ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF PARABOLIC DIFFERENTIAL INEQUALITIES

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1. Introduction. Let there be given a parabolic differential operator

$$P = \frac{\partial}{\partial t} + A\left(x, t, \frac{\partial}{\partial x}\right),$$

where A is a second order linear elliptic (<0) differential operator in an open set $\Omega \subset R^n$, having coefficients depending on $x \in \Omega$ and $t \in [0, \infty)$. Recently, Protter **(1)** investigated the asymptotic behaviour of functions $u(x, t)$ that satisfy the differential inequality

$$(1.1) \qquad \int_\Omega |Pu|^2\, dx \leqslant C_1(t) \int_\Omega |u|^2\, dx + C_2(t) \int_\Omega \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 dx.$$

Under suitable restrictions on the functions $C_i(t)$ and the coefficients of A, he proved that any solution of (1.1), subject to certain homogeneous boundary conditions, that vanishes sufficiently fast, as $t \to \infty$, must be identically zero in $\Omega \times [0, \infty)$. For example, conditions are given under which no solution of (1.1) can vanish faster than $e^{-\lambda t}$, $\forall \lambda > 0$, unless identically zero.

In this paper we shall consider similar problems from an abstract point of view, based on Hilbert space methods for solving parabolic differential equations (cf. **2**). Our results apply to certain inequalities for parabolic operators of arbitrary order, and, in particular, they overlap somewhat with those of Protter for the inequality (1.1).

2. Notation and assumptions. Let H and V be Hilbert spaces with V dense in H. We assume that the injection mapping from V into H is continuous. The inner product in H will be denoted by $\langle u, v\rangle$, and the corresponding norm $\langle u, u\rangle^{1/2}$ by $|u|$. For the norm in V we employ the notation $||u||$.

We introduce a real variable $t \geqslant 0$, and give on V a family of sesquilinear forms

$$(2.1) \qquad (u, v) \to a(t; u, v),$$

continuous on $V \times V$. The form (2.1) is sesquilinear if it is linear in u and conjugate-linear in v.

In order to define an operator generated by the form $a(t; u, v)$, we make the

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DEFINITION. $\mathfrak{D}[A(t)]$ *is the set of all* $u \in V$ *for which the conjugate-linear mapping* $v \to a(t; u, v)$ *is continuous on* V *in the norm topology induced by* H.

It is possible that $\mathfrak{D}[A(t)] = \{0\}$.

Since V is dense in H, the conjugate-linear mapping $v \to a(t; u, v)$ can be extended by continuity to all of H, and therefore,

$$a(t; u, v) = \langle A(t)u, v \rangle, \tag{2.2}$$

where $A(t)u \in H$. This defines an additive operator $A(t)$ from $\mathfrak{D}[A(t)]$ into H.

Assumption I. We can write

$$a(t; u, v) = a_0(t; u, v) + a_1(t; u, v),$$

where $\forall t \geqslant 0$, a_0 and a_1 are continuous sesquilinear forms on $V \times V$, which possess the following properties:

(i) For all u and $v \in V$, the function $t \to a_0(t; u, v)$ is continuously differentiable on $[0, \infty)$.

(ii) For every u and $v \in V$, $a_0(t; u, v) = \overline{a_0(t; v, u)}$, and there exist m and $\rho > 0$ such that

$$a_0(t; u, u) + \rho|u|^2 \geqslant m||u||^2.$$

(iii) For all u and $v \in V$, we have

$$|a_1(t; u, v)| \leqslant \phi_3(t)||u|| \, |v|,$$

where $\phi_3 \in L^2(0, \infty)$.

(iv) For all $u \in V$,

$$\left|\frac{da_0}{dt}\right| = |a_0'(t; u, u)| \leqslant \phi_4(t) \, ||u||^2,$$

where $\phi_4 \in L^1(0, \infty)$.

Letting μ and σ be arbitrary positive numbers, we set

$$b(t) = \exp\left\{ - \int_0^t \left[\mu\phi_2^2(\eta) + \frac{1}{m}\, \phi_4(\eta) + \sigma\phi_3^2(\eta) \right] d\eta \right\}, \tag{2.3}$$

where m is the constant appearing in (ii), and $\phi_2 \in L^2(0, \infty)$. We also put

$$k(t) = \int_0^t b^{-1}(\eta) \int_0^\eta b(\xi)\, \phi_1^2(\xi) d\xi d\eta, \tag{2.4}$$

where $\phi_1 \in L^{2, \mathrm{loc}}(0, \infty)$.

Whenever it is necessary, we may assume that $\phi_j(t) > 0$ in $[0, \infty)$; for instance, in the case of $\phi_4(t)$, we may always replace it by $\phi_4(t) + |\phi_4(t)| + (1 + t^2)^{-1}$.

Assumption II. There exists $\beta > 0$ such that

$$\phi_1 e^{-\beta k} \in L^2(0, \infty).$$

Finally, we make the

DEFINITION. *$Q(k)$ is the set of all functions $t \to u(t)$ from $t \geqslant 0$ into $\mathfrak{D}[A(t)]$ such that u is continuously differentiable in the topology of V, and*

$$\lim_{t\to\infty} e^{\beta k(t)}||u(t)|| = 0, \qquad \forall \beta > 0. \tag{2.5}$$

3. The basic inequality.

LEMMA. *Let I and II be satisfied. If $u \in Q(k)$, and has its support in the interval $\epsilon \leqslant t < \infty$ $(\epsilon > 0)$, then*

$$\beta \int_0^\infty be^{2\beta k}\phi_1^2|u|^2dt + m\mu \int_0^\infty be^{2\beta k}\phi_2^2||u||^2dt \leqslant \int_0^\infty be^{2\beta k}\left|\frac{du}{dt} + (A+\rho)u\right|^2 dt. \tag{3.1}$$

Proof. Because $u \in Q(k)$ and I and II are satisfied, the two integrals on the left in (3.1) are finite.

We may assume that $u(t)$ has compact support in the interval $0 < t < \infty$. For in the general case we can approximate $u(t)$ by the sequence $u_j(t) = \zeta_j(t)u(t)$, $\zeta_j(t)$ being a C^∞ scalar-valued function, equal to one for $t \leqslant j$, equal to zero for $t \geqslant j + 1$, and $0 \leqslant \zeta_j(t) \leqslant 1$ in between. As $j \to \infty$, the inequality (3.1) for $u_j(t)$ goes over into (3.1) for $u(t)$.

If we put $u(t) = z(t)e^{-\beta k(t)}$, then $z \in Q(k)$ and

$$\int_0^\infty be^{2\beta k}\left|\frac{du}{dt} + (A+\rho)u\right|^2 dt = \int_0^\infty b|z'(t) - \beta k'(t)z(t) + (A+\rho)z(t)|^2dt.$$

Putting

$$M = \int_0^\infty b|z'|^2dt$$

and

$$\begin{aligned} N &= 2\operatorname{Re}\int_0^\infty b\beta k'\langle z, z'\rangle dt + 2\operatorname{Re}\int_0^\infty b\langle (A+\rho)z, z'\rangle dt \\ &= N_1 + N_2, \end{aligned}$$

we see that the right side of (3.1) dominates $M + N_1 + N_2$.

Integrating N_1 by parts and taking into account the nature of the support of $z(t)$, we find that

$$N_1 = \beta\int_0^\infty |z|^2 \frac{d}{dt}(bk')dt.$$

But, according to (2.4), $(bk')' = b\phi_1^2$, and since $z = e^{\beta k}u$, N_1 becomes

$$N_1 = \beta\int_0^\infty be^{2\beta k}\phi_1^2|u|^2dt.$$

Also, in view of (2.2) and I, we have that

$$\begin{aligned}
N_2 &= 2\,\mathrm{Re}\int_0^\infty b\{a_0(t; z, z') + \rho\langle z, z'\rangle + a_1(t; z, z')\}dt \\
&= \int_0^\infty b\left\{\frac{d}{dt}[a_0(t, z, z) + \rho|z|^2]\right\}dt \\
&\qquad - \int_0^\infty ba_0'(t; z, z)dt + 2\,\mathrm{Re}\int_0^\infty ba_1(t; z, z')dt \\
&\geqslant - \int_0^\infty b'[a_0(t; z, z) + \rho|z|^2]dt \\
&\qquad - \int_0^\infty \phi_4 b||z||^2 dt - 2\int_0^\infty \phi_3 b|z'|\,||z||dt.
\end{aligned}$$

Therefore, by (2.3),

$$\begin{aligned}
N_2 \geqslant \int_0^\infty &\left(\mu\phi_2^2 + \frac{1}{m}\phi_4 + \sigma\phi_3^2\right)bm||z||^2 dt \\
&- \int_0^\infty b|z'|^2 dt - \int_0^\infty \phi_4 b||z||^2 dt - \int_0^\infty \phi_3^2 b||z||^2 dt.
\end{aligned}$$

Finally, taking $\sigma = m^{-1}$, we conclude that

$$M + N_1 + N_2 \geqslant \beta\int_0^\infty be^{2\beta k}\phi_1^2|u|^2 dt + \mu m\int_0^\infty \phi_2^2||z||^2 dt,$$

which implies (3.1).

4. The main theorems.

THEOREM 1. *Let I and II be satisfied, and let $F \in Q(k)$. If*

$$\left|\frac{dF}{dt} + A(t)F\right| \leqslant \phi_1(t)|F| + \phi_2(t)||F||, \qquad (t > 0),$$

then $F(t) = 0, \forall t \geqslant 0$.

Proof. First, we put $F(t) = e^{\rho t}f(t)$, where ρ is the constant appearing in (ii). Then f satisfies the inequality

$$\left|\frac{df}{dt} + (A + \rho)f\right| \leqslant \phi_1(t)|f| + \phi_2(t)||f||, \qquad (t > 0). \tag{4.1}$$

Thus it suffices to prove that $f(t) = 0, \forall t \geqslant 0$.

Let $g(t)$, $t \geqslant 0$, be a C^∞ scalar-valued function, equal to zero for $0 \leqslant t \leqslant \epsilon$, equal to one for $t \geqslant 2\epsilon$, and $0 \leqslant g(t) \leqslant 1$ in between. Set $u(t) = g(t)f(t)$. Then u satisfies all of the conditions of the lemma, and we obtain from (4.1) that

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$$\beta \int_{2\epsilon}^{\infty} be^{2\beta k}\phi_1^2|f|^2 dt + \mu m \int_{2\epsilon}^{\infty} be^{2\beta k}\phi_2^2||f||^2 dt$$
$$\leqslant \int_{\epsilon}^{2\epsilon} be^{2\beta k}\left|\frac{du}{dt} + (A+\rho)u\right|^2 dt + \int_{2\epsilon}^{\infty} be^{2\beta k}\left|\frac{df}{dt} + (A+\rho)f\right|^2 dt$$
$$\leqslant \int_{\epsilon}^{2\epsilon} be^{2\beta k}\left|\frac{du}{dt} + (A+\rho)u\right|^2 dt + 2\int_{2\epsilon}^{\infty} be^{2\beta k}\phi_1^2|f|^2 dt$$
$$+ 2\int_{2\epsilon}^{\infty} be^{2\beta k}\phi_2^2||f||^2 dt.$$

Taking $\mu = 2m^{-1}$ and $\beta > 2$, we obtain the inequality

$$(\beta - 2)\int_{2\epsilon}^{\infty} be^{2\beta k}\phi_1^2|f|^2 dt \leqslant \int_{\epsilon}^{2\epsilon} be^{2\beta k}\left|\frac{du}{dt} + (A+\rho)u\right|^2 dt.$$

Since $k(t)$ is non-decreasing, it follows that

$$(\beta - 2)\int_{2\epsilon}^{\infty} b\phi_1^2|f|^2 dt \leqslant \int_{\epsilon}^{2\epsilon} b\left|\frac{du}{dt} + (A+\rho)u\right|^2 dt.$$

Letting $\beta \to \infty$, we conclude that

$$\int_{2\epsilon}^{\infty} b\phi_1^2|f|^2 dt = 0, \qquad \forall\epsilon > 0.$$

Since we may assume that $\phi_1(t) > 0$ in $[0, \infty)$, this implies that $f(t) = 0$, $\forall t \geqslant 0$.

THEOREM 2. *Let I be satisfied, and let* $\phi_1 \in L^2(0, \infty)$. *If* $F \in Q(k)$, *for* $k(t) = t$, *and*

$$\left|\frac{dF}{dt} + A(t)F\right| \leqslant \phi_1(t)|F| + \phi_2(t)||F||, \qquad (t > 0), \tag{4.2}$$

then $F(t) = 0$, $\forall\, t \geqslant 0$.

Proof. Since $\phi_1 \in L^2(0, \infty)$, it follows that I is satisfied. This theorem will follow from Theorem 1 if we can show that $F \in Q(k)$, for $k(t) = t$, implies (2.5). From (2.3) we have that $b \leqslant 1$ and $b^{-1} \leqslant r < \infty$. Thus, according to (2.4),

$$k(t) \leqslant \beta r \int_0^t \int_0^\eta \phi_1^2(\xi)d\xi d\eta = \beta r \int_0^t (t-\xi)\phi_1^2(\xi)d\xi$$
$$= 0(\beta t),$$

which shows that (2.5) is satisfied.

THEOREM 3. *Let I be satisfied, and let* $\phi_1{}^2(t) = 0(t^{d-2})$, $d > 1$. *If* $F \in Q(k)$, *for* $k(t) = t^d$, *and satisfies* (4.2), *then* $F(t) = 0, \forall\, t \geqslant 0$.

Proof. The proof is so similar to that of Theorem 2 that it will be omitted (cf. **1**).

5. An example. In this section we specialize the theorems of §4 to the case considered by Protter.

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Let $H = L^2(\Omega)$, Ω being an open subset of R^n, and let V be the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$||u||^2 = \int_\Omega \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 dx.$$

Then V is dense in H, and the injection mapping from V into H is continuous.

We set

$$a_0(t; u, v) = \int_\Omega \sum_{i,j=1}^n \overline{\frac{\partial v}{\partial x_i}} a_{ij}(x, t) \frac{\partial u}{\partial x_j} dx,$$

where $a_{ij}(x, t) = a_{ji}(x, t) \in C^1(\bar{\Omega} \times [0, \infty))$, and

$$a_1(t; u, v) = \int_\Omega \sum_{i=1}^n \bar{v}\, b_i(x, t) \frac{\partial u}{\partial x_i} dx,$$

where $b_i(x, t) \in C^0(\bar{\Omega} \times [0, \infty))$. The operator A generated by the form $a = a_0 + a_1$ is

$$A(t)u = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x, t) \frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^n b_i(x, t) \frac{\partial u}{\partial x_i}.$$

In order that I be satisfied, it is necessary to assume that $A(t)$ is uniformly elliptic and that

$$\phi_3(t) = \sup_{i, x \in \bar{\Omega}} |b_i(x, t)| \in L^2(0, \infty)$$

and

$$\phi_4(t) = \sup_{i, j, x \in \bar{\Omega}} \left|\frac{\partial a_{ij}}{\partial t}(x, t)\right| \in L^1(0, \infty).$$

Theorems 1, 2, and 3 now apply to this example.

In **(1)** it is assumed that $\phi_3(t) = o(t^{-1/2})$ and $\phi_4(t) = o(t^{-1})$, and in the application of Theorem 2 to this example, it is assumed that $\phi_i(t) = o(t^{-1})$ $(i = 1, 2)$. Lemma 5 in **(1)** is stated incorrectly; instead of $A_0(t) = 0(t^{-1})$, it should read $A_0(t) = o(t^{-1})$.

It can be shown that our results apply to inequalities for parabolic operators of arbitrary order $2s$ in which no derivatives of order p, where $s < p \leqslant 2s - 1$, appear.

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