_R}{\overline{G} \otimes F_R} \arrow[r] & 0, \end{tikzcd}$$

where the left vertical arrow is obtained from the composition

$$\overline{G} \otimes F_R \to \overline{G} \otimes \Omega^1_{R/f^{-1}\mathcal{O}_Y} \otimes G_R \to \overline{G} \otimes \Omega^1_{R/f^{-1}\mathcal{O}_Y} \otimes G \to \overline{J}$$

and therefore induces the reduced Atiyah class (up to a shift) when passing to $D(X)$. The proposition then clearly follows. To construct the diagram, let $s : E_R \to P(E_R)$ denote the section in the construction of the reduced Atiyah class. Then the middle vertical map is the composition

$$\overline{E} \otimes E_R \xrightarrow{s \otimes \mathrm{id}} \overline{E} \otimes P(E) \to \frac{\alpha_*(\overline{E} \otimes P(E_R))}{\iota(\overline{G} \otimes F_R)}.$$

Since the composition $\overline{E} \otimes E_R \to \overline{E} \otimes P(E_R) \to \overline{E} \otimes E_R$ is the identity, we naturally obtain a commutative diagram of the desired form, and it remains only to show that the left vertical map is as desired. But in fact, we have a subdiagram

$$\begin{tikzcd} 0 \arrow[r] & \overline{G} \otimes F_R \arrow[r] \arrow[d] & \overline{G} \otimes E_R \arrow[r] \arrow[d] & \frac{\overline{G} \otimes E_R}{\overline{G} \otimes F_R} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L_{X/Y} \otimes \overline{J} \arrow[r] & \frac{\alpha_*(\overline{G} \otimes P(E_R))}{\iota(\overline{G} \otimes F_R)} \arrow[r] & \frac{\overline{G} \otimes E_R}{\overline{G} \otimes F_R} \arrow[r] & 0, \end{tikzcd}$$

the lower row of which is identified with the row

$$0 \to L_{X/Y} \otimes \overline{J} \to \alpha_*(\overline{G} \otimes G_R) \to \overline{G} \otimes G_R \to 0.$$

Under this identification, the middle map factors as

$$\overline{G} \otimes E_R \xrightarrow{id_{\overline{G}} \otimes s} \overline{G} \otimes P(G) \to \alpha_*(\overline{G} \otimes P(G)),$$

and it follows that the left vertical map indeed factors through the reduced Atiyah class, as desired. □

4. Definitions

We construct the Atiyah class for an algebraic stack and show that it is independent of various choices made in the construction. We then address the case of the reduced Atiyah class and of the Atiyah class of an exact sequence.

4.1. Construction of the Atiyah class

We define the Atiyah class for an object $E \in D^{\leq 0}_{qcoh}(\mathcal{X})$ *with vanishing cohomology groups in positive degree*. In Construction 5.6, we generalize this to arbitrary bounded above complexes, which uses the invariance of the Atiyah class under shifts (Corollary 5.5).

Construction 4.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and let $E \in D^{\leq 0}_{qcoh}(\mathcal{X})$. By truncating, we may assume that E is represented by a complex with nonzero terms only in negative degrees. We let $E_{W_\parallel}$ be the induced $\mathcal{O}_{W_\parallel}$-module, which we also regard as a simplicial module. Choose a diagram as in (2.3) and consider the setup of Situation 2.9. We let $R := P_{h^{-1}\mathcal{O}_{Y_\parallel}}(W_\parallel)$ be the free simplicial resolution. This is a simplicial ring on $W_\parallel$ with components $R_X = P_{g^{-1}\mathcal{O}_{Y_\bullet}}(\mathcal{O}_{X_\bullet})$ and $R_W = P_{h^{-1}\mathcal{O}_{Y_\bullet}}(\mathcal{O}_{W_\bullet})$. We regard $E_{W_\parallel}$ as an R-module via restriction of scalars. Consider the exact sequence of principal parts $\underline{P}^1_{R/\mathcal{O}_{Y_\parallel}}(E_{W_\parallel})$ associated to $E_{W_\parallel}$ and the ring map $h_\parallel^{-1}\mathcal{O}_{Y_\parallel} \to R$.

$$0 \to L_{W_\parallel/Y_\parallel} \otimes_{\mathcal{O}_{W_\parallel}} E_{W_\parallel} \to P^1_{R/h_\parallel^{-1}\mathcal{O}_{Y_\parallel}}(E_{W_\parallel}) \to E_{W_\parallel} \to 0.$$

It induces a map

$$\delta_{R/h_\parallel^{-1}\mathcal{O}_{Y_\parallel}}(E_{W_\parallel}) : E_{W_\parallel} \to \sigma L_{W_\parallel/Y_\parallel} \otimes_{\mathcal{O}_{W_\parallel}} E_{W_\parallel} \tag{4.1}$$

in $D^\Delta(R)$. By Lemma 2.2, this is just the restriction of a morphism in $D^\Delta(W_\parallel)$ and thus corresponds to a unique morphism $E_{W_\parallel} \to L_{W_\parallel/Y_\parallel} \otimes_{\mathcal{O}_{W_\parallel}} E_{W_\parallel}[1]$ in $D(\mathcal{O}_{W_\parallel})$ by the Dold–Kan correspondence. We have canonical natural isomorphisms $\mathrm{Cone}_{\mathcal{O}_{W_\parallel}}(E) \xrightarrow{\sim} E_{W_\bullet}[1]$ and $\mathrm{Cone}_{\mathcal{O}_{W_\parallel}}(L_{W_\parallel/Y_\parallel}) \xrightarrow{\sim} \eta_W^* L_{\mathcal{X}/\mathcal{Y}}[1]$ by Lemmas 2.10 and 2.11, respectively. Since E has quasi-coherent cohomology sheaves, the assumptions of Lemma 2.7 i) are satisfied, so that we obtain a morphism

$$E_{W_\bullet}[1] \to \eta_W^* L_{\mathcal{X}/\mathcal{Y}} \otimes E_{W_\bullet}[2]. \tag{4.2}$$

Applying η_{W*} and the shift $[-1]$ and multiplying by -1 (the sign is obtained from commuting two shift functors and is needed for compatibility with the usual Atiyah class), we obtain a morphism

$$\mathrm{at}_{E,\mathcal{X}/\mathcal{Y}} := \mathrm{at}_{E,\mathcal{X}/\mathcal{Y},X/Y} : E \to L_{\mathcal{X}/\mathcal{Y}} \otimes E[1]. \tag{4.3}$$

This is the *Atiyah class of E over $\mathcal{Y}$*. We will also write at_E if the morphism $f : \mathcal{X} \to \mathcal{Y}$ is understood. In Corollary 4.6, we show that the Atiyah class is independent of choice of diagram (2.3).

As a consequence of the construction, we have the following:

Lemma 4.2. *The morphism (4.3) is functorial in $E \in D^{\leq 0}_{qcoh}(\mathcal{X})$ for a fixed choice of diagram (2.3).*

Remark 4.3. i) In Construction 4.1, let F be an R-module together with an isomorphism $F \to E_{W_\parallel}$ in $D^\Delta(R)$. Then we may instead work with the exact sequence $\underline{P}_{R/h_\parallel^{-1}\mathcal{O}_{Y_\parallel}}(F)$ and define (4.1) equivalently as the map obtained as the composition

$$E_{W_\parallel} \xrightarrow{\sim} F \to \sigma \underline{P}^1_{R/\mathcal{O}_{Y_\parallel}}(F_{W_\parallel}) \xrightarrow{\sim} \sigma L_{W_\parallel/Y_\parallel} \otimes_{\mathcal{O}_{W_\parallel}} E_{W_\parallel}.$$

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ii) We may also replace $W_{\|}$ and $Y_{\|}$ by their analogues $W_{\wedge}$ and $Y_{\wedge}$, associated to the respective diagrams

$$X_\bullet \xleftarrow{s_\bullet} W_\bullet \xrightarrow{t_\bullet} X_\bullet, \text{ and } Y_\bullet \xleftarrow{=} Y_\bullet \xrightarrow{=} Y_\bullet$$

throughout Construction 4.1.

Well-definedness and compatibility with pullback

Suppose that we are given a map of diagrams (2.3) – that is, that we have a 2-commuting cube

$$\begin{tikzcd}
X' \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{X}' \arrow[dr, "A"] \arrow[dd] & \\
& X \arrow[rr] \arrow[dd] & & \mathcal{X} \arrow[dd] \\
Y' \arrow[rr] \arrow[dr] & & \mathcal{Y}' \arrow[dr] & \\
& Y \arrow[rr] & & \mathcal{Y},
\end{tikzcd} \tag{4.4}$$

whose front and back faces are as in (2.3). (This means, in particular, that for any two maps $X' \to \mathcal{Y}$ obtained by traveling along the edges of the cube, the two two-isomorphisms relating them by traversing the faces are identical.)

Lemma 4.4. *Let $E \in D^{\leq 0}_{qcoh}(\mathcal{X})$. Let*

$$\mathrm{at}_E := \mathrm{at}_{E,\mathcal{X}/\mathcal{Y},X/Y} \;\; \textit{and} \;\; \mathrm{at}_{A^*E} := \mathrm{at}_{A^*E,\mathcal{X}'/\mathcal{Y}',X'/Y'}\,.$$

Then the diagram

$$\begin{tikzcd}
A^*E \arrow[r, "A^*\mathrm{at}_E"] \arrow[d] & A^*L_{\mathcal{X}/\mathcal{Y}} \otimes A^*E[1] \arrow[d] \\
A^*E \arrow[r, "\mathrm{at}_{A^*E}"] & L_{\mathcal{X}'/\mathcal{Y}'} \otimes A^*E[1]
\end{tikzcd}$$

commutes.

Proof. By using the setup of Situation 2.9 and Construction 4.1 for the primed objects, we obtain a commutative cube of simplicial algebraic spaces

$$\begin{tikzcd}
W'_\bullet \arrow[rr, "t'_\bullet"] \arrow[dr, "s'_\bullet"'] \arrow[dd, "a_\bullet"'] & & X'_\bullet \arrow[dr] \arrow[dd, "b_\bullet"' near start] & \\
& X'_\bullet \arrow[rr] \arrow[dd, "b_\bullet"' near start] & & Y'_\bullet \arrow[dd] \\
W_\bullet \arrow[rr, "t_\bullet" near end] \arrow[dr, "s_\bullet"'] & & X_\bullet \arrow[dr] & \\
& X_\bullet \arrow[rr] & & Y_\bullet
\end{tikzcd}$$

as well as morphisms of topoi $a_{\|} : W'_{\|} \to W_{\|}$ and $Y'_{\|} \to Y_{\|}$, which fit in a 2-commutative square

$$\begin{tikzcd} W'_{\|} \arrow[r] \arrow[d] & Y'_{\|} \arrow[d] \\ W_{\|} \arrow[r] & Y_{\|}. \end{tikzcd}$$

By construction, the Atiyah class at_E corresponds via η_W^* to a map $\alpha : E_{W_\bullet} \to \eta_W^* L_{\mathcal{X}/\mathcal{Y}}[1] \otimes E_{W_\bullet}$, obtained as the shift of the map (4.2). Since the pullback A^* can be computed as $\eta_{W'*} a_\bullet^* \eta_W^*$, we have a natural commutative diagram

$$\begin{tikzcd} \eta_{W'}^* A^* E \arrow[r, "\eta_{W'}^* A^* \mathrm{at}_E"] \arrow[d] & \eta_{W'}^* A^* (L_{\mathcal{X}/\mathcal{Y}}[1] \otimes E) \arrow[d] \\ a_\bullet^* E_W \arrow[r, "a_\bullet^* \alpha"] & a_\bullet^* (\eta_W^* L_{\mathcal{X}/\mathcal{Y}}[1] \otimes E_W). \end{tikzcd}$$

Let $E' := \eta_{W'*}(a_\bullet^* E_W)$. We apply Construction 4.1 to the primed objects (i.e., with respect to the backside of (4.4) and E'). We obtain the ring $R' = P_{h_{\|}'^{-1}\mathcal{O}_{Y'_{\|}}}(\mathcal{O}_{W'_{\|}})$ and the map

$$\alpha' : E'_{W'_\bullet} \to \eta_{W'}^* L_{\mathcal{X}'/\mathcal{Y}'}[1] \otimes E'_{W'_\bullet},$$

as the shift of the map (4.2). By functoriality of simplicial resolutions, we have a natural map $a_{\|}^{-1} R \to R'$ of rings on $W'_{\|}$. Now the statement of the lemma is equivalent to the following:

Claim 4.5. *The diagram*

$$\begin{tikzcd} a_\bullet^* E_{W_\bullet} \arrow[r, "a_\bullet^* \alpha"] \arrow[d] & a_\bullet^* (\eta_W^* L_{\mathcal{X}/\mathcal{Y}}[1] \otimes E_{W_\bullet}) \arrow[d] \\ E'_{W'_\bullet} \arrow[r, "\alpha'"] & \eta_{W'}^* L_{\mathcal{X}'/\mathcal{Y}'}[1] \otimes E'_{W'_\bullet} \end{tikzcd}$$

in $D(W'_\bullet)$ commutes, where the vertical maps are induced by the natural isomorphism $a_\bullet^* E_{W_\bullet} \xrightarrow{\sim} E'_{W'}$ and the pullback map on cotangent complexes.

Proof. We may assume that E is represented by a complex of flat modules concentrated in degrees ≤ 0, so that we have an induced representative with flat terms for $E_{W_{\|}}$. Then the pullback $a_{\|}^* E_{W_{\|}} = a_{\|}^* E_{W_{\|}}$ can be computed termwise, and its components are given by $b_\bullet^* E_X$ and $a_\bullet^* E_W$, respectively.

By functoriality of the principal parts construction, we have a morphism of exact sequences of $a_{\|}^{-1}R$-modules $a_{\|}^{-1} \underline{P}^1_{\mathcal{O}_{W_{\|}}/h_{\|}^{-1}\mathcal{O}_{Y_{\|}}}(E_{W_{\|}}) \to \underline{P}^1_{\mathcal{O}_{W'_{\|}}/h'^{-1}_{\|}\mathcal{O}_{Y'_{\|}}}(a_{\|}^* E_{W_{\|}})$. We get the induced diagram of connecting maps in $D^\Delta(a_{\|}^{-1}R)$

$$\begin{tikzcd} a_{\|}^{-1}\delta_{R/h_{\|}^{-1}\mathcal{O}_{Y_{\|}}}(E_{W_{\|}}) : a_{\|}^{-1} E_{W_{\|}} \arrow[r] \arrow[d] & \sigma a_{\|}^{-1} L_{W_{\|}/Y_{\|}} \otimes_{a_{\|}^{-1}\mathcal{O}_{W_{\|}}} a_{\|}^{-1} E_{W_{\|}} \arrow[d] \\ \delta_{R'/h'^{-1}_{\|}\mathcal{O}_{Y'_{\|}}}(a_{\|}^* E_{W_{\|}}) : a_{\|}^* E_{W_{\|}} \arrow[r] & \sigma L_{W'_{\|}/Y'_{\|}} \otimes_{\mathcal{O}_{W'_{\|}}} a_{\|}^* E_{W_{\|}}. \end{tikzcd}$$

By adjunction, this corresponds to a diagram in $D^\Delta(R')$, which in turn corresponds to a diagram in $D(\mathcal{O}_{W_{\|}})$:

$$\begin{tikzcd} a_{\|}^* E_{W_\|} \arrow[r] \arrow[d] & a_{\|}^* L_{W_\|/Y_\|}[1] \otimes a_{\|}^* E_{W_\|} \arrow[d] \\ a_{\|}^* E_{W_\|} \arrow[r] & L_{W'_\|/Y'_\|}[1] \otimes a_{\|}^* E_{W_\|}. \end{tikzcd}$$

After applying $\mathrm{Cone}_{\mathcal{O}_{W'_\|}}$ and shifting, the upper line is canonically identified with $a_\bullet^* \alpha$ (this uses that we have a natural isomorphism $\mathrm{Cone}_{\mathcal{O}_{W'_\|}} \circ a_\|^* \simeq a_\bullet^* \circ \mathrm{Cone}_{\mathcal{O}_{W_\|}}$, which one can check by computing with K-flat complexes of $\mathcal{O}_{W_\|}$ modules). The lower line yields a map

$$\alpha'' : a_\bullet^* E_{W_\bullet} \to \eta_{W'}^* L_{\mathcal{X}'/\mathcal{Y}'}[1] \otimes a_\bullet^* E_{W_\bullet}.$$

To finish the proof of the claim, we need to show that under the isomorphism $a^* E_W \to E'_{W'}$, the maps α'' and α get identified. This follows from Lemma 2.13 and Remark 4.3 ii). □

□

Corollary 4.6. *The map* $\mathrm{at}_E : E \to L_{\mathcal{X}/\mathcal{Y}}[1] \otimes E$ *obtained from Construction 4.1 is independent of choice of diagram (2.3).*

Proof. Suppose we are given two choices

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ Y_1 \arrow[r] & \mathcal{Y} \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ Y_2 \arrow[r] & \mathcal{Y}. \end{tikzcd}$$

We need to show that $\mathrm{at}_{E,\mathcal{X}/\mathcal{Y},X_1/Y_1}$ and $\mathrm{at}_{E,\mathcal{X}/\mathcal{Y},X_2/Y_2}$ agree. By replacing $X_2 \to Y_2$ with the fiber product $X_1 \times_{\mathcal{X}} X_2 \to Y_1 \times_{\mathcal{Y}} Y_2$, we may without loss of generality assume that we have a 2-commuting diagram

$$\begin{tikzcd} & X_2 \arrow[dl] \arrow[dr] \arrow[dd] & \\ X_1 \arrow[rr] \arrow[dd] & & \mathcal{X} \arrow[dd] \\ & Y_2 \arrow[dl] \arrow[dr] & \\ Y_1 \arrow[rr] & & \mathcal{Y}. \end{tikzcd}$$

By adding in the identity morphisms on $\mathcal{X}$ and $\mathcal{Y}$, this gives a 2-commuting cube (4.4), and we obtain the desired equality from Lemma 4.4. □

Corollary 4.7. *The Atiyah class is compatible with pullback. More precisely, given a 2-commutative square*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "A"] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y}, \end{tikzcd}$$

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the natural diagram

$$\begin{tikzcd} A^*E \arrow[r, "A^* \mathrm{at}_E"] \arrow[d] & A^*L_{\mathcal{X}/\mathcal{Y}} \otimes A^*E[1] \arrow[d] \\ A^*E \arrow[r, "\mathrm{at}_{A^*E}"] & L_{\mathcal{X}'/\mathcal{Y}'} \otimes A^*E[1] \end{tikzcd}$$

commutes.

Proof. This follows immediately from Lemma 4.4, after completing the square to a cube as in (4.4). □

Compatibility with Illusie's definition

Proposition 4.8. *If $\mathcal{X}$ and $\mathcal{Y}$ are in fact algebraic spaces, then the morphism* at_E *agrees with Illusie's Atiyah class for E with respect to the map of ringed étale topoi $\mathcal{X}_{\mathrm{et}} \to \mathcal{Y}_{\mathrm{et}}$.*

Proof. In this case, we may choose $X = \mathcal{X}$ and $Y = \mathcal{Y}$ in diagram (2.3), so that we get $W = X$. Then $X_\bullet$ and $W_\bullet$ are constant strictly simplicial algebraic spaces with value X, and $Y_\bullet$ is the constant strictly simplicial algebraic space with value Y. In this case, we have the morphism of topoi $\Delta : W_{\|} \to X_\bullet$, where $\Delta^* M_X = (M_X, M_X, \mathrm{id}, \mathrm{id})$. The morphism c_E of Lemma 2.6 gives a natural transformation of functors $\mathrm{Cone}_{\mathcal{O}_{W_{\|}}} \circ \Delta^* \Rightarrow [-1]$, which induces to a natural isomorphism of functors from $D(\mathcal{O}_{X_\bullet})$ to itself. It follows from Construction 4.1 that the morphism $E_{W_{\|}} \to L_{W_{\|}/Y_{\|}}[1] \otimes_{\mathcal{O}_{W_{\|}}} E_{W_{\|}}$ obtained from (4.1) by applying the functors $D^\Delta(R) \to D^\Delta(\mathcal{O}_{W_{\|}}) \to D(W_{\|})$ is naturally identified with the pullback $\Delta^* \mathrm{at}_{E_{X_\bullet}, X_\bullet/Y_\bullet}$ of the Atiyah class for the map of topoi $X_\bullet \to Y_\bullet$. Thus, taking the cone, we see that (4.2) is identified with $\mathrm{at}_{E_{X_\bullet}, X_\bullet/Y_\bullet}[1]$. From the commutative diagram of topoi

$$\begin{tikzcd} X_{\bullet,\text{lis-et}} \arrow[r, "\pi_X"] \arrow[d, "\epsilon"] & X_{\text{lis-et}} \arrow[d, "\epsilon"] \\ X_{\bullet,\mathrm{et}} \arrow[r] & X_{\mathrm{et}}, \end{tikzcd}$$

we see that $\eta_{X*} \mathrm{at}_{E_{X_\bullet}, X_\bullet/Y_\bullet}$ is naturally identified with the pullback to the lisse-étale site of the usual Atiyah class for the map $X_{\mathrm{et}} \to Y_{\mathrm{et}}$, from which the result follows. □

As a first application of our definition, we compute the Atiyah class of the universal vector bundle on BGL_n over a chosen base field k.

Example 4.9. Let $\mathcal{V}$ denote the universal rank n locally free sheaf on BGL_n, which is the sheaf of sections of the vector bundle associated to the universal principal GL_n-bundle. We have the 2-cartesian diagram, where the map x from $\operatorname{Spec} k$ is the one corresponding to the trivial GL_n-torsor

$$\begin{tikzcd} GL_n \arrow[r, "t"] \arrow[d, "s"] & \operatorname{Spec} k \arrow[d, "x"] \\ \operatorname{Spec} k \arrow[r, "x"] \arrow[ur, Rightarrow, "\rho"] & BGL_n. \end{tikzcd}$$

There is a natural trivialization of the pullback $\mathcal{V}_{\operatorname{Spec} k}$. Then for every T-valued point $g = (g_{i,j})$ on GL_n, the 2-morphism ρ indicated in the diagram induces a natural pullback map $\rho^* : t^*\mathcal{V}_{\operatorname{Spec} k}|_T \xrightarrow{\sim} s^*\mathcal{V}_{\operatorname{Spec} k}|_T$. With respect to the given trivialization of $\mathcal{V}$ over $\operatorname{Spec} k$, this morphism is just given by $g^{-1} : \mathcal{O}_T^{\oplus n} \to \mathcal{O}_T^{\oplus n}$.

We now compute the Atiyah class of $\mathcal{V}$ using Construction 4.1, where we chose $X = Y = \operatorname{Spec} k$, so that we get $W = GL_n$. More precisely, we will calculate the restriction of $\mathrm{at}_{\mathcal{V}}$ to the etale site of W, so that we do not have to consider the associated strictly simplicial algebraic spaces. Thus, here we let $W_{\|}$

and $Y_{\|}$ be the topos associated to the diagrams

$$W \underset{t}{\overset{s}{\rightrightarrows}} X \text{ and } Y \rightrightarrows Y,$$

respectively, and $h_{\|} : W_{\|} \to Y_{\|}$ the natural map, and we let $\mathcal{V}_{W_{\|}}$ denote the locally free sheaf on $W_{\|}$ obtained by pullback. First, to calculate $\mathrm{at}_{\mathcal{V}_{W_{\|}}, W_{\|}/Y_{\|}}$, we do not need to take a simplicial resolution of $\mathcal{O}_{W_{\|}}$ over $h_{\|}^{-1}\mathcal{O}_{Y_{\|}}$ since the map $W_{\|} \to Y_{\|}$ is composed of smooth maps of algebraic spaces. Moreover, since $\mathrm{Cone}_{\mathcal{O}_{W_{\|}}}$ preserves mapping cones of complexes, we may calculate $\mathrm{Cone}_{\mathcal{O}_{W_{\|}}}(\mathrm{at}_{\mathcal{V}_{W_{\|}}, W_{\|}/Y_{\|}})$ by first applying $\mathrm{Cone}_{\mathcal{O}_{W_{\|}}}$ to the sequence of principal parts $P_{W_{\|}/Y_{\|}}(\mathcal{V}_{W_{\|}})$ and only then taking connecting homomorphisms. Writing out the pullback maps in the sequence of principal parts $P_{W_{\|}/Y_{\|}}(\mathcal{V}_{W_{\|}})$ gives the following diagram on W, where we use that $\Omega_{X/Y} = 0$:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & s^*\mathcal{V}_X \arrow[r] \arrow[d] & s^*\mathcal{V}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_{W/Y} \otimes \mathcal{V}_W \arrow[r] & P_{W/Y}(\mathcal{V}_W) \arrow[r] & \mathcal{V}_W \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & t^*\mathcal{V}_X \arrow[r] \arrow[u] & t^*\mathcal{V}_X \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Since $\mathcal{V}_W$ is pulled back from $X = Y = \operatorname{Spec} k$ via s, we have a natural splitting $P_{W/Y}(\mathcal{V}_W) = \Omega_{W/Y} \otimes s^*\mathcal{V}_X \oplus s^*\mathcal{V}_X$. With respect to this splitting, one calculates that the induced pullback map along t is given by

$$t^*\mathcal{V}_X \xrightarrow{(d\rho, \rho)} \Omega_{W/Y} \otimes s^*\mathcal{V}_X \oplus s^*\mathcal{V}_X.$$

It follows, that after applying $\mathrm{Cone}_{\mathcal{O}_{W_{\|}}}$, the resulting exact sequence has the following exact subsequence, and the termwise inclusions give quasi-isomorphisms:

$$0 \to \Omega_{W/Y} \otimes s^*\mathcal{V}_X \to \mathrm{Cone}(t^*\mathcal{V}_X \xrightarrow{d\rho} \Omega_{W/Y} \otimes s^*\mathcal{V}_X) \to t^*\mathcal{V}_X[1] \to 0.$$

It is a general fact that for an exact sequence of this form, the connecting map is canonically quasi-isomorphic to minus the shift of the map the cone is taken over for the middle term – that is, in our case, this gives

$$d\rho[1] : t^*\mathcal{V}_X[1] \to \Omega_{W/Y} \otimes s^*\mathcal{V}_X[1].$$

In particular, we find that there is a natural isomorphism $s^*x^*L_{BGL_n}[1] \simeq \Omega_{W/Y}$ and that using this identification and the trivialization of $\mathcal{V}_X$, we have that $s^*x^*\,\mathrm{at}_{\mathcal{V}}$ is given by

$$\mathcal{O}_{GL_n}^{\oplus n} \xrightarrow{d(T^{-1})\circ T} \Omega_{GL_n} \otimes \mathcal{O}_{GL_n}^{\oplus n}, \tag{4.5}$$

where $T = (T_{ij})$ is the universal matrix on GL_n. In particular, for $n = 1$, we have the following:

Example 4.10. By taking the trace in (4.5), we obtain the section

$$\mathcal{O}_{GL_n} \to \Omega_{GL_n}$$

given by $\mathrm{tr}(d(T^{-1})T)$. This is equal to

$$d(\det T^{-1}) \det T.$$

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In particular, for $n = 1$, we have $T = (t)$ as a 1×1 matrix, and the pullback of $\mathrm{at}_{\mathcal{L}} \otimes \mathcal{L}^{-1}$ – with $\mathcal{L}$ the universal rank 1 sheaf – to $\mathbb{G}_m = GL_1$ is given by

$$\mathcal{O}_{\mathbb{G}_m} \to \Omega_{\mathbb{G}_m},$$

sending 1 to $d(1/t)t = -dt/t$. We have the natural map $DET : BGL_n \to B\mathbb{G}_m$ given by associating to a locally free sheaf its determinant line bundle. By taking fiber products with a point, this induces a map $GL_n \to \mathbb{G}_m$, (which is again just the determinant) whose pullback map on differentials is given by $dt \mapsto d(\det T)$. In particular, we have $td(1/t) \mapsto \det T d(\det T^{-1})$. This shows that we have a natural identification $\mathrm{tr}(s^* x^* \mathrm{at}_{\mathcal{V}}) = s^* x^* (\mathrm{at}_{\det \mathcal{V}} \otimes \det \mathcal{V}^{-1})$. One can show that this isomorphism is part of a descent datum, and thus we have

$$\mathrm{tr}(\mathrm{at}_{\mathcal{V}}) = \mathrm{at}_{\det \mathcal{V}} \otimes \det(\mathcal{V})^{-1}.$$

4.2. Construction of the reduced Atiyah class

Construction 4.11. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $E \in D^-_{qcoh}(\mathcal{Y})$ and let $E_{\mathcal{X}} := f^* E$. Let $F \to E_{\mathcal{X}} \to G \xrightarrow{+1}$ be a distinguished triangle in $D^-_{qcoh}(\mathcal{X})$ such that $R\,\mathrm{Hom}^{-1}(F, G) = 0$. We let $X_\bullet, Y_\bullet$ and $W_\bullet$ be as in Situation 2.9 and let $R = P_{h^{-1}\mathcal{O}_{Y_\|}}(\mathcal{O}_{W_\|})$. Let $F_{W_\|}, E_{W_\|}$ and $G_{W_\|}$ denote the $\mathcal{O}_{W_\|}$-modules induced by $E_{\mathcal{X}}$ and G, respectively, and let $E_{Y_\|}$ denote the $\mathcal{O}_{Y_\|}$-module induced by E. From Construction 3.5, we obtain a morphism

$$\overline{\mathrm{at}}_{E_{Y_\|}} : F_{W_\|} \to L_{W_\|/Y_\|} \otimes G_{W_\|}.$$

Applying the sequence of maps of (2.6), we get

$$\overline{\mathrm{at}}_{E,\mathcal{X}/\mathcal{Y},G} : F \to L_{\mathcal{X}/\mathcal{Y}} \otimes G.$$

We also write $\overline{\mathrm{at}}_E$ if the rest of the data is clear.

Proposition 4.12. *The reduced Atiyah class is independent of the choice of cover X/Y.*

Proof. This is similar to Lemma 4.4 and left to the reader (and slightly simpler since E and G are assumed to be quasicoherent sheaves, rather than more general complexes). □

Proposition 4.13. *If $\mathcal{X}$ and $\mathcal{Y}$ are algebraic spaces, the reduced Atiyah class of Construction 4.11 agrees with the one for ringed topoi defined in §3.1.*

Proof. The proof is analogous to the proof of Proposition 4.8 and left to the reader. □

Corollary 4.14. *The following triangles commute:*

1)

$$\begin{tikzcd} F \arrow[r, "-\overline{\mathrm{at}}_E"] \arrow[dr, "\mathrm{at}_F"'] & L_{\mathcal{X}/\mathcal{Y}} \otimes G \arrow[d] \\ & L_{\mathcal{X}/\mathcal{Y}}[1] \otimes F, \end{tikzcd}$$

2)

$$\begin{tikzcd} G \arrow[d] \arrow[dr, "\mathrm{at}_G"] & \\ F[1] \arrow[r, "\overline{\mathrm{at}}_E[1]"'] & L_{\mathcal{X}/\mathcal{Y}}[1] \otimes G. \end{tikzcd}$$

In both cases, the vertical morphisms are induced from the connecting map $G \to F[1]$ of the given exact sequence.

Proof. From Remark 3.2, we get commutative triangles for the classical Atiyah classes $\mathrm{at}_{E_{W_\|}, W_\|/Y_\|, G_{W_\|}}$ and $\overline{\mathrm{at}}_{E_{Y_\|}, W_\|/Y_\|, G_{W_\|}}$. The result follows by passing to $D(\mathcal{X})$. □

4.3. Construction of the Atiyah class for an exact sequence

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and let $0 \to F \to E \to G \to 0$ be an exact sequence of quasicoherent sheaves on $\mathcal{X}$, which we denote by $\underline{E}$. Assume that F, E and G are dualizable as objects of the derived category of $\mathcal{X}$. We let $X_\|$, $Y_\|$ and $W_\|$ be as in Situation 2.9. We also let $R = P_{h^{-1}\mathcal{O}_{Y_\|}} \mathcal{O}_{W_\|}$, and we let $F_{W_\|}$, $E_{W_\|}$ and $G_{W_\|}$ denote the sheaves on $W_\|$ induced by F, E, G, respectively. Let

$$\alpha : \frac{E_{W_\|} \otimes E^\vee_{W_\|}}{F_{W_\|} \otimes G^\vee_{W_\|}}[-1] \to L_{W_\|/Y_\|}$$

be the Atiyah class of the exact sequence $\underline{E}_{W_\|}$ with respect to the morphism of topoi $W_\| \to Y_\|$. This is a morphism in $D(W_\|)$. By taking cones, shifting, and applying η_{W*}, we obtain a morphism

$$\mathrm{at}_{\underline{E},\mathcal{X}/\mathcal{Y}} : \frac{E \otimes E^\vee}{F \otimes G^\vee}[-1] \to L_{\mathcal{X}/\mathcal{Y}},$$

which we call the Atiyah class of the exact sequence $\underline{E}$. We also write $\mathrm{at}_{\underline{E}}$ if the morphism $f : \mathcal{X} \to \mathcal{Y}$ is understood.

From the construction and Corollary 3.12, we conclude the following:

Corollary 4.15. *The morphism $\mathrm{at}_{\underline{E}}$ is compatible with the usual Atiyah class (i.e., the diagram*

$$\begin{tikzcd} E^\vee \otimes E[-1] \arrow[d] \arrow[dr, "\mathrm{at}'_E"] & \\ \frac{E^\vee \otimes E}{G^\vee \otimes F}[-1] \arrow[r, "\mathrm{at}_{\underline{E}}"] & L_{\mathcal{X}/\mathcal{Y}}. \end{tikzcd}$$

commutes).

Similarly, we conclude immediately from Proposition 3.15:

Corollary 4.16. *We have a commutative diagram*

$$\begin{tikzcd} \frac{E^\vee \otimes E}{G^\vee \otimes F}[-1] \arrow[r, "-"] \arrow[dr, "\mathrm{at}_{\underline{E}}"'] & G^\vee \otimes F \arrow[d, "\overline{\mathrm{at}}'_E"] \\ & L_{\mathcal{X}/\mathcal{Y}}, \end{tikzcd}$$

where the horizontal map is the natural connecting homomorphism.

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5. Properties

5.1. Tensor compatibility of the Atiyah class

Let A be a ring, and B an A-algebra. For a B-module M, let $\underline{P}^1_{B/A}(M)$ denote the exact sequence

$$0 \to \Omega_{B/A} \otimes_B M \to P^1_{B/A}(M) \to M \to 0.$$

We will also write $\underline{P}^1_{B/A}(M)$ to denote the corresponding element of $\mathrm{Ext}^1_B(M, \Omega_{B/A} \otimes_B M)$.

Lemma 5.1. *Let M and N be B-modules and suppose that M is flat. Then in $\mathrm{Ext}^1_B(M \otimes_B N, \Omega_{B/A} \otimes_B M \otimes_B N)$ we have the following equality:*

$$\underline{P}_{B/A}(M) \otimes_B N + M \otimes_B \underline{P}_{B/A}(N) = \underline{P}_{B/A}(M \otimes N).$$

Here, we regard $M \otimes_B \underline{P}_{B/A}(N)$ as an extension of $M \otimes N$ by $\Omega_{B/A} \otimes_B M \otimes_B N$ via the symmetry isomorphism of the tensor product.

Proof. Recall that $P^1_{B/A}(M) = B \otimes_A M/(I^2_\Delta B \otimes_A M)$, where I_Δ is the kernel of the map of A-algebras $B \otimes_A B \longrightarrow B$; the B-module structure on $P^1_{B/A}(M)$ is given by action on the left side of the tensor product. Therefore, we have natural isomorphisms $P_{B/A}(M) \otimes_B N \simeq N \otimes_B P_{B/A}(M) \simeq N \otimes_A M/I_{N,M}$, where $I_{N,M} := I^2_\Delta(N \otimes_A M)$, and similarly with N and M reversed.

We then have that the left-hand side of the equality in the lemma is represented by the Baer sum of the following two exact sequences:

$$0 \to \Omega_{B/A} \otimes_B M \otimes_B N \xrightarrow{j_1} N \otimes_A M/I^2_{N,M} \xrightarrow{p_1} M \otimes_B N \to 0, \tag{5.1}$$

$$0 \to \Omega_{B/A} \otimes_B M \otimes_B N \xrightarrow{j_2} M \otimes_A N/I^2_{M,N} \xrightarrow{p_2} M \otimes_B N \to 0, \tag{5.2}$$

where

$$\begin{aligned} j_1(db \otimes m \otimes n) &= [bn \otimes m - n \otimes bm], \\ p_1([n \otimes m]) &= m \otimes n, \text{ and} \\ j_2(db \otimes m \otimes n) &= [bm \otimes n - m \otimes bn], \\ p_2([m \otimes n]) &= m \otimes n. \end{aligned}$$

We make the definition of the Baer sum explicit: We have submodules

$$Q \subset R \subset N \otimes_A M/I^2_{N,M} \oplus M \otimes N/I^2_{M,N},$$

given by

$$R = \{(a, b) \mid p_1(a) = p_2(b)\};$$

$$Q = \{(j_1(x), -j_2(x)) \mid x \in \Omega_{B/A} \otimes_B M \otimes_B N\}.$$

Then the Baer sum of (5.1) and (5.2) is given by

$$0 \to \Omega_{B/A} \otimes_B M \otimes_B N \xrightarrow{j_3} R/Q \xrightarrow{p_3} M \otimes_B N \to 0,$$

where the maps are defined by $p_3((a, b) + Q) = p_1(a)$, and $j_3(x) = (j_1(x), 0) + Q$.

Note that we also have (by exactness of the sequences (5.1) and (5.2))

$$Q = \{(\sum [n_i \otimes m_i], \sum [m_i \otimes n_i]) \mid \sum m_i \otimes n_i = 0 \text{ in } M \otimes_B N\}. \tag{5.3}$$

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We claim that this extension is isomorphic to $\underline{P}^1_{B/A}(M \otimes_B N)$. We first construct a map of B-modules $P^1_{B/A}(M \otimes_B N) \to R/Q$. By the universal property of the module of principal parts, this is equivalent to giving an A-linear degree one differential operator $M \otimes_B N \to R/Q$. We define

$$\begin{aligned} D : M \otimes_B N &\to R/Q, \\ \sum m_i \otimes n_i &\mapsto ([\sum n_i \otimes m_i], [\sum m_i \otimes n_i]) + Q. \end{aligned}$$

This is well defined since if $\sum m_i \otimes n_i = \sum m'_j \otimes n'_j$ in $M \otimes_B N$, then the difference

$$([\sum n_i \otimes m_i - \sum n'_j \otimes m'_j], [\sum m_i \otimes n_i - \sum m'_j \otimes n'_j])$$

lies in Q due to (5.3). The map D is clearly A-linear, and a first order differential operator, since for any $b \in B$, we have

$$D(b(m \otimes n)) - bD(m \otimes n) = ([n \otimes bm - bn \otimes m], 0) + Q,$$

and thus, the map $x \mapsto D(bx) - bD(x)$ is B-linear. Thus, we have the corresponding map of B-modules

$$\begin{aligned} P^1_{B/A}(N \otimes_B M) &\to R/Q \\ [b \otimes x] &\mapsto bD(x). \end{aligned}$$

A straightforward explicit computation shows that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega_{B/A} \otimes_B N \otimes_B M \arrow[r] \arrow[d, equal] & P^1_{B/A}(N \otimes_B M) \arrow[r] \arrow[d] & N \otimes_B M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Omega_{B/A} \otimes_B N \otimes_B M \arrow[r] & R/Q \arrow[r] & N \otimes_B M \arrow[r] & 0 \end{tikzcd}$$

commutes, which finishes the proof. □

Since taking tensor products, Kähler differentials, modules of principal parts and Baer sums commute with sheafification and are suitably functorial, we have as a consequence the following:

Corollary 5.2. *Lemma 5.1 holds when A and B are rings in a topos.*

Using Lemma 5.1, we can prove the tensor compatibility of the Atiyah class. Regard $E \otimes \mathrm{at}_F$ as a map $E \otimes F \to L_{\mathcal{X}/\mathcal{Y}} \otimes E \otimes F[1]$ via the composition

$$E \otimes F \xrightarrow{E \otimes \mathrm{at}_F} E \otimes (L_{\mathcal{X}/\mathcal{Y}} \otimes F[1]) \simeq E \otimes L_{\mathcal{X}/\mathcal{Y}} \otimes F[1] \simeq L_{\mathcal{X}/\mathcal{Y}} \otimes E \otimes F[1].$$

Here, the second map is the map defining the triangulated structure on $E \otimes -$, and the third map is the symmetry isomorphism exchanging E and $L_{\mathcal{X}/\mathcal{Y}}$.

Proposition 5.3. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks, and $E, F \in D^{\leq 0}_{qcoh}(\mathcal{O}_{\mathcal{X}})$. Then we have the equality*

$$\mathrm{at}_{E \otimes F} = \mathrm{at}_E \otimes F + E \otimes \mathrm{at}_F\ .$$

Proof. We use the setup of Construction 4.1. Without loss of generality, we may assume that E is represented by a complex of flat $\mathcal{O}_{\mathcal{X}}$-modules, so in particular, we may choose $E \otimes F \in D^{\leq 0}_{qcoh}(\mathcal{O}_{\mathcal{X}})$ to be represented by the usual tensor product of the complexes E and F. Let $E_R \to E_{W_\parallel}$ be a flat resolution of the R-module $E_{W_\parallel}$. By the construction of the Atiyah class in 4.1, we may use the exact sequences

$$\underline{P}^1_{R/h_\parallel^{-1}\mathcal{O}_{Y_\parallel}}(E_R) \text{ and } \underline{P}^1_{R/h_\parallel^{-1}\mathcal{O}_{Y_\parallel}}(E_R \otimes_R F_{W_\parallel})$$

to compute at_E and $\mathrm{at}_{E\otimes F}$, respectively. By Corollary 5.2, we have an equality of extensions of R-modules

$$\underline{P}^1_{R/h_{\|}^{-1}\mathcal{O}_{Y_\|}}(E_R) \otimes_R F_{W_\|} + E_R \otimes_R \underline{P}^1_{R/h_{\|}^{-1}\mathcal{O}_{Y_\|}}(F_{W_\|}) = \underline{P}^1_{R/h_{\|}^{-1}\mathcal{O}_{Y_\|}}(E_R \otimes_R F_{W_\|}).$$

After taking connecting maps in $D^{\Delta}(R)$ and using that extensions of scalars and the Dold-Kan correspondence are compatible with the symmetry isomorphisms of the derived tensor product, we get an equality of maps

$$\mathrm{at}_{E_{W_\|}\otimes F_{W_\|}} = \mathrm{at}_{E_{W_\|}} \otimes F_{W_\|} + E_{W_\|} \otimes \mathrm{at}_{F_{W_\|}}.$$

By Lemma 2.7, this implies the result after passing to $D^{\leq 0}(\mathcal{O}_{W_\bullet})$ and then to $D^{\leq 0}(\mathcal{X})$. □

If we assume that one of the Atiyah classes of E or F vanish, this simplifies to the following:

Corollary 5.4. *Suppose that* $\mathrm{at}_F = 0$ *(respectively that* $\mathrm{at}_E = 0$*), then we have* $\mathrm{at}_{E\otimes F} = \mathrm{at}_E \otimes F$ *(respectively* $\mathrm{at}_{E\otimes F} = E \otimes \mathrm{at}_F$*) after identifying the targets using symmetry of the tensor product.*

Corollary 5.5 (Shift invariance). *For $E \in D^{\leq 0}_{qcoh}(\mathcal{X})$, the following diagram commutes:*

$$\begin{tikzcd} E[1] \arrow[r, "\mathrm{at}_{E[1]}"] \arrow[dr, "\mathrm{at}_E[1]"'] & L_{\mathcal{X}/\mathcal{Y}}[1] \otimes (E[1]) \arrow[d, "\tau"] \\ & L_{\mathcal{X}/\mathcal{Y}}[1] \otimes E[1], \end{tikzcd}$$

where τ is the map defining the triangulated structure on the tensor product functor $L_{\mathcal{X}/\mathcal{Y}}[1] \otimes -$ (see [22, Tag 0G6A and Tag 0G6E]).

Proof. This follows from Proposition 5.3 applied to $F \otimes E$ with $F = \mathcal{O}_{\mathcal{X}}[1]$ and noting that $\mathrm{at}_F = 0$, since F is pulled back from $\mathcal{Y}$. □

Construction 5.6. Using Corollary 5.5, we extend the definition of at_E to any $E \in D^{-}_{qcoh}(\mathcal{X})$: Choose some large enough integer $N \geq 0$, so that $E[N] \in D^{\leq 0}_{qcoh}(\mathcal{X})$, so that $\mathrm{at}_{E[N]}$ is defined. Then define at_E via the commutative diagram

$$\begin{tikzcd} E[N] \arrow[r, "\mathrm{at}_{E[N]}"] \arrow[dr, dashed, "\mathrm{at}_E[N]"'] & L_{\mathcal{X}/\mathcal{Y}} \otimes (E[N]) \arrow[d, "\tau^{(N)}"] \\ & L_{\mathcal{X}/\mathcal{Y}}[1] \otimes E[N], \end{tikzcd}$$

where we take $\tau^{(N)} := \tau[N] \circ \cdots \circ \tau$. Explicitly, we have

$$\mathrm{at}_E := (\tau^{(N)} \circ \mathrm{at}_{E[N]})[-N].$$

Corollary 5.5 guarantees that this is independent of the choice of N.

Remark 5.7. In what follows, we will generally check properties of the Atiyah class for objects $E \in D^{\leq 0}$. In each case, one may check that the statement is appropriately invariant under shifts.

Another consequence is the following:

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Corollary 5.8. *The following diagram is a morphism of exact triangles:*

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "\mathrm{at}_F"] & E \arrow[r] \arrow[d, "\mathrm{at}_E"] & G \arrow[r] \arrow[d, "\mathrm{at}_G"] & F[1] \arrow[d, "\mathrm{at}_F[1]"] \\
L_{\mathcal{X}/\mathcal{Y}}[1] \otimes F \arrow[r] & L_{\mathcal{X}/\mathcal{Y}}[1] \otimes E \arrow[r] & L_{\mathcal{X}/\mathcal{Y}}[1] \otimes G \arrow[r] & L_{\mathcal{X}/\mathcal{Y}}[1] \otimes F[1].
\end{tikzcd}$$

Here, the lower row is obtained by applying the triangulated functor $L_{\mathcal{X}/\mathcal{Y}}[1] \otimes -$ to the upper row.

Proof. The commutativity of the first two squares follows from the functoriality of the Atiyah class (Lemma 4.2), while the commutativity of the last square follows from functoriality combined with Corollary 5.5. □

5.2. *Compatibility with traces*

We let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and consider Atiyah classes with respect to this morphism. In Proposition 5.9 and Corollary 5.10, we make global boundedness assumptions, but the results likely hold for arbitrary perfect complexes; cf. [11].

Proposition 5.9. *Let $F \to E \to G \to F[1]$ be a distinguished triangle of perfect complexes in $D^b_{qcoh}(\mathcal{X})$ and assume each of E, F, G has finite Tor-amplitude. Then*

$$\mathrm{tr}(\mathrm{at}_E) = \mathrm{tr}(\mathrm{at}_F) + \mathrm{tr}(\mathrm{at}_G).$$

Proof. By the shift invariance of the Atiyah class and Lemma 2.16, we may assume that the given distinguished triangle is represented by a short exact sequence of complexes

$$0 \to F \to E \to G \to 0,$$

where F, E, G have flat components and lie in $C^{\leq 0}(\mathcal{X})$.

We then use the setup of Situation 2.18 with $L = L_{\mathcal{X}/\mathcal{Y}}[1]$ and $R = \mathcal{O}_{\mathcal{X}}$ (this is justified by Corollary 5.8). Then by Proposition 2.20, we need to find a morphism

$$\frac{E \otimes E^\vee}{F \otimes G^\vee} \to L_{\mathcal{X}/\mathcal{Y}} \otimes \frac{E \otimes E^\vee}{F \otimes G^\vee}$$

making the diagram there commute. By the assumption on Tor-dimension, each term of the sequence

$$0 \to G^\vee \to E^\vee \to F^\vee \to 0$$

lies in $D^-_{qcoh}(\mathcal{X})$. We can therefore find an integer N, such that the natural maps from the good truncation $\tau_{\leq N}$ are quasi-isomorphisms for each term. Set $\overline{E} := (\tau_{\leq N} E)[N]$ and define $\overline{F}, \overline{G}$ in the same way, so that we obtain an exact sequence in $C^{\leq 0}(\mathcal{O}_{\mathcal{X}})$. We also let $\overline{E}_{W_{||}}, \overline{F}_{W_{||}}$ and $\overline{G}_{W_{||}}$ denote the induced complexes of $\mathcal{O}_{W_{||}}$-modules, which we regard as simplicial modules via the Dold–Kan correspondence and as R-modules by restriction of scalars. Choose resolutions of $F_{W_{||}}, E_{W_{||}}$ and $G_{W_{||}}$ by termwise flat R-modules, so that we have a morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & F_R \arrow[r] \arrow[d] & E_R \arrow[r] \arrow[d] & G_R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_{W_{||}} \arrow[r] & E_{W_{||}} \arrow[r] & G_{W_{||}} \arrow[r] & 0.
\end{tikzcd}$$

We then obtain a natural morphism of exact sequences of R-modules

$$\underline{P}^1_{R/h^{-1}\mathcal{O}_{Y_\|}}(F_R) \otimes_R \overline{G}_{W_\|} \to \underline{P}^1_{R/h^{-1}\mathcal{O}_{Y_\|}}(E_R) \otimes_R \overline{E}.$$

This is termwise an injection of complexes, so we get a quotient exact sequence $\underline{S}$ of the form

$$0 \to \Omega^1_{R/h^{-1}\mathcal{O}_{Y_\|}} \otimes_R \frac{E_R \otimes_R \overline{E}_{W_\|}}{F_R \otimes_R \overline{G}_{W_\|}} \to *** \to \frac{E_R \otimes \overline{E}_{W_\|}}{F_R \otimes \overline{G}_{W_\|}} \to 0.$$

Taking the connecting map of this sequence and passing back to $D(W_\|)$, we obtain a morphism α making the following square commute:

$$\begin{tikzcd} E_{W_\|} \otimes \overline{E}_{W_\|} \arrow[r] \arrow[d, "\mathrm{at}_{E_{W_\|}} \otimes \overline{E}_{W_\|}"] & \frac{E_{W_\|} \otimes \overline{E}_{W_\|}}{F_{W_\|} \otimes \overline{G}_{W_\|}} \arrow[d, "\alpha"] \\ L_{W_\|/Y_\|}[1] \otimes E_{W_\|} \otimes \overline{E}_{W_\|} \arrow[r] & L_{W_\|/Y_\|}[1] \otimes \frac{E_{W_\|} \otimes \overline{E}_{W_\|}}{F_{W_\|} \otimes \overline{G}_{W_\|}}. \end{tikzcd}$$

We claim that the diagram

$$\begin{tikzcd} F_{W_\|} \otimes \overline{F}_{W_\|} \arrow[r] \arrow[d, "\mathrm{at}_{F_{W_\|}} \otimes \overline{F}_{W_\|}"] & \frac{E_{W_\|} \otimes \overline{E}_{W_\|}}{F_{W_\|} \otimes \overline{G}_{W_\|}} \arrow[d, "\alpha"] \\ L_{W_\|/Y_\|}[1] \otimes F_{W_\|} \otimes \overline{F}_{W_\|} \arrow[r] & L_{W_\|/Y_\|}[1] \otimes \frac{E_{W_\|} \otimes \overline{E}_{W_\|}}{F_{W_\|} \otimes \overline{G}_{W_\|}} \end{tikzcd} \tag{5.4}$$

also commutes (and similarly with G in place of F in the left column). This follows from the observation that the exact sequence $\underline{P}^1_{R/h_\|^{-1}\mathcal{O}(Y_\|)}(F_R) \otimes \overline{F}_{W_\|}$ is isomorphic to the termwise cokernel of the map of exact sequences

$$\underline{P}^1_{R/h^{-1}\mathcal{O}_{Y_\|}}(F_R) \otimes_R \overline{G}_{W_\|} \to \underline{P}^1_{R/h^{-1}\mathcal{O}_{Y_\|}}(F_R) \otimes_R \overline{E}_{W_\|},$$

and thus includes into $\underline{S}$. Then we get the commutativity (5.4) by taking connecting morphisms and passing to $D(W_\|)$. The argument for G in place of F goes similarly. Taking $\eta_{W*} \circ \mathrm{Cone}_{\mathcal{O}_{W_\|}}(\alpha) \circ [-1]$ then gives a morphism

$$\frac{E \otimes E^\vee}{F \otimes G^\vee}[N] \to L_{\mathcal{X}/\mathcal{Y}} \otimes \frac{E \otimes E^\vee}{F \otimes G^\vee}[N].$$

Shifting by $-N$ gives the desired map in Proposition 2.20. ◻

Corollary 5.10. *Let $E \in D^-_{qcoh}(\mathcal{O}_\mathcal{X})$ be an object that can be represented by a finite length complex of locally free sheaves. Then we have an equality of maps $\mathcal{O}_\mathcal{X} \to L_{\mathcal{X}/\mathcal{Y}}[1]$:*

$$\mathrm{tr}(\mathrm{at}_E) = \frac{\mathrm{at}_{\det E}}{\det E}.$$

Proof. We argue by induction on the number k of nonzero components in a resolution by locally free sheaves. Without loss of generality, we may assume that E is given by such a resolution. If $k = 1$, the

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result follows from Example 4.10. If $k > 1$, we may write $k = k_1 + k_2$ for positive integers k_1, k_2 and can take bad truncations of E to get an exact sequence of complexes

$$0 \to F \to E \to G \to 0,$$

where F has k_1 nonzero components and G has k_2, and they are all locally free. Then by Proposition 5.9 and the induction hypothesis, on one hand, we have

$$\mathrm{tr}(\mathrm{at}_E) = \mathrm{tr}(\mathrm{at}_F) + \mathrm{tr}(\mathrm{at}_G) = \frac{\mathrm{at}_{\det F}}{\det F} + \frac{\mathrm{at}_{\det G}}{\det G}.$$

On the other hand, the determinant is multiplicative in exact sequences of perfect complexes, which gives $\det E = \det F \otimes \det G$. Using the tensor compatibility Proposition 5.3, we find

$$\frac{\mathrm{at}_{\det E}}{\det E} = \frac{\mathrm{at}_{\det F} \otimes \det G + \det F \otimes \mathrm{at}_{\det G}}{\det F \otimes \det G} = \frac{\mathrm{at}_{\det F}}{\det F} + \frac{\mathrm{at}_{\det G}}{\det G}. \qquad \square$$

5.3. Compatibility of Atiyah class and reduced Atiyah class

We prove Proposition 1.1. Consider the setup of Construction 4.11 and let $\mathcal{Y} \to \mathcal{Z}$ be a further morphism of algebraic stacks. Choose a diagram

$$\begin{tikzcd} X \arrow[r] & \mathcal{X}_Y \arrow[r] \arrow[d] & \mathcal{X}_Z \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ & Y \arrow[r] & \mathcal{Y}_Z \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ & & Z \arrow[r] & \mathcal{Z} \end{tikzcd}$$

in which all squares are cartesian, the horizontal morphisms are smooth and surjective, and X, Y and Z are algebraic spaces. We let $X_\bullet$, $Y_\bullet$ and $Z_\bullet$ be the strictly simplicial algebraic spaces associated to compositions along the horizontal rows, respectively. We also let $V := Y \times_{\mathcal{Y}_Z} Y$ with strictly simplicial algebraic space $V_\bullet$ associated to the morphism $V \to \mathcal{Y}$, and further $W := X \times_{\mathcal{X}_Y} X$ and $\widetilde{W} := X \times_{\mathcal{X}_Z} X$ with strictly simplicial algebraic spaces $W_\bullet$ and $\widetilde{W}_\bullet$ associated to the morphisms to $\mathcal{X}$. We have the following natural commutative diagram:

$$\begin{tikzcd} X_\bullet \arrow[d] & W_\bullet \arrow[l, "s_\bullet"'] \arrow[r, "t_\bullet"] \arrow[d] & X_\bullet \arrow[d] \\ X_\bullet \arrow[d] & \widetilde{W}_\bullet \arrow[l, "s_\bullet"'] \arrow[r, "t_\bullet"] \arrow[d] & X_\bullet \arrow[d] \\ Y_\bullet & V_\bullet \arrow[l, "s_\bullet"'] \arrow[r, "t_\bullet"] & Y_\bullet. \end{tikzcd}$$

Here, by abuse of notation, we use $s_\bullet$ and $t_\bullet$ to denote the morphisms given by (degreewise) projection to the first and second factor, respectively. We define diagrammatic topoi $W_{||}$, $\widetilde{W}_{||}$ and $V_{||}$ by the rows of this diagram and $Y_{||}$ and $Z_{||}$ associated to the constant diagram with values $Y_\bullet$ and $Z_\bullet$, respectively. Then we have morphisms

$$W_{||} \xrightarrow{j_{||}} \widetilde{W}_{||} \to V_{||} \to Y_{||} \to Z_{||}.$$

Denote the sheaves on either of these obtained by pulling back E, F or G, respectively, by a corresponding subscript. In particular, we have $E_{Y_{||}}$ on Y, which is Tor-independent with the morphism $W_{||} \to Y_{||}$, and

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we have an exact sequence

$$0 \to F_{W_\parallel} \to E_{W_\parallel} \to G_{W_\parallel} \to 0.$$

By construction, the reduced Atiyah class $\overline{\mathrm{at}}_{E,\mathcal{X}/\mathcal{Y},G}$ is obtained from $\overline{\mathrm{at}}_{E_{Y_\parallel},W_\parallel/Y_\parallel,G_\parallel}$ by applying $\mathrm{Cone}_{\mathcal{O}_{W_\parallel}}$ and descent to $D(\mathcal{X})$. Similarly, the Atiyah class $\mathrm{at}_{E,\mathcal{Y}/\mathcal{Z}}$ is obtained from $\mathrm{at}_{E_{V_\parallel},V_\parallel/Z_\parallel}$ by applying $\mathrm{Cone}_{\mathcal{O}_{V_\parallel}}$ and passing to $D(\mathcal{Y})$.

Let $\widetilde{f}_\parallel$ denote the morphism $\widetilde{W}_\parallel \to V_\parallel$. By Proposition 3.10, we have the anti-commutative diagram

$$\begin{tikzcd}
F_{\widetilde{W}_\parallel} \arrow[rr, "\overline{\mathrm{at}}_{E_{V_\parallel},\widetilde{W}_\parallel/V_\parallel,G_{\widetilde{W}_\parallel}}"] \arrow[d] & & L_{\widetilde{W}_\parallel/V_\parallel} \otimes G_{\widetilde{W}_\parallel} \arrow[d] \\
E_{\widetilde{W}_\parallel} \arrow[r, "\widetilde{f}_\parallel^* \mathrm{at}_{E_{V_\parallel},V_\parallel/Z_\parallel}"] & \widetilde{f}_\parallel^* L_{V_\parallel/Z_\parallel}[1] \otimes E_{\widetilde{W}_\parallel} \arrow[r] & \widetilde{f}_\parallel^* L_{V_\parallel/Z_\parallel}[1] \otimes G_{\widetilde{W}_\parallel}.
\end{tikzcd}$$

We also have a morphism $Y_\parallel \to V_\parallel$ induced by the morphism $Y_\bullet \to V_\bullet$, given by the diagonal of $Y_n \times_{\mathcal{Y}_{Z_n}} Y_n$ in degree n. This fits into the commutative diagram

$$\begin{tikzcd}
W_\parallel \arrow[r, "j_\parallel"] \arrow[d] & \widetilde{W}_\parallel \arrow[d] \\
Y_\parallel \arrow[r] & V_\parallel.
\end{tikzcd}$$

Here, the horizontal maps are Tor-independent to $E_{V_\parallel}$ and to $E_{\widetilde{W}_\parallel}$ and $G_{\widetilde{W}_\parallel}$, respectively; thus, we can apply Corollary 3.8. Moreover, the pullback $j_\parallel^* L_{\widetilde{W}_\parallel/V_\parallel} \to L_{W_\parallel/Y_\parallel}$ is a quasi-isomorphism (this follows, since the morphisms $Y_\bullet \to V_\bullet \leftarrow \widetilde{W}_\bullet$ are Tor-independent, which can be checked degreewise). In conclusion, by pulling back along $j_\parallel$, we obtain an anti-commutative diagram in $D(W_\parallel)$:

$$\begin{tikzcd}
F_{W_\parallel} \arrow[rr, "\overline{\mathrm{at}}_{E_{Y_\parallel},W_\parallel/Y_\parallel,G_{W_\parallel}}"] \arrow[dd] & & L_{W_\parallel/Y_\parallel} \otimes G_{W_\parallel} \\
 & & j_\parallel^* L_{\widetilde{W}_\parallel/V_\parallel} \otimes G_{W_\parallel} \arrow[u, "\sim"'] \arrow[d] \\
E_{W_\parallel} \arrow[r, "f_\parallel^* \mathrm{at}_{E_{V_\parallel},V_\parallel/Z_\parallel}"] & f_\parallel^* L_{V_\parallel/Z_\parallel}[1] \otimes E_{W_\parallel} \arrow[r] & f_\parallel^* L_{V_\parallel/Z_\parallel}[1] \otimes G_{W_\parallel},
\end{tikzcd}$$

where $f_\parallel$ is the morphism $W_\parallel \to V_\parallel$. By applying $\mathrm{Cone}_{\mathcal{O}_{W_\parallel}}$ and passing to $D(\mathcal{X})$, we conclude the following:

Proposition 5.11. *The diagram*

$$\begin{tikzcd}
F \arrow[rr, "\overline{\mathrm{at}}_{E,\mathcal{X}/\mathcal{Y},G}"] \arrow[d] & & L_{\mathcal{X}/\mathcal{Y}} \otimes G \arrow[d] \\
E_{\mathcal{X}} \arrow[r, "f^* \mathrm{at}_{E,\mathcal{Y}/\mathcal{Z}}"] & f^* L_{\mathcal{Y}/\mathcal{Z}}[1] \otimes E \arrow[r] & f^* L_{\mathcal{Y}/\mathcal{Z}}[1] \otimes G
\end{tikzcd}$$

anti-commutes.

If E, F, G are perfect complexes, this directly implies Proposition 1.1.

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5.4. Compatibilities of the Atiyah class for an exact sequence

Proposition 5.12. *The map* $\mathrm{at}_{\underline{E},\mathcal{X}/\mathcal{Y}}$ *is functorial in* $\mathcal{Y}$ *– that is, given a map* $\mathcal{Y} \to \mathcal{Z}$*, the composition*

$$\frac{E^\vee \otimes E}{G^\vee \otimes F}[-1] \xrightarrow{\mathrm{at}_{\underline{E},\mathcal{X}/\mathcal{Z}}} L_{\mathcal{X}/\mathcal{Z}} \to L_{\mathcal{X}/\mathcal{Y}}$$

equals $\mathrm{at}_{\underline{E},\mathcal{X}/\mathcal{Y}}$ *(at least assuming we make the same choices of N in the construction).*

Proof. We use the setup of §5.3. There we had the topoi

$$W_\| \to \widetilde{W}_\| \to V_\| \to Y_\| \to Z_\|$$

and the following morphisms of maps of topoi

$$W_\|/Y_\| \to \widetilde{W}_\|/V_\| \to \widetilde{W}_\|/Z_\|$$

whose cotangent complexes represent (after taking cones and shifting) $L_{\mathcal{X}/\mathcal{Y}}$, $L_{\mathcal{X}/\mathcal{Y}}$ and $L_{\mathcal{X}/\mathcal{Z}}$, respectively. The result now follows from Corollary 3.13 and Lemma 3.14. □

Proof of Proposition 1.3. The commutativity of the first square follows from the functoriality of the usual Atiyah class and Corollary 4.15. Commutativity of the second square follows from Proposition 5.12 and Corollary 4.16. Finally, the commutativity of the square involving connecting morphisms is Proposition 5.11. □

6. Deformation theoretic properties

In this section, we prove Theorem 1.4. The proof of Proposition 1.5 is similar, but not addressed here. For details see [3, §4]. In §§6.3–6.5, we use the following notation: X is a smooth projective variety over a base field k, and $\mathcal{M}$ is an open substack of the moduli stack of coherent sheaves on X.[4]

6.1. Deformations of morphisms to algebraic stacks

Let $\mathcal{Y}$ be an algebraic stack over a base scheme S and let T be a scheme over S. Here, we consider the problem of deforming maps from the scheme T to $\mathcal{Y}$. As a special case of [17, Theorem 1.5], we have the following:

Theorem 6.1. *Let* $g : T \to \mathcal{Y}$ *be a morphism and let* $j : T \hookrightarrow \overline{T}$ *be a square zero extension of T by a quasicoherent sheaf I. Then,*

1) *there is a natural obstruction class* $\omega(g, \overline{T}) \in \mathrm{Ext}^1(g^*L_\mathcal{Y}, I)$ *which vanishes if and only if there is an extension of g to a morphism* $\overline{g} : \overline{T} \to \mathcal{Y}$.
2) *if an extension of g to* $\overline{T}$ *exists, then the set of isomorphism classes of extensions naturally forms a torsor under* $\mathrm{Ext}^0(g^*L_\mathcal{Y}, I)$.
3) *for a fixed extension* $\overline{g}$ *of g, the set of automorphisms of* $\overline{g}$ *as an extension of g is canonically isomorphic to* $\mathrm{Ext}^{-1}(g^*L_\mathcal{Y}, I)$.

One can describe the characterizations in Theorem 6.1 explicitly.

Remark 6.2 (Obstructions). The morphism g induces a natural map $g^*L_\mathcal{Y} \to L_T$. Similarly, j induces a natural map $L_T \to L_{T/\overline{T}}$, and $L_{T/\overline{T}}$ is concentrated in degrees ≤ -1 with $h^{-1}(L_{T/\overline{T}})$ naturally isomorphic to I. The obstruction class $\omega(g, \overline{T})$ is then given by the composition

$$g^*L_\mathcal{Y} \to L_{T/\overline{T}} \to I[1].$$

[4]See, for example, [22, Theorem 08WC].

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This follows from the construction in [17, 4.8] and the construction of the obstruction class for topoi in [7, III 2.2].

Remark 6.3 (Deformations). For a given $g : T \to \mathcal{Y}$, let $\overline{T}$ be the trivial extension of T by I, given by taking the structure sheaf $\mathcal{O}_T \oplus I$ on T. Then there is a natural morphism $\overline{T} \to T$ corresponding to the inclusion $\mathcal{O}_T \oplus \{0\} \subset \mathcal{O}_T \oplus I$, giving rise to a canonical extension of g as the composition $\overline{T} \to T \to \mathcal{Y}$. Taking this as a base point, the torsor structure in Theorem 6.1 (2) induces a bijection between the set of isomorphism classes of extensions of g and the group $\mathrm{Ext}^0(g^* L_{\mathcal{Y}}, I)$. To describe this bijection explicitly, note that we have a natural isomorphism $h^0(i^* L_{\overline{T}/T}) \simeq I$. Now, for a given extension $\overline{g} : \overline{T} \to \mathcal{Y}$, consider the composition $\overline{g}^* L_{\mathcal{Y}} \to L_{\overline{T}} \to L_{\overline{T}/T}$. Up to given isomorphisms, this restricts to a map $\alpha_{\overline{g}} : g^* L_{\mathcal{Y}} \to I$ on T. The association $\overline{g} \mapsto \alpha_{\overline{g}}$ is the bijection in question. This follows from [17, 4.8] and the construction in [7, III 2.2].

Remark 6.4 (Automorphisms). Consider a fixed square zero extension $j : T \hookrightarrow \overline{T}$ with sheaf of ideals I, and an extension $\overline{g} : \overline{T} \to \mathcal{Y}$ of g. Let $\mathrm{Aut}(\overline{g})$ denote the automorphism group of $\overline{g}$ as an extension of g (i.e., the group of 2-isomorphisms of $\overline{g} : \overline{T} \to \mathcal{Y}$ that restrict to the identity 2-isomorphism when restricted to T). Let $\mathrm{Aut}_{\mathcal{Y}}(\overline{T})$ denote the group of automorphisms of $\overline{T}$ as an extension of T over $\mathcal{Y}$ whose elements are pairs (a, ϕ), where $a : \overline{T} \to \overline{T}$ is an automorphism satisfying $a \circ j = j$, and where ϕ is a 2-isomorphism $\phi : \overline{g} \circ a \Rightarrow \overline{g}$. Similarly, we let $\mathrm{Aut}_S(\overline{T})$ denote the group of automorphisms of $\overline{T}$ as an extension of T over S. Then we have a natural forgetful map $\mathrm{Aut}_{\mathcal{Y}}(\overline{T}) \to \mathrm{Aut}_S(\overline{T})$ whose Kernel is $\mathrm{Aut}(\overline{g})$. We have identifications $\mathrm{Ext}^0(L_{X/\mathcal{Y}}, I) \simeq \mathrm{Aut}_{\mathcal{Y}}(\overline{T})$, which are natural in $\mathcal{Y}$ (and, in particular, hold for S in place of $\mathcal{Y}$), and via the exact triangle $g^* L_{\mathcal{Y}/S} \to L_{T/S} \to L_{T/\mathcal{Y}} \xrightarrow{+1}$, we obtain the exact sequence

$$0 \to \mathrm{Ext}^{-1}(g^* L_{\mathcal{Y}/S}, I) \to \mathrm{Ext}^0(L_{T/\mathcal{Y}}, I) \to \mathrm{Ext}^0(L_{T/S}, I).$$

By what is said above, this gives an identification

$$\mathrm{Ext}^{-1}(g^* L_{\mathcal{Y}/S}, I) \simeq \mathrm{Aut}(\overline{g}).$$

For our purposes, a different characterization of the bijection $\mathrm{Ext}^{-1}(g^* L_{\mathcal{Y}/S}, I) \simeq \mathrm{Aut}(\overline{g})$ than the one given in Remark 6.4 will be needed. For the rest of this subsection, we consider the case where $\overline{T}$ is the trivial square zero extension of T by I and where $\overline{g}$ is the trivial extension of g. Let $y : Y \to \mathcal{Y}$ be a smooth cover by an algebraic space and assume that $g : T \to \mathcal{Y}$ factors through Y (this can always be arranged by passing to an étale cover of T, which is enough for our later application). We fix such a factorization $g_Y : T \to Y$ (with an implicit choice of 2-isomorphism $y \circ g_Y \Rightarrow g$). Form the Cartesian diagram

$$\begin{tikzcd} Z \arrow[r, "t"] \arrow[d, "s"] & Y \arrow[d] \\ Y \arrow[r] & \mathcal{Y}. \end{tikzcd}$$

We observe that Y naturally has the structure of a groupoid algebraic space with the space of morphisms given by Z and that we have a natural equivalence $[Y/Z] \xrightarrow{\sim} \mathcal{Y}$. Moreover, by definition of the 2-cartesian product, the set of automorphisms of the morphism $g : T \to \mathcal{Y}$ is in natural bijection to the set of maps $f : T \to Z$ satisfying $s \circ f = t \circ f = g_Y$. In particular, there is a morphism $e : T \to Z$ corresponding to the identity automorphism of g_Y.

Now let $\overline{g}_Y$ denote the composition $\overline{T} \to T \xrightarrow{g_Y} Y$, which is a lift of $\overline{g}$. Then we observe the following:

Lemma 6.5. *We have a natural bijection between* $\mathrm{Aut}(\overline{g})$ *and the set of morphisms* $\overline{f} : \overline{T} \to Z$ *satisfying* $s \circ \overline{f} = t \circ \overline{f} = \overline{g}_Y$ *and* $\overline{f} \circ j = e$. *In other words, the group of infinitesimal automorphisms of* $\overline{g}$ *is* *in bijection with the group of deformations of e to $\overline{T}$ that induce the trivial deformation of g_Y upon composition with either s or t.*

As a consequence of this Lemma, we have a canonical isomorphism

$$\operatorname{Aut}(g) \simeq \operatorname{Ker}\left(\operatorname{Hom}(e^*L_Z, I) \xrightarrow{(-s^*,t^*)} \operatorname{Hom}(e^*s^*L_Y, I) \oplus \operatorname{Hom}(e^*t^*L_Y, I)\right).$$

Let $z := y \circ s : Z \to \mathcal{Y}$. By Lemma 2.11, we have the natural isomorphism

$$z^*L_{\mathcal{Y}} \simeq \operatorname{Cone}(s^*L_Y \oplus t^*L_Y \to L_Z)[-1].$$

Using this, we get an identification

$$\begin{aligned} \operatorname{Aut}(\overline{g}) \simeq \operatorname{Ext}^{-1}(e^*z^*L_{\mathcal{Y}}, I) &= \operatorname{Ext}^{-1}(g^*L_{\mathcal{Y}}, I) \\ \varphi \mapsto \tau_\varphi. \end{aligned} \tag{6.1}$$

6.2. *Deformations of sheaves*

Let X, T be schemes over a common base field k. Let $T \hookrightarrow \overline{T}$ be a square zero extension defined by an ideal sheaf I. Let also E be a T-flat quasicoherent sheaf on $X \times T$. We consider the problem of extending E to a $\overline{T}$-flat coherent sheaf on $X \times \overline{T}$. Let $\pi : X \times T \to T$ denote the projection. By [7, IV Proposition 3.1.8], we have the following:

Theorem 6.6. 1) *There is a natural obstruction class $\omega^{sh}(E, \overline{T}) \in \operatorname{Ext}^2_{X\times T}(E, \pi^*I \otimes E)$ which vanishes if and only if there is an extension of E to a $\overline{T}$-flat sheaf on $X \times \overline{T}$.*

2) *If a $\overline{T}$-flat extension of E to $X \times \overline{T}$ exists, then the set of isomorphism classes of such extensions naturally forms a torsor under $\operatorname{Ext}^1_{X\times T}(E, \pi^*I \otimes E)$.*

3) *For a fixed $\overline{T}$-flat extension $\overline{E}$, the set of automorphisms of $\overline{E}$ which restrict to the identity on E is canonically isomorphic to $\operatorname{Hom}_{X\times T}(E, \pi^*I \otimes E)$.*

We make some of the natural maps implied in this theorem explicit.

Remark 6.7 (Obstructions). For a given $\overline{T}$, the obstruction class $\omega^{sh}(E, \overline{T})$ is given by the composition

$$E \xrightarrow{\mathrm{at}_{E,X\times T/S}} L_{X\times T}[1] \otimes E \to L_{X\times T/X\times\overline{T}}[1] \otimes E \to \pi^*I[2] \otimes E.$$

Here, the first map is the Atiyah class, the second map is induced from the naturality of cotangent complexes, and the last map is induced from the natural identification $\tau_{\geq 1}L_{T/\overline{T}} \simeq I[1]$. This is proven in [7, IV Proposition 3.1.8].

Remark 6.8 (Deformations). Let $\overline{T}$ be the trivial extension of T by I. Then there is a canonical flat extension of E to $X \times \overline{T}$ given by $E \oplus \pi^*I \otimes E$, with multiplication by I given by $j(e, 0) = (0, je)$ for local sections. The torsor structure on the space of extensions therefore gives rise to a bijection between $\operatorname{Ext}^1_{X\times T}(E, I \otimes E)$ and the set of extensions of E to $\overline{T}$. To describe this bijection, let $\nu \in \operatorname{Ext}^1_{X\times T}(E, \pi^*I \otimes E)$, corresponding to an extension

$$0 \to E \otimes I \xrightarrow{\mu} \overline{E} \xrightarrow{\rho} E \to 0.$$

We make this into an $\mathcal{O}_{X\times T} \oplus \pi^*I$-module, by defining the action of I on $\overline{E}$ on local sections as $jx := \mu(j \otimes \rho(x))$. One checks that this defines a $\overline{T}$-flat coherent sheaf on $X \times \overline{T}$ extending E. It is straightforward to see that this construction is invertible.

Remark 6.9 (Automorphisms). Let $\overline{T}$ be the trivial extension of T by I and let $\overline{E} = E \oplus \pi^*I \otimes E$ be the canonical flat extension of E to $\overline{T}$. For an element $a \in \operatorname{Hom}_{X\times T}(E, \pi^*I \otimes E)$, the map $\psi_a : \overline{E} \to \overline{E}$

given locally by $(x_1, j \otimes x_2) \mapsto (x_1, j \otimes x_2 + \varphi(x_1))$ is an automorphism of $\overline{E}$ which restricts to the identity on E. This gives the claimed bijection of automorphism groups for this choice of $\overline{E}$.

Remark 6.10. Suppose that X is a smooth projective variety and that T is of finite type over k. Then in particular, E has a finite length resolution by locally free sheaves. Then we have natural isomorphisms

$$\begin{aligned}\operatorname{Ext}^i_{X\times T}(E, \pi^* I \otimes E) &\simeq \operatorname{Ext}^i_{X\times T}(\mathcal{O}_{X\times T}, \pi^* I \otimes E \otimes E^\vee)\\ &\simeq \operatorname{Ext}^i_T(\mathcal{O}_T, I \otimes R\pi_*(E \otimes E^\vee))\\ &\simeq \operatorname{Ext}^i_T(R\pi_*(E \otimes E^\vee), I).\end{aligned}$$

Here, all tensor products and duals are taken in the derived sense. The first isomorphism is due to the fact that E is dualizable, the second one is push-pull adjunction and the projection formula, and the third one uses that $R\pi_*(E \otimes E^\vee)$ is dualizable.

6.3. *Comparison of obstruction classes*

Now let $X, \mathcal{M}$ be as specified in the beginning of the section with universal sheaf $\mathcal{E}$ on $\mathcal{M} \times X$. Our goal is to show that the map $\mathrm{At}_{\mathcal{E}} : R\pi_*(\mathcal{E} \otimes \mathcal{E}^\vee)^\vee[-1] \to L_{\mathcal{M}}$ is surjective on h^{-1}. By the arguments of [2, §4],[5] it is enough to show that for every map from an affine scheme $g : T \to \mathcal{M}$ and any quasicoherent sheaf I on T, the map

$$(g^* \mathrm{At}_{\mathcal{E}})^* : \operatorname{Ext}^1_T(g^* L_{\mathcal{M}}, I) \to \operatorname{Ext}^2_T(g^* R\pi_*(\mathcal{E} \otimes \mathcal{E}^\vee)^\vee, I)$$

given by composition with the Atiyah class is injective. In fact, it is enough to show that for any such g and I and any square zero extension $T \hookrightarrow \overline{T}$, there exists an extension of g to $\overline{T}$ if and only if the image of the obstruction class $\omega(g, \overline{T})$ under this map vanishes. By Remark 6.2, the obstruction class $\omega(g, \overline{T})$ is obtained as the composition of the natural maps

$$g^* L_{\mathcal{M}} \to L_T \to L_{T/\overline{T}} \to I[1].$$

However, let $E = g^*\mathcal{E}$, so that we have a commutative diagram

$$\begin{tikzcd} g^* R\pi_*(\mathcal{E} \otimes \mathcal{E}^\vee)^\vee[-1] \arrow[r, "g^* \mathrm{At}_{\mathcal{E}}"] \arrow[d, "\sim"] & g^* L_{\mathcal{M}} \arrow[d] \\ R\pi_*(E \otimes E^\vee)^\vee[-1] \arrow[r, "\mathrm{At}_E"] & L_T, \end{tikzcd}$$

where the left vertical map is the canonical base change isomorphism. We find that the image of $\omega(g, \overline{T})$ under $(g^* \mathrm{At}_{\mathcal{E}})^*$ is up to the given isomorphism equal to the composition

$$R\pi_*(E \otimes E^\vee)^\vee[-1] \xrightarrow{\mathrm{At}_E} L_T \to L_{T/\overline{T}} \to I[1].$$

By the definition of At_E, this corresponds to the morphism

$$E \xrightarrow{\mathrm{at}_E} L_{T\times X}[1] \otimes E \to L_{T\times X/\overline{T}\times X}[1] \otimes E \to \pi^* I[2] \otimes E.$$

By Remark 6.7, this is exactly the obstruction to the existence of a $\overline{T}$-flat extension of E to $\overline{T} \times X$ and therefore an obstruction to the existence of an extension of $\overline{g}$ by the universal property of $\mathcal{M}$. This shows what we needed.

[5]Or see [1, Proposition 8.4 and Corollary 8.5] for a proof using higher categorical language.

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6.4. Comparison of deformation spaces

We show that the map $\mathrm{At}_{\mathcal{E}} : R\pi_*(\mathcal{E} \otimes \mathcal{E}^\vee)^\vee[-1] \to L_{\mathcal{M}}$ is an isomorphism on h^0. By the arguments of [2, §4], it is enough to show that for every map from an affine scheme $g : T \to \mathcal{M}$ and any quasicoherent sheaf I on T, the map

$$(g^* \mathrm{At}_{\mathcal{E}})^* : \mathrm{Ext}^0_T(g^* L_{\mathcal{M}}, I) \to \mathrm{Ext}^1_T(g^* R\pi_*(\mathcal{E} \otimes \mathcal{E}^\vee)^\vee, I)$$

given by composition with the Atiyah class is an isomorphism. This follows from the following stronger statement:

Lemma 6.11. *Let $\overline{T}$ be the trivial extension of T by I and let $\overline{g} : \overline{T} \to \mathcal{M}$ be any extension of g. Let $\overline{E}$ be the corresponding extension of E. Then the class $\alpha_{\overline{g}}$ of Remark 6.3 is mapped by $(g^* \mathrm{At}_{\mathcal{E}})^*$ to the class of*

$$\mathrm{Ext}^1_T(g^* R\pi_*(\mathcal{E} \otimes \mathcal{E}^\vee)^\vee, I) \simeq \mathrm{Ext}^1_T(E, I \otimes E)$$

corresponding to the extension $\overline{E}$ via Remark 6.8.

Proof. By Remark 6.3, we have that $(g^* \mathrm{At}_{\mathcal{E}})^* \alpha_{\overline{g}}$ is equal to the composition

$$g^* R\pi_*(\mathcal{E} \otimes \mathcal{E}^\vee)^\vee[-1] \xrightarrow{g^* \mathrm{At}_{\mathcal{E}}} g^* L_{\mathcal{M}} \xrightarrow{\overline{g}^*|_T} j^* L_{\overline{T}} \to j^* L_{\overline{T}/T} \to I.$$

The composition of the first two maps is just $j^* \mathrm{At}_{\overline{E}}$. Therefore, by definition of At, this corresponds under adjunction to the morphism

$$E \xrightarrow{j^* \mathrm{at}_{\overline{E}}} L_{\overline{T} \times X / T \times X}[1] \otimes E \to \pi^* I[1] \otimes E.$$

Thus, we are reduced to showing that this morphism agrees with the class $\beta_{\overline{E}}$ in $\mathrm{Ext}^1(E, \pi^* I \otimes E)$. This is shown in Lemma 6.12 below. □

Lemma 6.12. *Let T be an affine scheme and I a quasicoherent sheaf on T and let $j : T \to \overline{T}$ be the trivial square zero thickening of T with ideal sheaf I. Let E be a coherent sheaf on $T \times X$ and $\overline{E}$ an extension of E to $\overline{T}$ such that the induced map $I \otimes_T E \to I\overline{E} \subset \overline{E}$ is an isomorphism (see [7, IV 3.1]). Let $\beta \in \mathrm{Ext}^1_{T \times X}(E, \pi^* I \otimes E)$ be the corresponding extension class. Then the composition*

$$E \xrightarrow{j^* \mathrm{at}_{\overline{E}}} j^* L_{\overline{T} \times X / T \times X}[1] \otimes E \to \pi^* I[1] \otimes E$$

equals β.

Proof. Let $r : \overline{T} \to T$ be the projection coming from the inclusion $\mathcal{O}_T = \mathcal{O}_T \oplus \{0\} \subset \mathcal{O}_T \oplus I$. Let $R = P_{r^{-1}\mathcal{O}_T}(\mathcal{O}_{\overline{T}})$ be the standard simplicial resolution. Then we have the following commutative diagram of R-modules with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_{R/r^{-1}\mathcal{O}_T} \otimes_R \overline{E} \arrow[r] \arrow[d] & P^1_{R/r^{-1}\mathcal{O}_T}(\overline{E}) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_{\overline{T}/T} \otimes_{\overline{T}} \overline{E} \arrow[r] & P^1_{\overline{T}/T}(\overline{E}) \arrow[r] & E \arrow[r] & 0.
\end{tikzcd}$$

The upper row is used to define the Atiyah class, so by taking connecting map in $D^\Delta(R)$ and passing to $D(\overline{T})$, we get the commutative diagram

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$$\begin{tikzcd}
\overline{E} \arrow[r, "\mathrm{at}_{\overline{E},\overline{T}/T}"] \arrow[dr] & L_{\overline{T}/T} \otimes_{\overline{T}} \overline{E} \arrow[d] \\
& \Omega_{\overline{T}/T} \otimes_{\overline{T}} \overline{E},
\end{tikzcd}$$

where the diagonal map is the connecting map associated to the sequence of principal parts $\underline{P}_{\overline{T}/T}(\overline{E})$. This reduces us to showing that the restriction of $\underline{P}_{\overline{T}/T}(\overline{E})$ along j is equal to the extension β corresponding to E via the identification $j^*\Omega^1_{\overline{T}/T} = I$. This follows from a straightforward calculation using that $P^1_{\overline{T}/T}(E) = (\mathcal{O}_T \oplus I) \otimes_{\mathcal{O}_T} \overline{E}$. □

6.5. Comparison of automorphism groups

We show that the map $\mathrm{At}_{\mathcal{E}} : R\pi_*(\mathcal{E} \otimes \mathcal{E}^\vee)^\vee[-1] \to L_{\mathcal{M}}$ induces an isomorphism on h^{-1}. It is enough to show that for every map from an affine scheme $g : T \to \mathcal{M}$ and any quasicoherent sheaf I on T, the map

$$(g^* \mathrm{At}_{\mathcal{E}})^* : \mathrm{Ext}^{-1}_T(g^*L_{\mathcal{M}}, I) \to \mathrm{Ext}^0(g^*R\pi_*(\mathcal{E} \otimes \mathcal{E}^\vee)^\vee, I)$$

given by composition with the Atiyah class is an isomorphism. This follows from the following more precise statement:

Lemma 6.13. *Let $\overline{T}$ be the trivial extension of T by I and let $\overline{g} : \overline{T} \to \mathcal{M}$ be the trivial extension of g to $\overline{T}$. Let $\overline{E}$ be the corresponding extension of E. Let φ be an automorphism of $\overline{g}$ extending id_g, which we view as an automorphism γ_φ of $\overline{E}$ via the universal property of $\mathcal{M}$. Then the element τ_φ corresponding to φ via (6.1) is mapped to the element of* $\mathrm{Hom}_{X\times T}(E, \pi^*I \otimes E)$ *corresponding to φ via Remark 6.9.*

Proof. We will first make two reductions: First, the statement can be checked étale locally on T; therefore, we can choose a smooth cover Y and assume that g factors through Y, so that we are in the situation of Lemma 6.5. Then, in particular, φ corresponds to a morphism $f : \overline{T} \to Z$ which is the image of τ_φ in $\mathrm{Hom}(e^*L_Z, I)$ Second, the image of τ_φ is via adjunction identified with the composition

$$E \xrightarrow{g^*\,\mathrm{at}_{\mathcal{E}}} (g \times \mathrm{id}_X)^* L_{\mathcal{M}\times X/X}[1] \otimes E \xrightarrow{\pi^*\tau_\varphi \otimes E} \pi^* I \otimes E,$$

where the first morphism is just At_E, and the second morphism is the one induced by the infinitesimal automorphism $\varphi \times \mathrm{id}$ of the extension $\overline{g} \times \mathrm{id}_X$ of $g \times \mathrm{id}_X$. Since τ_φ factors through a map $\tau'_\varphi : h^1(e^*L_{\mathcal{M}}) \to I$, we can rewrite this composition as

$$E \xrightarrow{(\tau_{\geq 0} \otimes \mathrm{id}_E)\circ g^*\,\mathrm{at}_{\mathcal{E}}} h^0((g \times \mathrm{id}_X)^* L_{\mathcal{M}\times X/X}[1]) \otimes E \xrightarrow{g^*\tau'_\varphi \otimes \mathrm{id}_E} \pi^* I \otimes E.$$

This reduces the problem to understanding the maps $(\tau_{\geq 0} \otimes id_E) \circ g^*\,\mathrm{at}_{\mathcal{E}}$ and $g^*\tau'_\varphi \otimes \mathrm{id}_E$. By Lemma 6.14 below, the former is given as follows: For a section m of E_Z, write $m = \sum x_i s^* n_i = \sum x'_j t^* n'_j$ using the isomorphisms $s^*E_Y \simeq E_Z \simeq t^*E_Y$. Then $e \mapsto \sum dx_i \otimes s^*e_i - \sum dx'_j \otimes \varphi(t^*e'_j)$ in $\Omega_Z \otimes E_Z/(s^*(\Omega_Y \otimes_Y E_Y) + t^*(\Omega_Y \otimes_Y E_Y))$. In particular, if $m = s^*n = \sum x'_j t^* n'_j$, then $s^*y^*h^0(\mathrm{at}_{\mathcal{E}})(m) = -\sum dx'_j \otimes \varphi(t^*n'_j)$. Pulling back along g gives

$$g^*(\tau_{\geq 0} \otimes \mathrm{id}_E \circ \mathrm{at}_{\mathcal{E}})(g^*m) = -\sum dx_j \otimes \varphi(g^*m).$$

The morphism $\tau'_\varphi : \mathrm{Coker}(s^*\Omega_Y \oplus t^*\Omega_Y \to \Omega_Z) \mid_T \to \Omega_{\overline{T}/T} \mid_T = I$ sends dx to $df(x)$, where $df : \Omega_Z \to I$ is the derivation describing f as a deformation of e (s.t. $f = e + I \otimes e + df : e^{-1}\mathcal{O}_Z \to \mathcal{O}_T \oplus I$). In conclusion, we get that the composition is given by the map $g^*n \mapsto \sum -df(x_i) \otimes f^*\varphi(n_i)$

if $s^*n = \sum x_i t^* n_i$. However, we have the automorphism γ_φ of $\overline{E}$ given by the composition $\overline{E} \simeq f^*t^*E_Y \xrightarrow{f^*\varphi} f^*s^*E_Y \simeq \overline{E}$. We compute

$$\gamma_\varphi(\overline{g}^*n) = f^*\varphi\big(\sum x_i t^*n_i\big) = \sum f^*x_i f^*\varphi(t^*n_i)$$

whenever $s^*n = \sum x_i t^* n_i$. Since φ restricts to the identity automorphism on T, we know that γ_φ is of the form $\mathrm{id}_{\overline{E}} + \rho$, where $\rho : E \to I \otimes E$ is a morphism. We claim that $\rho(g^*n) = dx_i \otimes f^*\varphi(t^*n_i)$ whenever $s^*n = \sum x_i t^*n_i$. But we have

$$\rho(g^*n) = (f-e)^*\varphi\big(\sum x_i \otimes t^*n_i\big) = \big(0, \sum dfx_i \otimes \varphi(t^*n_i)\big). \qquad \square$$

Lemma 6.14. *In the situation of Lemma 6.5, suppose that $E \in D^{\le 0}_{qcoh}(\mathcal{Y})$. Then the composition $s^*y^*E \xrightarrow{s^*y^*\,\mathrm{at}_E} s^*y^*L_{\mathcal{Y}}[1] \otimes s^*y^*E \to s^*y^*h^1(L_{\mathcal{Y}}) \otimes s^*y^*E$ agrees up to natural isomorphisms with the connecting homomorphism δ, obtained by applying the Snake Lemma to the following diagram:*

$$\begin{tikzcd}
0 \arrow[r] & {\begin{smallmatrix} s^*(\Omega^1_Y \otimes E_Y)\oplus \\ (t^*\Omega^1_Y \otimes E_Y)\end{smallmatrix}} \arrow[r] \arrow[d] & s^*P^1_Y(E_Y) \oplus t^*P^1_Y(E_Y) \arrow[r] \arrow[d] & s^*E_Y \oplus t^*E_Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_Z \otimes E_Z \arrow[r] & P^1_Z(E_Z) \arrow[r] & E_Z \arrow[r] & 0.
\end{tikzcd}$$

Proof. Let $R := P_k(\mathcal{O}_Z)$ and $Q = P_k(\mathcal{O}_Y)$ be the standard simplicial resolutions and s^*_R and t^*_R the pullback functors from Q-modules to R-modules induced by s and t, respectively. Then, we have a commutative diagram of R-modules

$$\begin{tikzcd}
0 \arrow[r] & {\begin{smallmatrix} s^*_R(L_Y \otimes E_Y)\oplus \\ t^*_R(L_Y \otimes E_Y)\end{smallmatrix}} \arrow[r] \arrow[d, "\alpha"] & s^*_R P^1_Q(E_Y) \oplus t^*_R P^1_Q(E) \arrow[r] \arrow[d, "\beta"] & s^*E_Y \oplus t^*E_Y \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & L_Z \otimes E_Z \arrow[r] \arrow[d] & P^1_R(E_Z) \arrow[r] \arrow[d] & E_Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(\alpha) \arrow[r] & C(\beta) \arrow[r] & C(\gamma) \arrow[r] & 0.
\end{tikzcd}$$

By the construction of the Atiyah class, we have natural isomorphisms $C(\alpha) \simeq s^*y^*L_{\mathcal{Y}}[1] \otimes E$ and $C(\gamma) \simeq s^*y^*E[1]$ with respect to which the connecting map associated to the lower row is naturally identified with $s^*y^*\,\mathrm{at}_E[1]$ in $D^\Delta(R)$. The diagram maps to the similar diagram of $\mathcal{O}_Z$-modules

$$\begin{tikzcd}
0 \arrow[r] & {\begin{smallmatrix} s^*(\Omega_Y \otimes E_Y)\oplus \\ t^*(\Omega_Y \otimes E_Y)\end{smallmatrix}} \arrow[r] \arrow[d, "\alpha'"] & s^*P^1_Y(E_Y) \oplus t^*P^1_Y(E_Y) \arrow[r] \arrow[d, "\beta'"] & s^*E_Y \oplus t^*E_Y \arrow[r] \arrow[d, "\gamma'"] & 0 \\
0 \arrow[r] & \Omega_Z \otimes E_Z \arrow[r] \arrow[d] & P^1_Z(E_Z) \arrow[r] \arrow[d] & E_Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \dfrac{\Omega_Z}{s^*\Omega_Y \oplus t^*\Omega_Y} \otimes E_Z \arrow[r] & * \arrow[r] & C(\gamma') \arrow[r] & 0,
\end{tikzcd}$$

where the last row is obtained by taking cones of the vertical morphisms and then pushing out along $C(\alpha') \to \Omega_Z/(s^*\Omega_Y \oplus t^*\Omega_Y) \otimes E_Z$. The morphism $C(\alpha) \to \Omega_Z/(s^*\Omega_Y \oplus t^*\Omega_Y) \otimes E_Z$ here is identified with the truncation morphism $L_{\mathcal{Y}}[1] \to h^1(L_{\mathcal{Y}})$ tensored with E_Z. By pulling back the lower row along the natural map $s^*y^*E[1] \to C(\gamma')$ obtained from the diagonal map $s^*y^*E \to s^*E_Y \oplus t^*E_Y$, we get an extension

$$0 \to \frac{\Omega_Z}{s^*\Omega_Y + t^*\Omega_Y} \otimes E_Z \to * \to s^*y^*E[1] \to 0.$$

That the induced connecting homomorphism agrees with the one coming from the Snake Lemma follows from Lemma 6.15. □

Lemma 6.15. *Suppose that we have a commutative diagram in an abelian category with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & D \arrow[r] & E \arrow[r] & F \arrow[r] & 0. \end{tikzcd}$$

Form the exact sequence

$$\underline{C} : 0 \to C(\alpha) \to C(\beta) \to C(\gamma) \to 0$$

and let $j : C(\alpha) \to \operatorname{Coker}(\alpha)$ *and* $i : \operatorname{Ker}(\gamma)[1] \to C(\gamma)$ *denote the natural maps. Then the connecting homomorphism associated to the sequence of complexes of A-modules*

$$j_*i^*\underline{C} = i^*j_*\underline{C} : 0 \to \operatorname{Coker}(\alpha) \to M \to \operatorname{Ker}(\gamma)[1] \to 0$$

is exactly minus the shift by 1 *of the connecting morphism* $\delta : \operatorname{Ker}(\gamma) \to \operatorname{Coker}(\alpha)$ *in the Snake Lemma.*

Proof. One checks by a direct computation that M is just the cone over δ with the two maps being the canonical inclusion and projection. It follows by the usual considerations of mapping cones that the connecting morphism of the sequence $j_*i^*\underline{C}$ is $-\delta$. □

Remark 6.16. To conclude, we point out that the conclusions of §6.3–§6.5 still hold when $\mathcal{M}$ is open in a moduli stack of perfect complexes, and $\mathcal{E}$ is the universal complex, using the results of [6]. We assume here that the complexes being parametrized have vanishing negative Ext-groups universally on $\mathcal{M}$.

The arguments of 6.3 and 6.4 go through with Theorem 3.3 and Corollary 3.4 of [6] in place of Remark 6.7 and Remark 6.8, respectively. The arguments of 6.5 go through unchanged for complexes.

Competing interest. The author has no competing interest to declare.

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