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# $L^p$-$L^q$ ESTIMATES OFF THE LINE OF DUALITY

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**Abstract**

Theorems 1 and 2 are known results concerning $L^p$-$L^q$ estimates for certain operators wherein the point $(1/p, 1/q)$ lies on the line of duality $1/p + 1/q = 1$. In Theorems 1′ and 2′ we show that with mild additional hypotheses it is possible to prove $L^p$-$L^q$ estimates for indices $(1/p, 1/q)$ off the line of duality. Applications to Bochner-Riesz means of negative order and uniform Sobolev inequalities are given.

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## 1. Introduction

This paper is concerned with $L^p$-$L^q$ estimates for certain operators on $\mathbb{R}^n$. To explain our work we begin by stating two well-known principles as Theorems 1 and 2:

THEOREM 1. *With notation as in* [14], *suppose* $\{T_z\}$ *is an analytic family of operators of admissible growth on* $\mathbb{R}^n$ *satisfying the estimates*

(i) $$\|T_z f\|_2 \le C_z \|f\|_2, \qquad \operatorname{Re} z = 0,$$

(ii) $$\|T_z f\|_\infty \le C_z \|f\|_1, \qquad \operatorname{Re} z = -\lambda.$$

*Here* $C_z$ *is some non-negative function satisfying*

$$\log C_z \le K e^{k|\operatorname{Im} z|}, \qquad \operatorname{Re} z = 0, -\lambda$$

*for some* $K > 0$ *and* $k < \pi$. *Then if* $0 < \alpha < \lambda$ *and* $1/p - 1/p' = \alpha/\lambda$ *(where, always,* $1/p + 1/p' = 1$*) there is* $C = C(\alpha)$ *such that* $\|T_{-\alpha} f\|_{p'} \le C\|f\|_p$.





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THEOREM 2. *Suppose* $0 < \alpha < 1$ *and* $1/p - 1/p' = \alpha$. *Consider a one-parameter family* $\{U(t)\}$ *of operators on* $\mathbb{R}^n$ *satisfying* $\|U(t)f\|_{p'} \le C|t|^{\alpha-1}\|f\|_p$ *and a bounded function* $a(t)$ *on* $\mathbb{R}$. *Under suitable measurability conditions on* $\{U(t)\}$ *and* $a(t)$, *the operator* $S$ *defined on* $\mathbb{R}^{n+1}$ *by*

$$Sg(s,x) = \int_{-\infty}^{\infty} a(t)U(t)g(s-t,\cdot)(x)\,dt, \qquad (s,x) \in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1},$$

*satisfies the estimate* $\|Sg\|_{p'} \le C_2\|g\|_p$ *for functions* $g$ *on* $\mathbb{R}^{n+1}$.

Theorem 1 is of course a special case of Stein's interpolation theorem [14]. It has many applications in harmonic analysis. Perhaps the best known are to the problem of obtaining Fourier transform restriction estimates of the form

$$\left(\int_\Sigma |\widehat{f}|^2 d\mu\right)^{1/2} \le C\|f\|_p$$

where $\mu$ is a measure on a hypersurface $\Sigma$ in $\mathbb{R}^n$. The first of these is Stein's result in the paper [18]. Another class of applications of Theorem 1 concerns estimates of the form $\|\mu * f\|_q \le C\|f\|_p$ where $\mu$ is as above. Two early examples of this are in [9, 15]. Of the several later ones we mention [3, 10, 12].

Theorem 2 is a useful device introduced by Strichartz in the proof of [16, Theorem 1]. The proof is based on convolution properties of the fractional integration kernels $|t|^{\alpha-1}$ on $\mathbb{R}$. Two later applications may be found in [8, 10].

A feature common to Theorems 1 and 2 is that their conclusions give $L^p$-$L^q$ estimates only when $q = p'$, or when $(1/p, 1/q)$ lies on the line of duality $1/p + 1/q = 1$. Our purpose is to show that with mild additional hypotheses it is possible to obtain $L^p$-$L^q$ estimates along a segment through $(1/p, 1/p')$ and perpendicular to the line of duality. To this end we will prove Theorems 1′ and 2′ below.

THEOREM 1′. *With notation as in Theorem* 1, *suppose* $\{T_z\}$ *satisfies the additional hypothesis*

(iii) $\qquad |T_z^* T_z f| \le C_z |T_{2\operatorname{Re} z} f|$ *(pointwise)* *if* $\; -\mu < \operatorname{Re} z < 0$

*for some positive* $\mu$ *with* $\mu \le \lambda/2$. *Then if* $0 < \alpha < \lambda$ *there is* $C = C(\alpha, p)$ *such that*

$$\|T_{-\alpha} f\|_q \le C\|f\|_p$$

*provided that* $1/p - 1/q = \alpha/\lambda$ *and either*

(A) $$\frac{\lambda+\alpha}{2\lambda} \le \frac{1}{p} \le \frac{\lambda+2\alpha}{2\lambda} \qquad \text{if } 0 < \alpha < \mu, \qquad \text{or}$$

(B) $$\frac{\lambda+\alpha}{2\lambda} \le \frac{1}{p} < \frac{\lambda^2+\alpha\lambda-2\alpha\mu}{2\lambda(\lambda-\mu)} \qquad \text{if } \mu \le \alpha < \lambda.$$

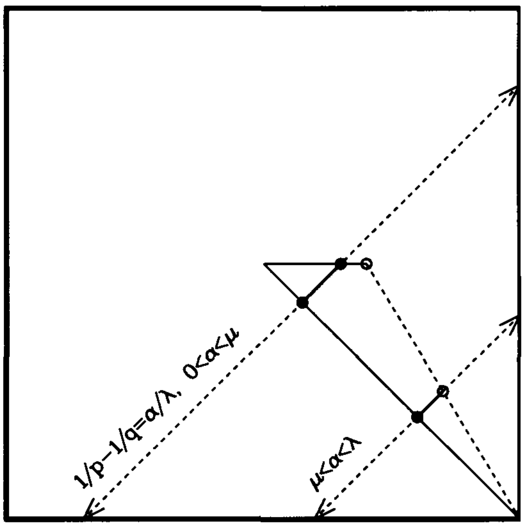


FIGURE 1

Thus $T_{-\alpha}$ is a bounded operator from $L^p$ to $L^q$ if the point $(1/p, 1/q)$ lies (1) on the line $1/p - 1/q = \alpha$ and (2) inside the triangle bounded by the points $P_1 = (1/2, 1/2)$, $P_2 = (1, 0)$, and $P_3 = (1/2 + \mu/\lambda, 1/2)$, excluding the segment $P_2P_3$; see Figure 1. If $\|T_z f\|_q \leq C\|f\|_p$ implies $\|T_z f\|_{p'} \leq C\|f\|_{q'}$, then the ranges of $1/p$ in (A) and (B) become $1/2 \leq 1/p \leq (\lambda + 2\alpha)/2\lambda$, and $(\lambda - 2\mu + \alpha)/2(\lambda - \mu) < 1/p < (\lambda^2 + \alpha\lambda - 2\alpha\mu)/[2\lambda(\lambda - \mu)]$.

THEOREM 2$'$. *With notation as in Theorem 2, suppose additionally that $\{U(t)\}$ is actually a one-parameter group of operators satisfying $U(-t) = U(t)^*$ and that*

$$\|U(t)f\|_\infty \leq C|t|^{(\alpha-1)/\alpha}\|f\|_1.$$

*Then there is $C = C(p)$ such that $\|Sg\|_q \leq C\|g\|_p$ for functions $g$ on $\mathbb{R}^{n+1}$ provided that*

$$\frac{1}{p} - \frac{1}{q} = \alpha \qquad \textit{and} \qquad \frac{1}{2-\alpha} < \frac{1}{p} < \frac{1+\alpha-\alpha^2}{2-\alpha}.$$

Theorems 1$'$ and 2$'$ are not, strictly speaking, new mathematics—their (somewhat parallel) proofs are based on methods present in [5, 11], respectively. They seem, however, to be useful tools which can be applied to produce, at the least, some

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interesting extensions of known results. For a slight and technical improvement of Theorem 1′, see the note after its proof in Section 2.

Our paper is organized as follows: Section 2 contains the proofs of Theorems 1′ and 2′, Section 3 is an application of Theorem 1′ to Bochner-Riesz means of negative order, and Section 4 contains applications of Theorems 1′ and 2′ to obtain generalizations of the results in [7, 8] on uniform Sobolev inequalities.

## 2. Proofs of Theorems 1′ and 2′

The proof of Theorem 1′ is an easy consequence of complex interpolation and the so-called method of $T^*T$: it follows from Theorem 1 that

$$(1) \qquad \|T_{-\alpha}f\|_{p'} \le C(\alpha)\|f\|_p \quad \text{if} \quad \frac{1}{p} = \frac{\lambda+\alpha}{2\lambda}, \quad 0<\alpha<\lambda.$$

Hölder's inequality and (iii) then imply

$$\langle T_z f, T_z f\rangle = \langle f, T_z^* T_z f\rangle \le C_z\|f\|_p\|T_{2\operatorname{Re} z}f\|_{p'} \le C_z\|f\|_p^2$$

and so

$$(2) \qquad \|T_z f\|_2 \le C_z\|f\|_p \quad \text{if} \quad \frac{1}{p} = \frac{\lambda-2\operatorname{Re} z}{2\lambda}, \quad -\mu<\operatorname{Re} z<0.$$

(This is the method of $T^*T$.) In particular,

$$(3) \qquad \|T_{-\alpha}f\|_2 \le C\|f\|_p \quad \text{if} \quad \frac{1}{p} = \frac{\lambda+2\alpha}{2\lambda}, \quad 0<\alpha<\mu.$$

By the Riesz-Thorin theorem, (1) and (3) yield (A). To obtain (B) apply analytic interpolation to (ii) and (2) with $\operatorname{Re} z = -\beta$, $0<\beta<\mu$. The result is $\|T_{-\alpha}f\|_q \le C(\alpha)\|f\|_p$ if

$$\frac{1}{p} = \frac{\lambda^2+\alpha\lambda-2\alpha\beta}{2\lambda(\lambda-\beta)}, \quad \frac{1}{p}-\frac{1}{q} = \frac{\alpha}{\lambda}, \quad \text{and} \quad \beta<\alpha<\lambda.$$

This proves (B) since for fixed $\alpha \ge \mu$, $1/p$ varies from $(\lambda+\alpha)/(2\lambda)$ to $(\lambda^2+\alpha\lambda-2\alpha\mu)/[2\lambda(\lambda-\mu)]$ as $\beta$ varies from 0 to $\mu$. Thus Theorem 1′ is established.

NOTE. The conclusion of Theorem 1′ is still true if (iii) holds only for a set of $\operatorname{Re} z$ whose closure contains $-\mu$. For then, by interpolation, (2) is still true. This observation is actually useful in some of the applications.

Let $p$ and $p'$ be as in Theorem 2 and put $\gamma = (1-\alpha)/\alpha$. The starting point for the proof of Theorem 2′ is an analog of the dual of (2), the inequality

$$\left[\int_{-\infty}^{\infty} \| |t|^{\gamma(1/2-1/p')} U(t) f \|_{p'}^{p'} dt/|t| \right]^{1/p'} \leq C \|f\|_2 \tag{4}$$

for functions $f$ on $\mathbb{R}^n$. This will be established, again by the method of $T^*T$, at the end of this section. Interpolating analytically, in the mixed norm setting, between (4) and the hypothesis

$$\sup_t |t|^{\gamma} \|U(t)f\|_{\infty} \leq C \|f\|_1$$

of Theorem 2′ shows that if

$$\left(\frac{1}{r}, \frac{1}{s}\right) = \left(\frac{1}{2-\alpha}, \frac{(1-\alpha)^2}{2-\alpha}\right),$$

then

$$\left[\int_{-\infty}^{\infty} \| |t|^{\gamma(1/r-1/s)} U(t) f \|_{s}^{s} dt/|t| \right]^{1/s} \leq C \|f\|_r.$$

Interpolating this with the hypothesis of Theorem 2 shows that if $1/p - 1/q = \alpha$ and $1/(2-\alpha) < 1/p < (1+\alpha)/2$, then there is $b \in (q, \infty)$ such that

$$\left[\int_{-\infty}^{\infty} \| |t|^{\gamma(1/p-1/q)} U(t) f \|_{q}^{b} dt/|t| \right]^{1/b} \leq C \|f\|_p. \tag{5}$$

Now suppose $g$ and $h$ are functions on $\mathbb{R}^{n+1}$ and $p, q$ are as above. Then

$$\begin{aligned}
|\langle Sg, h\rangle| &= \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} a(t) U(t) g(s-t, \cdot)(x) h(s,x)\, dx\, ds\, dt \right| \\
&\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\langle U(t) g(s-t, \cdot), h(s, \cdot)\rangle|\, ds\, |a(t)|\, dt \\
&\leq M \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\langle U(t) g(s, \cdot), h(s+t, \cdot)\rangle|\, ds\, dt
\end{aligned}$$

if $M$ is a bound for $|a(t)|$. Thus

$$|\langle Sg, h\rangle| \leq M \int_{-\infty}^{\infty} \|U(t) g(s, \cdot)\|_q \|h(s+t, \cdot)\|_{q'}\, ds\, dt$$

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$$= M \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \| |t|^{\gamma(1/p-1/q)} U(t) g(s,\cdot)\|_q \|h(s+t,\cdot)\|_{q'} |t|^{-\gamma(1/p-1/q)+1} \frac{dt}{|t|}\, ds$$

$$\leq M \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \| |t|^{\gamma(1/p-1/q)} U(t) g(s,\cdot)\|_q^b \frac{dt}{|t|}\right]^{1/b}$$

$$\cdot \left[\int_{-\infty}^{\infty} \|h(s+t,\cdot)\|_{q'}^{b'} |t|^{-b'\gamma(1/p-1/q)+b'} \frac{dt}{|t|}\right]^{1/b'} ds$$

$$\leq CM \int_{-\infty}^{\infty} \|g(s,\cdot)\|_p \left(\int_{-\infty}^{\infty} \|h(s+t,\cdot)\|_{q'}^{b'} |t|^{-b'\gamma(1/p-1/q)+b'-1}\, dt\right)^{1/b'} ds$$

by (5) with $f(x) = g(s,x)$. Therefore Hölder's inequality yields

$$(6)\quad |\langle Sg, h\rangle| \leq CM\left(\int_{-\infty}^{\infty} \|g(s,\cdot)\|_p^p\, ds\right)^{1/p}$$

$$\cdot \left(\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \|h(s+t,\cdot)\|_{q'}^{b'} |t|^{-b'\gamma(1/p-1/q)+b'-1}\, dt\right]^{p'/b'} ds\right)^{1/p'}.$$

Now if $h \in L^{q'}(\mathbb{R}^{n+1})$, then, as a function of $s$, $\|h(s,\cdot)\|_{q'}^{b'} \in L^{q'/b'}(\mathbb{R})$, and $q'/b' > 1$ since $b > q$. Since $\gamma = (1-\alpha)/\alpha$ implies

$$\frac{b'}{q'} - \left[-b'\gamma\left(\frac{1}{p} - \frac{1}{q}\right) + b' - 1\right] = \frac{b'}{p'} + 1,$$

the $L^{q'/b'}$-$L^{p'/b'}$ estimates for one-dimensional fractional integration combine with (6) to give $|\langle Sg, h\rangle| \leq CM\|g\|_p\|h\|_{q'}$, with the norms taken over $\mathbb{R}^{n+1}$. This yields the conclusion of Theorem 2′ for $1/(2-\alpha) < 1/p < (1+\alpha)/2$. The hypotheses of Theorem 2′ assure that the same is true for the adjoint of $S$, and so the proof of Theorem 2′ will be complete when (4) is established.

To this end, define an operator $T$ taking functions $f$ on $\mathbb{R}^n$ into functions on $\mathbb{R}^{n+1}$ by the rule $Tf(s,x) = U(s)f(x)$. Since $\gamma(1/2 - 1/p')\,p' = 1$, (4) is just the statement that $T$ is bounded from $L^2(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^{n+1})$. Thus, as in the proof of Theorem 1′, it suffices to show that $TT^*$ is bounded from $L^p(\mathbb{R}^{n+1})$ to $L^{p'}(\mathbb{R}^{n+1})$. A computation shows that, for $g$ on $\mathbb{R}^{n+1}$,

$$T^*g(x) = \int_{-\infty}^{\infty} U^*(t) g(t,\cdot)(x)\, dt.$$

It follows that

$$TT^*g(s,x) = U(s)T^*g(x) = \int_{-\infty}^{\infty} U(s-t)g(t,\cdot)(x)dt.$$

Therefore, basically repeating the proof of Theorem 2,

$$\|TT^*g\|_{p'} \le \left\| \int_{-\infty}^{\infty} \|U(s-t)g(t,\cdot)\|_{p',x}dt \right\|_{p',s} \le C \left\| \int_{-\infty}^{\infty} |s-t|^{\alpha-1}\|g(t,\cdot)\|_{p,x}dt \right\|_{p',s}.$$

Here the last inequality follows from the hypothesis of Theorem 2. By one-dimensional fractional integration, since $1/p + (1-\alpha) = 1/p' + 1$, the last term is dominated by

$$\left\| \|g(s,\cdot)\|_{p,x} \right\|_{p,s} = \|g\|_p.$$

This establishes (4).

## 3. Bochner-Riesz means of negative order

The Bochner-Riesz operators $T_\alpha$ are defined on $\mathbb{R}^n$, $n \ge 2$, for $-(n+1)/2 \le \alpha \le 0$ by

$$\widehat{T_\alpha f}(x) = \frac{(1-|x|^2)_+^\alpha}{\Gamma(\alpha+1)} \widehat{f}(x)$$

or, equivalently, by $T_\alpha f = K_\alpha * f$ where

$$K_\alpha(x) = 2^{\alpha+\frac{n}{2}}\pi^{\frac{n}{2}} J_{\alpha+\frac{n}{2}}(|x|)/|x|^{\alpha+\frac{n}{2}}.$$

Their $L^p$-$L^q$ boundedness has been studied in [1, 13, 2]. Necessary conditions for $T_\alpha : L^p \to L^q$, given in [1], are

$$\text{(7)} \qquad \frac{n-1-2\alpha}{2n} < \frac{1}{p}, \qquad \frac{1}{q} < \frac{n+1+2\alpha}{2n}, \qquad \frac{-2\alpha}{n+1} \le \frac{1}{p} - \frac{1}{q}.$$

We will summarize the sufficient conditions of [1, 13, 2] as Theorem 3 below. To facilitate the statement of this result, we label some points in $[0,1]\times[0,1]$ (see Figure 2): let

$$A = \left(1, \frac{n+1+2\alpha}{2n}\right), \qquad A' = \left(\frac{n-1-2\alpha}{2n}, 0\right),$$

$$B = \left(\frac{n+1+2\alpha}{2n} - \frac{2\alpha}{n+1}, \frac{n+1+2\alpha}{2n}\right), \quad B' = \left(\frac{n-1-2\alpha}{2n}, \frac{n-1-2\alpha}{2n} + \frac{2\alpha}{n+1}\right),$$

$$C = \left(\frac{n+3}{2n+2}, \frac{n+1+2\alpha}{2n}\right), \qquad C' = \left(\frac{n-1-2\alpha}{2n}, \frac{n-1}{2n+2}\right),$$

$$D = \left(\frac{n+1-2\alpha}{2n+2}, \frac{n+1+2\alpha}{2n+2}\right).$$

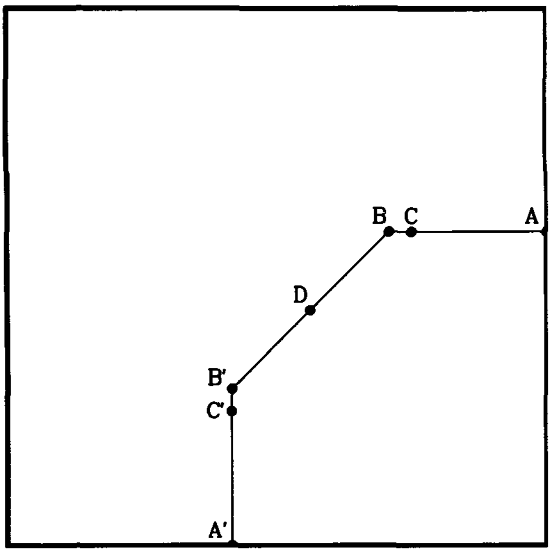


FIGURE 2

Let $N = N(\alpha, n)$ denote the set of points $(1/p, 1/q)$ in $[0, 1] \times [0, 1]$ which satisfy the necessary conditions (7).

THEOREM 3. *If $(1/p, 1/q) \in N$, then there exists a constant $C = C(p, q, \alpha, n)$ such that $\|T_\alpha f\|_q \leq C\|f\|_p$ provided that one of the following holds:*

(1) $n = 2$ *and* $\alpha = -1$;
(2) $n = 2, -3/2 \leq \alpha \leq 0,$ *and* $1/p - 1/q > -2\alpha/(n+1)$;
(3) $n \geq 3, -(n+1)/2 \leq \alpha \leq -1/2,$ *and* $1/p - 1/q > -2\alpha/(n+1)$;
(4) $n \geq 3, -1/2 < \alpha \leq 0,$ *and* $(1/p, 1/q)$ *lies strictly below the lines joining $D$ to $C$ and $D$ to $C'$*;
(5) $(1/p, 1/q) = D$.

Parts (1)-(3) are proved in [1] and [13], and (4) is in [2]. Part (5) is a well-known consequence of Theorem 1. But Theorem 1′ is as easily applicable and gives the following stronger result.

THEOREM 4. *Fix $n \geq 2$. There is $C = C(p, q, \alpha, n)$ such that $\|T_\alpha f\|_q \leq C\|f\|_p$ provided that $(1/p, 1/q)$ is on the open segment $BB'$ and either*

(A) $$-\frac{1}{2} < \alpha < 0 \quad \textit{and} \quad \frac{1}{2} \leq \frac{1}{p} \leq \frac{n+1-4\alpha}{2n+2}, \qquad \textit{or}$$

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(B) $$-\frac{n+1}{2} \leq \alpha \leq -\frac{1}{2}.$$

PROOF. Define $\{T_z\}$ by

$$\widehat{T_z f}(x) = \frac{(1-|x|^2)_+^z}{\Gamma(z+1)} \widehat{f}(x).$$

Clearly $\|T_z f\|_2 \leq C_z \|f\|_2$ if $\operatorname{Re} z = 0$. Well-known asymptotic estimates for Bessel functions give $\|T_z f\|_\infty \leq C_z \|f\|_1$ if $\operatorname{Re} z = -(n+1)/2$. Since

$$T_z^* T_z = \frac{\Gamma(1+2\operatorname{Re} z)}{|\Gamma(1+z)|^2} T_{2\operatorname{Re} z} \quad \text{if} \quad -\frac{1}{2} < \operatorname{Re} z < 0,$$

Theorem 1′ applies with $\lambda = (n+1)/2$ and $\mu = 1/2$ and yields (A) and (B).

Here are two additional observations:

(1) If $n \geq 3$ and $-1/2 < \alpha < 0$, boundedness at some additional interior points of $N$ can be obtained by interpolating (4) of Theorem 3 and (A) of Theorem 4.

(2) Let $Rf = \widehat{f}|_{S^{n-1}}$ denote the restriction of the Fourier transform of $f$ to the unit sphere. Then $R^*R$ is a multiple of $T_{-1}$. Thus taking $\alpha = -1$ in Theorem 4 gives a 'restriction' estimate which complements a result of [13]:

COROLLARY 5. *If* $1/p - 1/q = 2/(n+1)$ *and* $(n+1)/2n < 1/p < (n^2 - 1 + 4n)/[2n(n+1)]$, *then there is* $C = C(p, n)$ *such that*

$$\left\| \int_{S^{n-1}} e^{ix\cdot\xi} \widehat{f}(\xi) d\sigma(\xi) \right\|_q \leq C \|f\|_p.$$

This is (1) of Theorem 3 when $n = 2$, where it follows from the two-dimensional restriction theorem

$$\|Rf\|_{L^q(S^1)} \leq C \|f\|_{L^p(\mathbb{R}^2)} \quad \text{if} \quad q = p'/3 \quad \text{and} \quad 1 \leq p < 4/3$$

of [4, 19]. Similarly, Corollary 5 would follow from the $n$-dimensional restriction conjecture

$$\|Rf\|_{L^q(S^{n-1})} \leq C \|f\|_{L^p(\mathbb{R}^n)} \qquad \text{if } q = (n-1)p'/(n+1) \text{ and } 1 \leq p < 2n/(n+1).$$

## 4. Uniform Sobolev inequalities

We begin by recalling some of the principal results from [7, 8]. Here is the requisite notation: $Q(\xi)$ will denote the form on $\mathbb{R}^n$, $n \geq 3$, given for some $j = 1, \ldots, n-1$ by

$$Q(\xi) = -\xi_1^2 - \ldots - \xi_j^2 + \xi_{j+1}^2 + \ldots + \xi_n^2,$$

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$\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$ is the Laplacian on $\mathbb{R}^n$, $i\partial/\partial t + \Delta$ is the Schrödinger operator on $\mathbb{R}^{n+1}$, and $L(D) = \langle \mathbf{a}, \nabla_x \rangle + b$ is an arbitrary first order operator on $\mathbb{R}^n$ with constant complex coefficients.

THEOREM 6. ([7, Theorem 2.1]) *Suppose $n \geq 3$ and $1/p - 1/p' = 2/n$. There is a constant $C = C(n)$ such that whenever $P(D)$ is a constant coefficient operator on $\mathbb{R}^n$ with complex coefficients and principal part $Q(D)$, then*

$$\|u\|_{p'} \leq C\|P(D)(u)\|_p,$$

*for all $u \in \mathscr{S}(\mathbb{R}^n)$.*

THEOREM 7. ([7, Theorem 2.2]) *Suppose $n \geq 3$, $1/p - 1/q = 2/n$, and $(n+1)/2n < 1/p < (n+3)/2n$. There exists $C = C(p, n)$ such that whenever $P(D)$ is a constant coefficient operator on $\mathbb{R}^n$ with complex coefficients and principal part $\Delta$, then*

$$\|u\|_q \leq C\|P(D)(u)\|_p, \qquad u \in \mathscr{S}(\mathbb{R}^n).$$

THEOREM 8. ([8, Theorem 1]) *If $n \geq 1$ and $1/p - 1/p' = 2/(n+2)$, then there is $C = C(n)$ which does not depend on $L(D)$ such that*

$$\|u\|_{p'} \leq C\left\|\left(i\frac{\partial}{\partial t} + \Delta + L(D)\right)u\right\|_p, \qquad u \in \mathscr{S}(\mathbb{R}^{n+1}).$$

Comparison of Theorem 7 with Theorems 6 and 8 raises an obvious question which we answer with Theorems 6′ and 8′:

THEOREM 6′. *Suppose $n \geq 3$, $1/p - 1/q = 2/n$ and $n/(2n-2) < 1/p < (n^2+2n-4)/(2n^2-2n)$. There is a constant $C = C(p, n)$ such that*

$$\|u\|_q \leq C\|P(D)(u)\|_p, \qquad u \in \mathscr{S}(\mathbb{R}^n),$$

*for all $P(D)$ as in Theorem 6.*

THEOREM 8′. *Suppose $n \geq 1$, $1/p - 1/q = 2/(n+2)$ and $(n+2)/(2n+2) < 1/p < (n^2+6n+4)/(2n^2+6n+4)$. There is a constant $C = C(p, n)$ which does not depend on $L(D)$ such that*

$$\|u\|_q \leq C\left\|\left(i\frac{\partial}{\partial t} + \Delta + L(D)\right)u\right\|_p, \qquad u \in \mathscr{S}(\mathbb{R}^{n+1}).$$

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Taking $P(D)$ in Theorem 6′ to be the wave operator on $\mathbb{R}^n$ gives an estimate already obtained in [5, 11]. Examples in [5] therefore show that the hypotheses on $p$ and $q$ in Theorem 6′ cannot be weakened. Analogous examples show the same for Theorem 8′, even when $L(D) = 0$. As will be clear to the reader familiar with the paper [7], several other results there, including the estimates for the Klein-Gordon operator and the unique continuation theorems, have analogs 'off the line of duality'.

To explain the proofs of Theorems 6′ and 8′ we recall the proofs of Theorems 6 and 8, which are broadly similar. Let $\alpha = 2/n$ for Theorem 6 and $\alpha = 2/(n+2)$ for Theorem 8. After some reductions there is a clever argument based on Littlewood-Paley theory which yields, so long as $1/p - 1/q = \alpha$, $L^p$-$L^q$ Sobolev inequalities as consequences of certain other $L^p$-$L^q$ estimates. These estimates, [7, Lemma 2.1] and [8, Lemmas 1 and 2], are established when $q = p'$, that is, when $(1/p, 1/q)$ is on the line of duality. Our only contribution is to note that they actually hold for the ranges of $p$ and $q$ given in Theorems 6′ and 8′. The next three paragraphs contain sketches of the arguments.

After some Fourier transform estimates, [7, Lemma 2.1] is simply an application of Theorem 1, quite analogous to that which established (5) of Theorem 3. Again, the additional hypotheses of Theorem 1′ are easily verified and the result is that [7, Lemma 2.1] holds with $(1/p, 1/p')$ replaced by $(1/p, 1/q)$ as in Theorem 6′.

We will consider in detail only [8, Lemma 2]. The operator $S$ in question can be written

$$Sg(s,x) = \int_{-\infty}^{\infty} a(t)U(t)g(s-t,\cdot)(x)\,dt, \qquad (s,x) \in \mathbb{R}^{n+1},$$

where $U(t)$ is the Fourier multiplier operator on $\mathbb{R}^n$ with symbol $e^{it|\xi|^2}$. Recalling that $\alpha = 2/(n+2)$, [8, (8)] is the hypothesis

$$\|U(t)f\|_{p'} \leq C|t|^{\alpha-1}\|f\|_p, \quad f \text{ on } \mathbb{R}^n, \qquad 1/p - 1/p' = \alpha$$

of Theorem 2, and [8, Lemma 2] follows. But since $(\alpha-1)/\alpha = -n/2$, the additional hypothesis

$$\|U(t)f\|_\infty \leq C|t|^{(\alpha-1)/\alpha}\|f\|_1$$

of Theorem 2′ results from a homogeneity argument and the fact that $e^{i|\xi|^2}$ is an $L^1$-$L^\infty$ multiplier on $\mathbb{R}^n$. Theorem 2′ then gives the estimate

$$\|Sg\|_q \leq C(p)\|g\|_p$$

for $p$ and $q$ in Theorem 8′.

The result [8, Lemma 1] is a restriction result of [17] (as is [7, Lemma 2.1(a)]). It can be appropriately generalized either by reasoning analogous to that immediately above or, presumably, by another application of Theorem 1′.

A concluding remark: it seems likely that an argument similar to our generalization of Theorem 8 may yield an extension of the results of [6] on time-dependent Schrödinger operators.

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