ON THE SPACES OF CERTAIN CLASSES OF ENTIRE FUNCTIONS

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1. Introduction

Let ρ and d be two positive numbers. We shall be concerned in this paper with two classes of entire functions, namely:

(i) the class $C(\rho, d)$ of all entire functions of order ρ and type not exceeding d — we note that this class includes all entire functions of order less than ρ; and

(ii) the class $C(\rho)$ of all entire functions of order not greater than ρ.

A metric topology for $C(\rho, d)$ has been introduced by Ganapathy Iyer in [**10**, Part II]. The topological space thus obtained will be called $\Gamma(\rho, d)$. The relevant results of his are catalogued in § 2 below. In Part I of this paper we consider the question of proper bases in $\Gamma(\rho, d)$. The idea of a proper basis was first proposed by Ganapathy Iyer [8, p. 880] in his study of the space of all entire functions. Arsove ([**1**, p. 266]; [**2**, p. 45]) has modified the definition of a proper basis so as to make it possess more characteristic properties of the "natural" basis $\{e_n\}$, where $e_n \equiv z^n$, $n = 0, 1, 2 \cdots$. Using a theorem of Ganapathy Iyer [**8**, Theorem 6] he has obtained a characterisation [**1**, Theorem 4] of the linearly homeomorphic images of the space (of all entire functions) into itself as those closed sub-spaces admitting proper bases. In [**2**] he obtains further important information on proper bases and in [**3**] he extends the basis theory to the space of functions analytic at the origin. The aim of Part I of this paper is to obtain analogous results for $\Gamma(\rho, d)$. We adapt, throughout, Arsove's definition of proper bases and his terminology for linear combinations interpreted in the infinite series sense [**2**], [**3**] as explained in § 4 below.

Theorem 1 below is the analogue of [**8**, Theorem 6]. We obtain therein a necessary and sufficient condition in order that there may exist a continuous linear mapping of the space into itself carrying the fundamental basis $\{e_n\}$ into a given sequence $\{\alpha_n\}$ of entire functions belonging to the space. This enables us to establish as in [**2**, p. 45], a characterisation of proper bases in terms of conditions on their growth and, also, the same interrelationship between automorphisms and proper bases as in [2, Theorem 3]. In § 6, we take up the question of determining sufficient conditions under which functions of the form $\alpha_n = z^n(1+\lambda_n(z))$, where $\lambda_n(0) = 0$ and the λ_n's belong to the space, constitute proper bases. Such bases have been called Pincherle bases by Arsove [2]. Ganapathy Iyer's answer [7, Theorem 8] to this question for his space of all entire functions has been improved by Arsove [2, Theorem 4]. The same problem for $\Gamma(\rho, d)$ is answered by Theorem of this paper. We also set up, as an immediate corollary to Theorem 4, a general method of constructing proper Pincherle bases from certain entire functions belonging to the space.

In Part II, we first introduce a topology for $C(\rho)$ and make it into a metric space $\Gamma(\rho)$ on the lines of Ganapathy-Iyer's work [10, Part II] for $C(\rho, d)$. The proofs being exactly similar, only the results are detailed. We then discuss the question of proper bases in $\Gamma(\rho)$ and obtain results analogous to those already obtained in Part I.

In Part III, we consider two algebraic structures on $\Gamma(\rho)$: namely, $\Gamma_N(\rho)$, where multiplication is the natural multiplication $\alpha\beta$ and $\Gamma_C(\rho)$, under termwise multiplication of the coefficient sequences. We note that both these are topological algebras and we obtain characterisations of the general automorphisms — linearly homeomorphic mappings of the space onto itself with preservation of multiplication — in these algebras.

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PART I: PROPER BASES IN $\Gamma(\varrho, d)$

2. The Space $\Gamma(\varrho, d)$

In this section we detail Ganapathy Iyer's results pertaining to the topology of $\Gamma(\rho, d)$. The proofs are contained in his paper [10, Part II].

Let $\alpha = \alpha(z) = \Sigma a_n e_n$, where $e_n \equiv z^n$, $n = 0, 1, 2, \cdots$, be in $C(\rho, d)$. For each $\delta > 0$, the convergent series [1]

$$||\alpha; d+\delta|| = |a_0| + \sum_{n=1}^{\infty} |a_n| \left\{ \frac{n}{(d+\delta)e\rho} \right\}^{n/\rho}$$

defines a norm on $C(\rho, d)$. Denote the corresponding normed space by $\Gamma(\rho, d, \delta)$. As δ decreases, the norm increases and the topology becomes weaker (in the sense of Alexandroff-Hopf). The lattice product of these normed topologies on $C(\rho, d)$ for all $\delta > 0$ is denoted by $\Gamma(\rho, d)$. It is metri-

[1] It may be more specific to use the notation $||\alpha; \rho; d+\delta||$, instead of $||\alpha; d+\delta||$, but we shall stick to the latter for brevity. See Footnote 3.

sable with the metric $||\alpha - \beta||$ where

$$||\alpha|| = \sum_{p=1}^{\infty} \frac{1}{2^p} \frac{||\alpha; d+1/p||}{1+||\alpha; d+1/p||}.$$

Under this metric, $\Gamma(\rho, d)$ becomes a complete linear metric space. The sequence of partial sums of the series for α, converges to α in $\Gamma(\rho, d)$. Convergence in $\Gamma(\rho, d)$ is equivalent to uniform convergence relative to the function $\exp\{(d+\delta)|z|^\rho\}$, for each $\delta > 0$. In other words, the statement $\alpha_n \to \alpha$ in $\Gamma(\rho, d)$ implies and is implied by the following property in the language of classical analysis: for every $\delta > 0$, given a positive ε, it is possible to choose an integer $N(\varepsilon)$, independent of z, such that

$$|\alpha_n(z) - \alpha(z)| \leqq \varepsilon \exp\{(d+\delta)|z|^\rho\}, \quad \text{for} \quad n \geqq N(\varepsilon).$$

3. Continuous linear transformations of $\Gamma(\varrho, d)$ into itself

Let $T(\delta_1 \to \delta_2)$ stand for a continuous linear transformation from $\Gamma(\rho, d, \delta_1)$ into $\Gamma(\rho, d, \delta_2)$. The family of all such transformations for a fixed pair δ_1 and δ_2 may be denoted by $F(\delta_1 \to \delta_2)$. Naturally we denote a continuous linear transformation of $\Gamma(\rho, d)$ into itself by $T(0 \to 0)$ and the family of such transformations by $F(0 \to 0)$. The main result of this section can now be stated as

THEOREM 1. *A necessary and sufficient condition that there exists a* $T = T(0 \to 0)$ *with* $T(e_n) = \alpha_n$, $n = 0, 1, 2, \cdots$, *where* $\alpha_n \in \Gamma(\rho, d)$, *is that, for each* $\delta > 0$,

$$\limsup_{n\to\infty} \frac{||\alpha_n; d+\delta||^{1/n}}{n^{1/\rho}} < \frac{1}{(de\rho)^{1/\rho}}. \tag{3.1}$$

In order to prove this, we first establish two lemmas on linear operators — which are also of independent interest — corresponding to the analogous results of Ganapathy Iyer [7, Lemma 1] and [8, Theorem 5].

LEMMA 1. *If* $||\alpha|| \geqq k > 0$, *then we must have,*

$$||\alpha; d+\delta|| \geqq k/(2-k),$$

for some $\delta = \delta_0$, *where* $0 < \delta_0 \leqq 1$, *and therefore for all values of* $\delta \leqq \delta_0$.

PROOF. The sequence of norms $||\alpha; d+1/p||$ increases for increasing values of p. Choose p_0 such that

$$\sum_{p_0+1}^{\infty} \frac{1}{2^p} \frac{||\alpha; d+1/p||}{1+||\alpha; d+1/p||} \leqq k/2.$$

Then we have,

$$||\alpha|| \leqq \frac{||\alpha; d + 1/p_0||}{1 + ||\alpha; d + 1/p_0||} \left\{ \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{p_0}} \right\} + \frac{k}{2}$$
$$\leqq \frac{||\alpha; d + 1/p_0||}{1 + ||\alpha; d + 1/p_0||} + \frac{k}{2}$$

which gives the required result.

REMARK. A consequence of this lemma is that, if a series converges in $\Gamma(\rho, d, \delta)$ for each $\delta > 0$, then it converges in $\Gamma(\rho, d)$. We know that the converse is true, because $\Gamma(\rho, d)$ is weaker than all the $\Gamma(\rho, d, \delta)$'s. Thus convergence in $\Gamma(\rho, d)$ is equivalent to convergence in all of the $\Gamma(\rho, d, \delta)$'s. This property will often be used in the sequel without further mention.

LEMMA 2.

$$F(0 \to 0) = \prod_{\delta_2 > 0} \left\{ \sum_{\delta_1 > 0} F(\delta_1 \to \delta_2) \right\}.$$

In other words, if T be a linear transformation of $\Gamma(\rho, d)$ into itself, in order that it may also be continuous, it is necessary and sufficient that, to each $\delta_2 > 0$, there exists some $\delta_1 > 0$ such that $T \in F(\delta_1 \to \delta_2)$.

The proof is parallel to that of the corresponding theorem of Ganapathy Iyer [**8**, p. 879]. We only note that we make use of Lemma 1, a theorem of Banach [**4**, p. 54, Theorem 1] and the definition of the metric for $\Gamma(\rho, d)$.

REMARK. This Lemma is contained in a general result on locally convex linear topological spaces by Bourbaki [**5**, Chapter II, Prop. 9, Cor.]. But in conformity with the general trend of this paper we note the above proof based on the particular metric introduced.

PROOF OF THEOREM 1. $T \in F(0 \to 0)$ with $T(e_n) = \alpha_n$. It follows therefore from Lemma 2 that there exists a $\delta_1 = \delta_1(\delta) > 0$ such that $T \in F(\delta_1 \to \delta)$ for each $\delta > 0$. A theorem of Banach [**4**, p. 54, Theorem 1] now shows that there exists a $K = K(\delta)$ such that

$$||T(e_n); d + \delta|| \leqq K||e_n; d + \delta_1||,$$

which implies

$$||\alpha_n; d + \delta|| \leqq K \left\{ \frac{n}{(d + \delta_1)e\rho} \right\}^{n/\rho}$$

So

$$\frac{||\alpha_n; d + \delta||^{1/n}}{n^{1/\rho}} < \frac{1}{(de\rho)^{1/\rho}} \text{ for all large } n.$$

Conversely, let (α_n) satisfy (3.1). Let $\alpha = \Sigma a_n e_n \in \Gamma(\rho, d)$. This implies

$$\limsup_{n \to \infty} |a_n|^{1/n} n^{1/\rho} \leqq (de\rho)^{1/\rho}.$$

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So, given $\eta > 0$, we have, for $n \geqq n_0 = n_0(\eta)$

$$|a_n|^{1/n} n^{1/\rho} \leqq \{(d+\eta)e\rho\}^{1/\rho};$$

and, taking $\eta' > \eta$, we can find $n_0' = \eta_0'(\eta')$ from (3.1) such that, for $n \geqq n_0'$,

$$\frac{||\alpha_n; d+\delta||^{1/n}}{n^{1/\rho}} \leqq \frac{1}{\{(d+\eta')e\rho\}^{1/\rho}}.$$

So, if $n \geqq \max(n_0, n_0')$,

$$|a_n| \, ||\alpha_n; d+\delta|| \leqq \left\{ \frac{(d+\eta)e\rho}{(d+\eta')e\rho} \right\}^{n/\rho};$$

and, since $\eta' > \eta$, the series of norms $\Sigma |a_n| \, ||\alpha_n; d+\delta||$ is convergent, for each $\delta > 0$. Therefore, $\Sigma a_n \alpha_n(z)$ converges to an element in $\Gamma(\rho, d)$. Define $T(\alpha) = \Sigma a_n \alpha_n$, for $\alpha \in \Gamma(\rho, d)$. Then $T(e_n) = \alpha_n$. We have only to prove the continuity of T. Given $\delta > 0$, $\delta' > 0$, we have, for all large n depending on δ and δ',

$$\frac{||\alpha_n; d+\delta||^{1/n}}{n^{1/\rho}} \leqq \frac{1}{\{(d+\delta')e\rho\}^{1/\rho}},$$

and so

$$||\alpha_n; d+\delta|| \leqq K \left\{ \frac{n}{(d+\delta')e\rho} \right\}^{n/\rho}, \quad \text{with} \quad K = K(\delta)$$

and the inequality is true for all $n > 0$. Now

$$\begin{aligned} ||T(\alpha); d+\delta|| &\leqq \Sigma |a_r| \, ||\alpha_n; d+\delta|| \\ &\leqq K\Sigma |a_n| \left\{ \frac{n}{(d+\delta')e\rho} \right\}^{n/\rho} \\ &= K \, ||\alpha; d+\delta'||. \end{aligned}$$

Hence, by the theorem of Banach already referred to, $T \in F(\delta' \to \delta)$ for all $\delta > 0$ and $\delta' > 0$ and this proves, in virtue of Lemma 2 that $T \in F(0 \to 0)$.

4. Proper bases and their characterisation

Let $\{\alpha_n\}$, $n = 0, 1, 2, \cdots$ be a sequence of entire functions in $\Gamma(\rho, d)$. If $\sum_{n=0}^{\infty} c_n \alpha_n = 0$ implies $c_n = 0$ for all sequences $\{c_n\}$ of complex numbers for which $\Sigma c_n \alpha_n$ converges in $\Gamma(\rho, d)$, the sequence $\{\alpha_n\}$ will be called linearly independent. We shall say that $\{\alpha_n\}$ spans a subspace Γ_0 of $\Gamma(\rho, d)$ provided that Γ_0 consists of all linear combinations $\sum_{n=0}^{\infty} c_n \alpha_n$ for which $\Sigma c_n \alpha_n$ converges in $\Gamma(\rho, d)$. A sequence $\{\alpha_n\}$ which is linearly independent and spans a closed subspace Γ_0 of $\Gamma(\rho, d)$ will be said to be a basis in Γ_0. Clearly $\{e_n\}$ is a basis in $\Gamma(\rho, d)$. We now define a proper basis as a basis $\{\alpha_n\}$ in a subspace Γ_0 of $\Gamma(\rho, d)$, which possesses, in addition, the property:

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For all sequences $\{c_n\}$ of complex numbers, $\Sigma c_n \alpha_n$ converges in $\Gamma(\rho, d)$ if and only if $\Sigma c_n e_n$ converges in $\Gamma(\rho, d)$.

We know that "$\Sigma c_n e_n$ converges in $\Gamma(\rho, d)$" is equivalent to saying

$$\limsup_{n\to\infty} |c_n|^{1/n} n^{1/\rho} \leqq (de\rho)^{1/\rho}. \tag{4.1}$$

We shall now characterize proper bases in terms of growth conditions on $\{\alpha_n\}$ and for that purpose we first prove two lemmas, using arguments parallel to those of Arsove ([**2**], [**3**]).

LEMMA 3. *The following three properties are equivalent:*

(A) $$\limsup_{n\to\infty} \frac{||\alpha_n; d+\delta||^{1/n}}{n^{1/\rho}} < 1/(de\rho)^{1/\rho}, \quad \text{for each } \delta > 0.$$

(B) *For all sequences $\{c_n\}$ of complex numbers "$\Sigma c_n e_n$ converges in $\Gamma(\rho, d)$"* implies *"$\Sigma c_n \alpha_n$ converges in $\Gamma(\rho, d)$".*

(C) *For all sequences $\{c_n\}$ of complex numbers, "$\Sigma c_n e_n$ converges in $\Gamma(\rho, d)$" implies "$c_n \alpha_n$ tends to zero in $\Gamma(\rho, d)$".*

PROOF. It is clear that (B) $\Rightarrow$ (C). We have already proved, in the course of the sufficiency part of the proof of Theorem 1, that (A) $\Rightarrow$ (B). We have therefore only to show that (C) $\Rightarrow$ (A).

To prove this, we assume that (C) is true and (A) is not. The latter means that, for a particular δ, say δ', there exists a sequence $\{n_k\}$ of positive integers, such that

$$\frac{||\alpha_n; d+\delta'||^{1/n}}{n^{1/\rho}} \geqq \frac{1}{\{(d+1/k)e\rho\}^{1/\rho}}, \text{ for all } n = n_k. \tag{4.2}$$

We shall define a sequence $\{c_n\}$ by

$$c_n = \begin{cases} 1/||\alpha_n; d+\delta'|| & \text{when } n = n_k, \\ 0 & \text{when } n \neq n_k. \end{cases} \tag{4.3}$$

So we have,

$$|c_{n_k}|^{1/n_k} n_k^{1/\rho} = \frac{n_k^{1/\rho}}{||\alpha_{n_k}; d+\delta'||^{1/n_k}} \leqq \{(d+1/k)e\rho\}^{1/\rho}$$

in virtue of (4.2). Therefore,

$$\limsup_{k\to\infty} |c_{n_k}|^{1/n_k} n_k^{1/\rho} \leqq (de\rho)^{1/\rho}.$$

But $c_n = 0$ when $n \neq n_k$ and, consequently,

$$\limsup_{n\to\infty} |c_n|^{1/n} n^{1/\rho} \leqq (de\rho)^{1/\rho}.$$

Thus $\{c_n\}$ defined by (4.3) satisfies (4.1). So by the hypothesis (C), $c_n \alpha_n$

should tend to zero in $\Gamma(\rho, d)$. On the contrary, we have, for all $n = n_k$,

$$\|c_n \alpha_n; d + \delta'\| = |c_{n_k}| \, \|\alpha_{n_k}; d + \delta'\| = 1.$$

So $c_n \alpha_n$ does not tend to zero in $\Gamma(\rho, d, \delta')$ and this contradiction establishes that (C) $\Rightarrow$ (A).

LEMMA 4. *The following three properties are equivalent*:

$$(\alpha)\,^2 \qquad \lim_{\delta\to\infty} \left\{ \liminf_{n\to\infty} \frac{\|\alpha_n; d + \delta\|^{1/n}}{n^{1/\rho}} \right\} \geqq \frac{1}{(de\rho)^{1/\rho}}.$$

(β) *For all sequences $\{c_n\}$ of complex numbers, "$\Sigma c_n \alpha_n$ converges in $\Gamma(\rho, d)$" implies "$\Sigma c_n e_n$ converges in $\Gamma(\rho, d)$".*

(γ) *For all sequences $\{c_n\}$ of complex numbers, "$c_n \alpha_n$ tends to zero in $\Gamma(\rho, d)$" implies "$\Sigma c_n e_n$ converges in $\Gamma(\rho, d)$.*

PROOF. It is clear that $(\gamma) \Rightarrow (\beta)$. We shall prove that $(\beta) \Rightarrow (\alpha)$ and $(\alpha) \Rightarrow (\gamma)$.

First, we suppose that (β) is true and (α) is not. The latter means that

$$\lim_{\delta\to 0} \left\{ \liminf_{n\to\infty} \frac{\|\alpha_n; d + \delta\|^{1/n}}{n^{1/\rho}} \right\} < \frac{1}{(de\rho)^{1/\rho}}. \tag{4.4}$$

Since $\|\alpha_n; d + \delta\|$ increases as δ decreases, it follows that, for each $\delta > 0$,

$$\liminf_{n\to\infty} \frac{\|\alpha_n; d + \delta\|^{1/n}}{n^{1/\rho}} < \frac{1}{(de\rho)^{1/\rho}}. \tag{4.5}$$

If now η be a fixed small positive number, we can find, for each $r > 0$, in virtue of (4.4) and (4.5), a positive number n_r such that, we have for all r, $n_{r+1} > n_r$, and,

$$\frac{\|\alpha_{n_r}; d + 1/r\|^{1/n_r}}{n_r^{1/\rho}} \leqq \frac{1}{\{(d + \eta)e\rho\}^{1/\rho}}. \tag{4.6}$$

We choose a positive number $\eta_1 < \eta$ and define a sequence $\{c_n\}$ by

$$c_n = \begin{cases} \left\{\dfrac{(d + \eta_1)e\rho}{n}\right\}^{n/\rho} & \text{when } n = n_r, \\ 0 & \text{when } n \neq n_r. \end{cases} \tag{4.7}$$

Then, for any $\delta > 0$,

$$\sum |c_n| \, \|\alpha_n; d + \delta\| = \sum |c_{n_r}| \, \|\alpha_{n_r}; d + \delta\|.$$

Given $\delta > 0$, omit from the above series those terms (finite in number) which correspond to those n_r for which $1/r$ is greater than δ. The remainder

[2] Note that the symbol α, without parentheses, stands for $\alpha(z)$, a member of the class of entire functions in question.

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of the series is dominated by $\Sigma|c_{n_r}|\,||\alpha_{n_r}; d+1/r||$. Now, in virtue of (4.6) and (4.7), we see that

$$\sum |c_{n_r}|\,||\alpha_{n_r}; d+1/r|| \text{ is dominated by}$$

$$\sum\left\{\frac{(d+\eta_1)e\rho}{n_r}\right\}^{n_r/\rho}\cdot\left\{\frac{n_r}{(d+\eta)e\rho}\right\}^{n_r/\rho}. \quad \text{This is}$$

$$=\sum\left\{\frac{d+\eta_1}{d+\eta}\right\}^{n_r/\rho},$$

which is a convergent series, since $\eta_1<\eta$. Thus $\{c_n\}$ as defined by (4.7), is a sequence for which $\Sigma c_n\alpha_n$ converges in $\Gamma(\rho, d, \delta)$ for each $\delta>0$, and therefore converges in $\Gamma(\rho, d)$. So, by our hypothesis (β), we must have (4.1) for $\{c_n\}$. On the contrary, we have,

$$\begin{aligned}&\limsup_{n\to\infty}|c_n|^{1/n}n^{1/\rho}\\ &=\limsup_{r\to\infty}\left\{\frac{(d+\eta_1)e\rho}{n_r}\right\}^{1/\rho}\times n_r^{1/\rho}\\ &=\{(d+\eta_1)e\rho\}^{1/\rho}>(de\rho)^{1/\rho}.\end{aligned}$$

This contradiction proves that $(\beta)\Rightarrow(\alpha)$.

To prove $(\alpha)\Rightarrow(\gamma)$ we assume that (α) is true and consider sequences $\{c_n\}$ of complex numbers for which $c_n\alpha_n\to 0$ in $\Gamma(\rho, d)$. We must now show that every such sequence $\{c_n\}$ satisfies (4.1). If one such sequence $\{c_n\}$ does not satisfy (4.1), we shall then obtain a contradiction, using the hypothesis (α).

Let there be a sequence $\{c'_n\}$ for which

$$c'_n\alpha_n\to 0 \qquad \text{in } \Gamma(\rho, d); \tag{4.8}$$

and

$$\limsup_{n\to\infty}|c'_n|^{1/n}n^{1/\rho}>(de\rho)^{1/\rho}. \tag{4.9}$$

The latter means that there exists a sequence (n_k) of positive integers and a number $\lambda>0$ such that,

$$|c'_n|^{1/n}n^{1/\rho}\geqq\{(d+\lambda)e\rho\}^{1/\rho} \quad \text{for all } n=n_k. \tag{4.10}$$

Choose a positive number η such that $\lambda>(3\eta)/2$. The validity of (α) now implies that, we can find a $\delta=\delta(\eta)$, so that

$$\liminf_{n\to\infty}\frac{||\alpha_n; d+\delta||^{1/n}}{n^{1/\rho}}\geqq\frac{1}{\{(d+\eta)e\rho\}^{1/\rho}}.$$

This means that there exists an $N=N(\eta)$ such that

$$(4.11) \qquad \frac{\|\alpha_n; d+\delta\|^{1/n}}{n^{1/\rho}} \geqq \frac{1}{\{(d+3\eta/2)e\rho\}^{1/\rho}} \quad \text{for all } n \geqq N.$$

Therefore,

$$\begin{aligned}
&\max \|c'_n\alpha_n; d+\delta\| \\
&= \max |c'_n|\, \|\alpha_n; d+\delta\| \\
&\geqq \max |c'_{n_k}|\, \|\alpha_{n_k}; d+\delta\| \\
&\geqq \left\{\frac{(d+\lambda)e\rho}{n_k}\right\}^{n_k/\rho} \times \left\{\frac{n_k}{(d+3\eta/2)e\rho}\right\}^{n_k/\rho}
\end{aligned}$$

in virtue of (4.10) and (4.11). This last expression is greater than 1, since $\lambda > 3\eta/2$. $c'_n\alpha_n$ does not therefore tend to zero in $\Gamma(\rho, d, \delta)$ at least for this δ and this contradicts (4.8). We have thus proved that $(\alpha) \Rightarrow (\gamma)$.

Our definition of proper basis is that (B) and (β) are the conditions to be satisfied by a basis $\{\alpha_n\}$ to be proper. So we have, combining Lemmas 3 and 4, a characterisation of proper bases in the form of

THEOREM 2. *A basis $\{\alpha_n\}$ in a closed subspace Γ_0 of $\Gamma(\rho, d)$ is proper if and only if conditions (A) and (α) hold.*

5. Proper bases and Linear homeomorphisms in $\Gamma(\rho, d)$

One aspect of the importance of proper bases in the study of $\Gamma(\rho, d)$ is that it enables us to characterize linear homeomorphisms in $\Gamma(\rho, d)$ in terms of proper bases as in

THEOREM 3. *If T is a linear homeomorphic mapping of $\Gamma(\rho, d)$ into itself, then $T(e_n)$ is a proper basis in some closed subspace Γ_0 of $\Gamma(\rho, d)$. Conversely, if $\{\alpha_n\}$ is a proper basis in a closed subspace Γ_0 of $\Gamma(\rho, d)$, then there exists a linear homeomorphic mapping T of $\Gamma(\rho, d)$ onto Γ_0 such that $T(e_n) = \alpha_n$, $n = 0, 1, 2, \cdots$.*

The argument is essentially the same as in [**3**, p. 241, Theorem 2] of Arsove, except for the fact that our condition (A) plays the role of the condition (α) [**3**, p. 237] of his paper. We note also that a suitable combination of mappings leads to the following interrelationship between proper bases and automorphisms — linear homeomorphic mappings of $\Gamma(\rho, d)$ onto itself: If $\{\alpha^1_n\}$ and $\{\alpha^2_n\}$ are proper bases in $\Gamma(\rho, d)$, there exists an automorphism T of $\Gamma(\rho, d)$ such that $T(\alpha^1_n) = \alpha^2_n$, $n = 0, 1, 2, \cdots$ and, conversely, if T is an automorphism of $\Gamma(\rho, d)$ and $\{\alpha^1_n\}$ is a proper basis in $\Gamma(\rho, d)$, then $\{\alpha^2_n\}$, where $\alpha^2_n = T\alpha^1_n$, $n = 0, 1, 2, \cdots$ is also a proper basis in $\Gamma(\rho, d)$.

6. Proper Pincherle bases in $\Gamma(\varrho, d)$

A Pincherle basis in $\Gamma(\rho, d)$ is a basis $\{\alpha_n\}$ in $\Gamma(\rho, d)$ of the form

$$\alpha_n(z) = z^n\{1 + \lambda_n(z)\}, \tag{6.1}$$

where $\lambda_n(0) = 0$. Obviously $\lambda_n(z)$ is also in $\Gamma(\rho, d)$. So we may write

$$\lambda_n(z) = \sum_{k=0}^{\infty} h_{nk} z^k, \qquad n = 0, 1, 2, \cdots, \tag{6.2}$$

with each $h_{0k} = 0$, where, for each n,

$$\limsup_{k\to\infty} |h_{nk}|^{1/k} k^{1/\rho} \leqq (de\rho)^{1/\rho}.$$

It is easy to see that, since

$$||\alpha_n; d + \delta|| \geqq \left\{\frac{n}{(d+\delta)e\rho}\right\}^{n/\rho}, \quad \text{for each } \delta > 0,$$

$\{\alpha_n\}$ satisfies the condition (α) without any further hypothesis, and so, from Theorem 2, for a Pincherle basis to be proper, it is necessary and sufficient that it satisfies condition (A). Given functions of the Pincerle form, we shall now obtain sufficient conditions on the growth of $\lambda_n(z)$ in order that $\{\alpha_n\}$ form a proper basis. This result is contained in

THEOREM 4. *If $\{\alpha_n\}$ as defined in* (6.1) *and* (6.2} *satisfies,*

$$\limsup_{(n+k)\to\infty} |h_{nk}|^{1/(n+k)} (n+k)^{1/\rho} \leqq (de\rho)^{1/\rho}, \tag{6.3}$$

then it constitutes a proper basis in $\Gamma(\rho, d)$.

PROOF. First we note that $\{\alpha_n\}$ satisfies (A) and therefore, if it is a basis in $\Gamma(\rho, d)$, it is also a proper basis. To see this, we have, for each $\delta' > 0$, in virtue of the hypothesis (6.3),

$$|h_{nk}| \leqq \left\{\frac{(d+\delta')e\rho}{n+k}\right\}^{(n+k)/\rho} \quad \text{for all } (n+k) \geqq N, \tag{6.4}$$

where $N = N(\delta')$ is independent of n and k. So, for each $\delta > 0$, and for a fixed n,

$$\begin{aligned} ||\alpha_n; d+\delta|| &= \left\{\frac{n}{(d+\delta)e\rho}\right\}^{n/\rho} + \sum_{k=1}^{\infty} |h_{nk}| \left\{\frac{n+k}{(d+\delta)e\rho}\right\}^{(n+k)/\rho} \leqq \left\{\frac{n}{(d+\delta)e\rho}\right\}^{n/\rho} \\ &\quad + \sum_{\substack{k \\ (n+k)<N}} |h_{nk}| \left\{\frac{n+k}{(d+\delta)e\rho}\right\}^{(n+k)/\rho} + \sum_{\substack{k \\ (n+k)\geqq N}} \left\{\frac{(d+\delta')e\rho}{(d+\delta)e\rho}\right\}^{(n+k)/\rho}, \end{aligned}$$

for some positive $\delta' < \delta$.

The last sum on the right being the sum of a convergent series, we have, for all $n \geqq N$,

$$||\alpha_n; d+\delta|| \leqq \left\{\frac{n}{(d+\delta)e\rho}\right\}^{n/\rho} + \mu,$$

for each $\delta > 0$, μ being a finite constant depending only on d, δ', δ, ρ. It follows that

$$\limsup_{n\to\infty} \frac{||\alpha_n; d+\delta||^{1/n}}{n^{1/\rho}} < \frac{1}{(de\rho)^{1/\rho}} \quad \text{for each } \delta > 0.$$

This completes the proof that $\{\alpha_n\}$ satisfies condition (A). So to complete the proof, it remains to show that $\{\alpha_n\}$ is a basis in $\Gamma(\rho, d)$. But the α_n's are clearly linearly independent and so it is enough to show that $\{\alpha_n\}$ spans $\Gamma(\rho, d)$.

Let $f(z) = \Sigma a_n e_n \in \Gamma(\rho, d)$. Form the equations

$$a_0 = c_0; \quad a_n = c_n + \sum_{k=1}^{n} c_{n-k} h_{n-k,k}. \tag{6.5}$$

These equations determine c_n uniquely in terms of the a_n's and yield

$$f(z) = \sum a_n e_n = \sum c_n \alpha_n,$$

provided we can justify the step by showing that $\Sigma |c_n|\ ||\alpha_n; d+\delta||$ is convergent for each $\delta > 0$. This will be the purpose of the following argument.

Fix $\delta > 0$ and write $|||f|||$ to denote $||f; d+\delta||$. Putting $\beta_n(z) = z^n \lambda_n(z)$, $n = 1, 2, \cdots$, it is clear that the convergence of $\sum_{n=1}^{\infty} |c_n|\ |||\alpha_n|||$ will follow from that of

$$\sum_{n=1}^{\infty} |c_n|\ |||z^n||| + \sum_{n=1}^{\infty} |c_n|\ |||\beta_n|||.$$

Since

$$|c_n| \leqq |a_n| + \sum_{k=1}^{n} |c_{n-k}|\ |h_{n-k,k}|, \tag{6.6}$$

we see that the series $\sum_{n=1}^{\infty} |c_n|\ |||z^n|||$ is dominated by

$$\sum_{n=1}^{\infty} |a_n|\ |||z^n||| + \sum_{n=1}^{\infty} \left\{|||z^n||| \sum_{k=1}^{\infty} |c_{n-k}|\ |h_{n-k,k}|\right\},$$

which is equal to

$$\sum_{n=1}^{\infty} |a_n|\ |||z^n||| + \sum_{n=1}^{\infty} \left\{|c_n| \sum_{k=n+1}^{\infty} |h_{n,k-n}|\ |||z^k|||\right\}$$
$$= \sum_{n=1}^{\infty} |a_n|\ |||z^n||| + \sum_{n=1}^{\infty} |c_n|\ |||\beta_n|||.$$

Since $\Sigma a_n z^n \in \Gamma(\rho, d)$, the above shows that, for the required convergence of $\Sigma |c_n|\ |||\alpha_n|||$, we need only prove the convergence of $\Sigma |c_n|\ |||\beta_n|||$.

Now choose a $\delta' < \delta$ and two positive numbers N' and N'' such that

$$(6.7) \qquad |a_n| \leqq \left\{\frac{(d+\delta')e\rho}{n}\right\}^{n/\rho} \quad \text{for all } n \geqq N' = N'(\delta')$$

and

$$(6.8) \qquad n > 2^\rho(d+\delta')e\rho \quad \text{for all } n \geqq N'' = N''(\delta').$$

We note that (6.7) is possible since $\Sigma a_n e_n \in \Gamma(\rho, d)$. Choose $N_0 = \max\{N, N', N''\}$, where $N = N(\delta')$ is as defined in (6.4). So $N_0 = N_0(\delta')$. The inequalities (6.6), (6.7) and (6.4) now give, for $n \geqq N_0$,

$$|c_n| \leqq \left\{\frac{(d+\delta')e\rho}{n}\right\}^{n/\rho} + \left\{\frac{(d+\delta')e\rho}{n}\right\}^{n/\rho} \sum_{k=1}^{n} |c_{n-k}|.$$

Now define positive numbers (d_n) as

$$d_0 = |a_0|;\ d_n = 1 + \sum_{k=1}^{n} d_{n-k}, \quad n \geqq 1.$$

This gives us

$$d_n - d_{n-1} = d_{n-1}, \quad n \geqq 2;$$

from which we get

$$d_n = 2^{n-1}|d_1| = 2^{n-1}(1+|a_0|).$$

So

$$(6.9) \qquad \frac{|c_n|\, n^{n/\rho}}{\{(d+\delta')e\rho\}^{n/\rho}} \leqq d_n = 2^{n-1}(1+|a_0|), \quad n \geqq N_0.$$

Now

$$\sum_{n=1}^{\infty} |c_n|\, |||\beta_n|||$$
$$= \sum_{n=1}^{\infty} |c_n| \sum_{k=1}^{\infty} |h_{nk}| \left\{\frac{n+k}{(d+\delta)e\rho}\right\}^{(n+k)/\rho}.$$

We shall split this double summation as

$$\sum_{n=1}^{N_0-1}\sum_{k=1}^{N_0-1} + \sum_{n=1}^{N_0-1}\sum_{k=N_0}^{\infty} + \sum_{n=N_0}^{\infty}\sum_{k=1}^{\infty}.$$

The first series is clearly finite. The second series is dominated, because of (6.4), by the convergent series

$$N_0|C| \times \sum_{k=N_0}^{\infty} \left(\frac{d+\delta'}{d+\delta}\right)^{(n+k)/\rho}, \quad \text{where } C = \max_{n=1 \text{ to } N_0-1} |c_n|.$$

The third series is dominated by

$$\sum_{n=N_0}^{\infty} 2^{n-1}(1+|a_0|)\left\{\frac{(d+\delta')e\rho}{n}\right\}^{n/\rho} \sum_{k=N_0}^{\infty}\left(\frac{d+\delta'}{d+\delta}\right)^{(n+k)/\rho}$$

in virtue of (6.9) and (6.4). This latter series is now equivalent to

$$K\cdot\sum_{n=N_0}^{\infty}\left\{\frac{2^{\rho}(d+\delta')e\rho}{n}\right\}^{n/\rho}\left(\frac{d+\delta'}{d+\delta}\right)^{(n+N_0)/\rho}$$

where $K = K(\delta, \delta')$, since $\delta' < \delta$. This again, for the same reason, is dominated by

$$K\sum_{n=N_0}^{\infty}\left\{\frac{2^{\rho}(d+\delta')e\rho}{n}\right\}^{n/\rho}$$

and this is a convergent series, because of (6.8). This completes the proof of the Theorem.

7. Construction of Proper Pincherle bases in $\Gamma(\varrho, d)$.

A direct application of Theorem 4 gives a general method of construction of proper Pincherle bases from certain entire functions belonging to $C(\rho, d)$. We express it in the form of

COROLLARY 7.1. *Let* $\phi(z) = \sum_{k=0}^{\infty} t_k z^k \in C(\rho, d)$. *Further, let, for each* $\delta > 0$ *and for* $k \neq 0$,

$$\limsup_{(n+k)\to\infty}\left|\frac{t_{n+k}}{t_n}\right|^{1/(n+k)}(n+k)^{1/\rho} \leqq (de\rho)^{1/\rho}.$$

Then the sequence $\{\alpha_n\}$ *defined by*

$$\alpha_n = \frac{1}{t_n}\{\phi(z) - \sum_{k=0}^{n-1} t_k z^k\}$$

constiitutes a proper Pincherle basis in $\Gamma(\rho, d)$.

PROOF. It suffices to note that, in the notation of Theorem 4, $h_{nk} = t_{n+k}/t_n$, for all $n = 0, 1, 2, \cdots$ and $k = 1, 2, \cdots$.

This Corollary is the analogue, for $\Gamma(\rho, d)$, of Arsove's theorem [**2**, p. 49, Theorem 6] for the space Γ of all entire functions. Following Arsove, we make use of the interrelationship between proper bases and automorphisms in $\Gamma(\rho, d)$ (cf. remark at the end of § 5) and obtain the following corollary, thereby constructing another proper Pincherle basis in $\Gamma(\rho, d)$. Since the arguments are precisely the same as those of Arsove [**2**, pp. 50, 51], we omit them.

COROLLARY 7.2. *Under the hypothesis of Corollary* 7.1, *the sequence* $\{\beta_n\}$ *defined by*

$$\beta_n(z) = z^n\left\{1 - \frac{t_{n+1}}{t_n}z\right\}$$

is a proper Pincherle basis in $\Gamma(\rho, d)$.

We give below two examples of proper Pincherle bases in $\Gamma(\rho, d)$, where $\rho \geqq 1$, illustrative of Corollaries 7.1 and 7.2.

EXAMPLE 1. The sequence $\{\alpha_n\}$ defined by

$$\alpha_n = (n!)^n \left[\sum_{k=0}^{\infty} \frac{z^{(n+k)}}{\{(n+k)!\}^{(n+k)}} \right].$$

EXAMPLE 2. The sequence $\{\beta_n\}$ defined by

$$\beta_n = z^n \left\{ 1 - \frac{z}{(n+1)^n \cdot (n+1)!} \right\}.$$

The second example follows from the first and Corollary 7.2. To see the truth of the first, we need only verify, in virtue of Corollary 7.1, the hypothesis on t_{n+k}/t_n, where $t_n = 1/(n!)^n$. It is no loss of generality to assume $n \geqq 3$; for the other particular cases may be directly verified. If $n \geqq 3$, then,

$$\begin{aligned} \frac{t_{n+k}}{t_n} &= \frac{(n!)^n}{\{(n+k)!\}^{(n+k)}} \\ &= \frac{1}{(n+1)^n(n+2)^n \cdots (n+k)^n} \times \frac{1}{\{(n+k)!\}^k}. \end{aligned}$$

So we have,

$$\left| \frac{t_{n+k}}{t_n} \right|^{1/(n+k)} (n+k)^{1/\rho}$$

$$(7.1) \qquad = \left\{ \frac{1}{(n+1)(n+2)\cdots(n+k-1)} \right\}^{n/(n+k)} \times \frac{1}{(n+k)^{n/(n+k)-1/\rho}} \times \frac{1}{\{(n+k)!\}^{k/(n+k)}}.$$

Now Stirling's expression for $(n+k)!$, when $(n+k)$ is large, gives

$$(n+k)! = (2\pi)^{1/2}(n+k)^{n+k+1/2} e^{-(n+k)} \exp\left\{ \frac{1}{12(n+k)} + 0\left(\frac{1}{(n+k)^2} \right) \right\}.$$

Using this in (7.1), we see that, when $(n+k)$ is large, the right side of (7.1) is asymptotically equivalent to

$$\left\{ \frac{1}{(n+1)\cdots(n+k-1)} \right\}^{n/(n+k)} \times \frac{1}{(2\pi)^{k/2(n+k)}} \times \frac{e^k}{(n+k)^{n/(n+k)+k+k/2(n+k)-1/\rho}} \times \left\{ 1 + \frac{1+\varepsilon_{(n+k)}}{12(n+k)} \right\}^{-k/(n+k)},$$

where $\varepsilon_{(n+k)}$ tends to zero as $n+k$ tends to infinity.

This expression is less than or equal to

$$\frac{1}{n^{n(k-1)/(n+k)}} \times \frac{1}{(2\pi)^{k/2(n+k)}} \times \frac{1}{n^{n/(n+k)+k/2(n+k)}} \times \left(\frac{e}{n}\right)^{k-1/\rho}$$
$$\times\, e^{1/\rho} \times \left\{\left(1+\frac{1+\varepsilon_{(n+k)}}{12(n+k)}\right)^{(n+k)}\right\}^{-k/(n+k)^2}$$

which tends to

$$\frac{1}{n^{(2kn+k)/2(n+k)}} \times \left(\frac{e}{n}\right)^{k-1/\rho} \times \frac{e^{1/\rho}}{(2\pi)^{k/2(n+k)}} \times (e^{1/12})^{-k/(n+k)^2}.$$

This last expression tends to zero as $k \to \infty$ for any $n \geqq 3$ and as $n \to \infty$ for any $k \geqq 1/\rho$. Since $\rho \geqq 1$, the hypothesis of Corollary 7.1 is easily satisfied.

PART II: THE SPACE $\Gamma(\varrho)$

8. Topology for the class $C(\varrho)$

A function $\alpha = \Sigma a_n e_n \in C(\rho)$, $e_n \equiv z^n$, $n = 0, 1, 2, \cdots$ is characterised by

$$\limsup_{n\to\infty} \frac{n \log n}{\log(1/|a_n|)} \leqq \rho.$$

This is equivalent to the condition

$$(8.1) \qquad |a_n|^{1/n} n^{1/(\rho+\delta)} \to 0 \quad \text{for each } \delta > 0.$$

So the expression[3]

$$||\alpha; \rho+\delta|| = |a_0| + \sum_{n=1}^{\infty} |a_n| n^{n/(\rho+\delta)}$$

is convergent for each $\delta > 0$ and defines a norm on $C(\rho)$. We call the normed linear space thus obtained as $\Gamma(\rho, \delta)$. As δ decreases, $\Gamma(\rho, \delta)$ becomes weaker. The lattice product of these topologies for $\delta > 0$ is denoted by $\Gamma(\rho)$. It is therefore weaker than each of the $\Gamma(\rho, \delta)$'s. It is metrisable with the metric

$$||\alpha - \beta||, \quad \alpha = \Sigma a_n e_n, \quad \beta = \Sigma b_n e_n,$$

where

$$(8.2) \qquad ||\alpha|| = \sum_{p=1}^{\infty} \frac{1}{2^p} \cdot \frac{||\alpha; \rho + 1/p||}{1 + ||\alpha; \rho+1/p||}.$$

[3] The scrupulous use of the letters ρ for order and d for type will help to distinguish between $||\alpha; \rho+\delta||$ defined here and $||\alpha; d+\delta||$ defined in § 2. When α is considered as a member of $C(\rho, d)$, we write the corresponding sequence of norms as $||\alpha; d+\delta||$, where, though the order ρ is not mentioned, the use of the letter d is sufficiently decisive. (See Footnote 1). When we write $||\alpha; \rho+\delta||$, we shall think of α only as a member of $C(\rho)$.

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Now Ganapathy Iyer's arguments in [**10**, Part II] lead to the following theorems pertaining to the space $\Gamma(\rho)$ analogous to his results for the space $\Gamma(\rho, d)$ — some of which were referred to in § 2 above — and also to his earlier results for the space Γ of all entire functions:

THEOREM 5. *The space $\Gamma(\rho)$ is a complete linear metric space.*

We omit the proof as it is similar to that of the corresponding theorem of Ganapathy Iyer [**10**, Theorem 6].

We denote the dual space of continuous linear functionals on $\Gamma(\rho)$ as $\Gamma^*(\rho)$ and of those on $\Gamma(\rho, \delta)$ as $\Gamma^*(\rho, \delta)$. Since a functional continuous in any topology is also continuous in a weaker topology, $\Gamma^*(\rho, \delta) \subset \Gamma^*(\rho)$ for each $\delta > 0$. So we have,

$$\sum_{\delta>0} \Gamma^*(\rho, \delta) \subset \Gamma^*(\rho). \tag{8.3}$$

That the reverse inclusion is also true is shown by the following Lemma.

LEMMA 5. *Let Γ_0 be a subspace of $\Gamma(\rho)$. Let $f(\alpha)$ be a linear functional defined and continuous on Γ_0 in the topology of $\Gamma(\rho)$. Then $f(\alpha)$ will be continuous on Γ_0 regarded as a subspace of $\Gamma(\rho, \delta)$ for some $\delta > 0$.*

PROOF. Suppose that $f(\alpha)$ is not continuous on Γ_0 regarded as a subspace of $\Gamma(\rho, \delta)$ for any $\delta > 0$. Then, by Banach's theorem [**4**, p. 54, Theorem 1], we can, for each positive integer p, find an element $\alpha_p \varepsilon \Gamma_0$ such that

$$||\alpha_p; \rho + 1/p|| \leqq 1/p \quad \text{and} \quad |f(\alpha_p)| \geqq 1.$$

Now, we have, for all $n \leqq p$,

$$\frac{||\alpha_p; \rho + 1/n||}{1 + ||\alpha_p; \rho + 1/n||} \leqq \frac{1/p}{1 + 1/p} = \frac{1}{1 + p}.$$

When p is sufficiently large, the above gives

$$||\alpha_p|| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{||\alpha_p; \rho + 1/n||}{1 + ||\alpha_p; \rho + 1/n||} \leqq \frac{1}{1 + p} + \varepsilon.$$

Hence $||\alpha_p|| \to 0$ as $p \to \infty$, whereas we already have $|f(\alpha_p)| \geqq 1$ for all p. This means that $f(\alpha)$ is not continuous on Γ_0 in the topology of $\Gamma(\rho)$. This contradiction proves the Lemma.

Combining Lemma 5 and (8.3) we have

THEOREM 6. *The set $\Gamma^*(\rho)$ of continuous linear functionals on $\Gamma(\rho)$ is the union of the sets $\Gamma^*(\rho, \delta)$ for all $\delta > 0$.*

$\Gamma(\rho)$ being a locally convex space, the Hahn-Banach theorem on the extension of linear functionals is valid in $\Gamma(\rho)$, but it may also be directly proved as in [**7**, p. 89, Theorems 3 and 4]. We state it as

THEOREM 7. (i) *Let f be a continuous linear functional defined on $\Gamma_0 \subset \Gamma(\rho)$.*

Then there exists a continuous linear functional g over the whole of $\Gamma(\rho)$ *and equal to* f *on* Γ_0. (ii) *Let* $\Gamma_0 \subset (\rho)$ *and let* $\alpha \in \Gamma(\rho)$ *such that* $\alpha \notin \overline{\Gamma}_0$. *Then there exists a continuous linear functional* $f \in \Gamma^*(\rho)$ *such that* $f(\alpha) = 1$ *and* $f(\beta) = 0$ *for all* $\beta \in \Gamma_0$.

The general form of continuous linear functionals is given by

THEOREM 8. (i) *A functional* $f \in \Gamma^*(\rho, \delta)$ *is of the form*

$$f(\alpha) = \Sigma c_n a_n, \quad \alpha = \Sigma a_n e_n \in \Gamma(\rho, \delta),$$

where

$$|c_n|/n^{n/\rho+\delta)} \text{ is bounded;} \tag{8.4}$$

and conversely. (ii) *Every* $f \in \Gamma^*(\rho)$ *is of the form*

$$f(\alpha) = \Sigma c_n a_n, \quad \alpha = \Sigma a_n e_n \in \Gamma(\rho),$$

where

$$|c_n|/n^{n/\rho} \to 0.$$

PROOF. The second part follows from the first, in virtue of Theorem 6. To prove (i), suppose that $f \in \Gamma^*(\rho, \delta)$ with $f = \Sigma c_n a_n$ and $\alpha = \Sigma a_n e_n$. Then there is a K such that

$$|f(\alpha)| \leqq K\,||\alpha;\quad \rho + \delta||.$$

Taking $\alpha = e_n$ here, we get (8.4). Conversely if (8.4) be true, then there exists a K such that $|f(\alpha)| \leqq K||\alpha; \rho + \delta||$ and so f is continuous in $\Gamma(\rho, \delta)$.

9. Convergence in $\Gamma(\varrho)$

It is well-known [**6**] that convergence in the space Γ of all entire functions topologised according to the metric

$$||\alpha - \beta|| = \max\,[|a_0 - b_0|;\quad |a_n - b_n|^{1/n},\ n \geqq 1]$$

is equivalent to uniform convergence on compact sets. Convergence in $\Gamma(\rho, d)$ is equivalent to uniform convergence relative to $\exp\{(\alpha + \delta)\,|z|^\rho\}$ for all $\delta > 0$, as explained in § 2. We shall presently obtain a similar interpretation of convergence in $\Gamma(\rho)$.

As a preliminary, the following properties of convergence in $\Gamma(\rho)$ may be noted:

(i) The topology induced on $C(\rho)$ by Γ is stronger than $\Gamma(\rho)$; because, if a sequence of elements in $\Gamma(\rho)$ converges in $\Gamma(\rho)$, then it also converges in Γ.

(ii) Convergence in $\Gamma(\rho)$ is equivalent to convergence in $\Gamma(\rho, \delta)$ for each $\delta > 0$; because, we can prove the analogue of Lemma 1, (Part I) for $\Gamma(\rho)$.

(iii) The sequence of partial sums of $\alpha = \Sigma a_n e_n$ converges to α in $\Gamma(\rho)$;

because, the proof by Ganapathy Iyer [**10**, Theorem 10] of the corresponding result for $\Gamma(\rho, d)$ holds here with obvious modifications.

To obtain a classical interpretation of convergence in $\Gamma(\rho)$, we first introduce a new set of norms in $C(\rho)$. For each positive δ, the expression

$$\max_z \{\exp(-|z|^{(\rho+\delta)})\, |\alpha(z)|\}$$

defines a norm on $C(\rho)$ and, since

$$\begin{aligned} &\max_r |a_n| r^n \exp\{-r^{\rho+\delta}\} \\ &\leqq |a_n| \left\{\frac{n}{e(\rho+\delta)}\right\}^{n/(\rho+\delta)} \\ &\leqq |a_n|\, n^{n/(\rho+\delta)}, \quad \text{if } \rho \geqq 1/e, \text{ for each } \delta > 0; \\ \text{and } &> |a_n|\, n^{n/(\rho+\delta)}, \quad \text{if } \rho < 1/e, \text{ for each } \delta\text{ [4] in } 0 < \delta < 1/e - \rho, \end{aligned}$$

the topology introduced by the above norm is comparable with $\Gamma(\rho, \delta)$ for each ρ. The lattice product, for all positive δ's, (sufficiently small, if $\rho < 1/e$), of these normed topologies, which grow weaker as δ decreases, can be shown to be a complete linear metric space, whose topology is therefore stronger than $\Gamma(\rho)$ if $\rho \geqq 1/e$ and weaker than $\Gamma(\rho)$ if $\rho < 1/e$. It now follows from a theorem of Banach, [**4**, p. 41, Theorem 6] that the two metrics are equivalent and the interpretation of convergence in $\Gamma(\rho)$ may therefore be given by

THEOREM 9. *Let $\{\alpha_n\}$ be a sequence of elements in $\Gamma(\rho)$. The statement $\alpha_n \to \alpha$ is equivalent to the statement that, for every $\delta > 0$, the sequence $\{\alpha_n(z)\}$ converges to $\alpha(z)$ uniformly over the whole finite complex plane relative to the function* $\{\exp(|z|^{(\rho+\delta)})\}$.

We have, finally, by arguments similar to those of Ganapathy Iyer in [**7**, p. 89],

THEOREM 10. *Weak and strong convergence in $\Gamma(\rho)$ are equivalent.*

10. Continuous Linear transformations of $\Gamma(\varrho)$ into itself

Denoting, as in § 3, by $T(0 \to 0)$ a continuous linear transformation of $\Gamma(\rho)$ into itself, we obtain the following theorem giving the equivalent growth condition on a given $\{\alpha_n\}$ in order that $T(e_n) = \alpha_n$, $n = 0, 1, 2, \cdots$. Since the proof is parallel to that of the analogous theorem for $\Gamma(\rho, d)$ (Theorem 1, Part I), we merely give the statement.

THEOREM 11. *A necessary and sufficient condition that there exists a $T = T(0 \to 0)$ with $T(e_n) = \alpha_n$, $n = 0, 1, 2, \cdots$ is that, for each $\delta > 0$,*

$$\limsup_{n\to\infty} \frac{\log ||\alpha_n; \rho+\delta||}{n \log n} < 1/\rho.$$

[4] This restriction of δ from above does not matter because we are concerned only with the lattice product of $\Gamma(\rho, \delta)$'s as δ tends to zero.

11. Proper bases in $\Gamma(\varrho)$

A proper basis in $\Gamma(\rho)$ is defined as in § 4, namely:

A proper basis is a basis $\{\alpha_n\}$ in a subspace Γ_0 of $\Gamma(\rho)$, which possesses, in addition, the property:

For all sequences $\{c_n\}$ of complex numbers, $\Sigma c_n \alpha_n$ converges in $\Gamma(\rho)$, if and only if, $\Sigma c_n e_n$ converges in $\Gamma(\rho)$.

We note that the statement "$\Sigma c_n e_n$ converges in $\Gamma(\rho)$" is equivalent to saying

$$\lim_{n\to\infty} |c_n|^{1/n} n^{1/(\rho+\delta)} = 0 \quad \text{for each } \delta > 0. \tag{11.1}$$

We are now in a position to obtain the characterisation of proper bases in $\Gamma(\rho)$ in terms of growth conditions on $\{\alpha_n\}$ as in § 4, the methods being quite similar. We therefore state, without proof, the following two Lemmas and Theorem 12 which are respectively the analogues of Lemma 3, Lemma 4 and Theorem 2 of Part I.

LEMMA 6. *The following three properties are equivalent*:

(A′) $$\limsup_{n\to\infty} \frac{\log ||\alpha_n; \rho+\delta||}{n \log n} < 1/\rho \quad \text{for each } \delta > 0.$$

(B′) *For all sequences $\{c_n\}$ of complex numbers, "$\Sigma c_n e_n$ converges in $\Gamma(\rho)$" implies "$\Sigma c_n \alpha_n$ converges in $\Gamma(\rho)$".*

(C′) *For all sequences $\{c_n\}$ of complex numbers, "$\Sigma c_n e_n$ converges in $\Gamma(\rho)$" implies "$c_n \alpha_n \to 0$ in $\Gamma(\rho)$".*

LEMMA 7. *The following three properties are equivalent*:

(α') $$\lim_{\delta\to 0} \left\{ \liminf_{n\to\infty} \frac{\log||\alpha_n; \rho+\delta||}{n \log n} \right\} \geqq 1/\rho.$$

(β') *For all sequences $\{c_n\}$ of complex numbers, "$\Sigma c_n \alpha_n$ converges in $\Gamma(\rho)$" implies "$\Sigma c_n e_n$ converges in $\Gamma(\rho)$".*

(γ') *For all sequences $\{c_n\}$ of complex numbers, "$c_n \alpha_n \to 0$ in $\Gamma(\rho)$" implies "$\Sigma c_n e_n$ converges in $\Gamma(\rho)$".*

THEOREM 12. *A basis $\{\alpha_n\}$ in a subspace Γ_0 of $\Gamma(\rho)$ is proper if and only if conditions* (A′) *and* (α') *hold.*

It may be remarked that the comments about the interrelationship between proper bases and linear homeomorphisms made in § 5 are valid in $\Gamma(\rho)$ in analogous fashion.

12. Proper Pincherle bases in $\Gamma(\varrho)$

A Pincherle basis in $\Gamma(\rho)$ is a basis $\{\alpha_n\}$ in $\Gamma(\rho)$ of the form

$$\alpha_n(z) = z^n\{1+\lambda_n(z)\}, \tag{12.1}$$

where $\lambda_n(0) = 0$. Obviously $\lambda_n(z) \in \Gamma(\rho)$. So we may write

$$\lambda_n(z) = \sum_{k=0}^{\infty} h_{nk} z^k, \qquad n = 0, 1, 2, \cdots, \tag{12.2}$$

with each $h_{0k} = 0$, where, for all n,

$$\lim_{k\to\infty} |h_{nk}|^{1/k} k^{1/(\rho+\delta)} = 0, \text{ for each } \delta > 0.$$

It is easy to see that, since

$$||\alpha_n; \rho + \delta|| \geqq n^{n/(\rho+\delta)} \quad \text{for each } \delta > 0,$$

$\{\alpha_n\}$ satisfies condition (α') without any further hypothesis. So, from Theorem 12, in order that a Pincherle basis may be proper, a necessary and sufficient condition is that it satisfies (A'). As in the case of $\Gamma(\rho, d)$, we obtain the following theorem, giving sufficient conditions on the growth of λ_n, for a given $\{\alpha_n\}$ to be a proper Pincherle basis. We omit the proof, because of its similarity with that of Theorem 4.

THEOREM 13. *If $\{\alpha_n\}$ as defined in* (12.1) *and* (12.2) *satisfies, for each $\delta > 0$ the condition*

$$\lim_{(n+k)\to\infty} |h_{nk}|^{1/(n+k)} (n+k)^{1/(\rho+\delta)} = 0,$$

then, it constitutes a proper Pincherle basis in $\Gamma(\rho)$.

Moreover, we obtain the following two corollaries, which set up a general method, as in the earlier case, of constructing proper Pincherle bases from certain functions belonging to $\Gamma(\rho)$.

COROLLARY 12.1. *Let $\phi(z) = \sum_{k=0}^{\infty} t_k z^k \in \Gamma(\rho)$. If*

$$\alpha_n = \frac{1}{t_n}\left[\phi(z) - \sum_{k=0}^{n-1} t_k z^k\right]$$

and if

$$\lim_{(n+k)\to\infty} \left|\frac{t_{n+k}}{t_n}\right|^{1/(n+k)} (n+k)^{1/(\rho+\delta)} = 0$$

for each $\delta > 0$, then $\{\alpha_n\}$ is a proper Pincherle basis in $\Gamma(\rho)$.

COROLLARY 12.2. *Under the hypothesis of the previous corollary, if*

$$\beta_n = z^n\left(1 - \frac{t_{n+1}}{t_n} z\right),$$

then $\{\beta_n\}$ is a proper Pincherle basis in $\Gamma(\rho)$.

EXAMPLE. A scrutiny of examples 1 and 2 of § 7 will show that we have actually proved, for the sequences considered there, the conditions of Corollaries 12.1 and 12.2 respectively, so that we may state: $\{\alpha_n\}$ and $\{\beta_n\}$ as defined in Examples 1 and 2 of § 7 are proper Pincherle bases in $\Gamma(\rho)$, provided $\rho \geqq 1$.

PART III: ALGEBRAS IN $\Gamma(\rho)$

13. The Algebra $\Gamma_N(\rho)$

We know that if $\alpha \in \Gamma(\rho)$ and $\beta \in \Gamma(\rho)$, then $\alpha\beta = \alpha(z)\beta(z)$ is also in $\Gamma(\rho)$. $\Gamma(\rho)$ is thus closed under natural multiplication. Since convergence in $\Gamma(\rho)$ is equivalent to uniform convergence relative to $\{\exp(|z|^{\rho+\delta})\}$, for each $\delta > 0$, it follows that the natural multiplication is continuous in the topology of $\Gamma(\rho)$. So we get a topological algebra, which we may call $\Gamma_N(\rho)$. This is a commutative algebra with e_0 as unit element. Repeating the arguments that Ganapathy Iyer [**9**, p. 646] uses for his algebra $\Gamma(N)$ of all entire functions, we now obtain

THEOREM 14. *The general automorphism of $\Gamma_N(\rho)$ — a linearly homeomorphic mapping of the algebra onto itself with preservation of multiplication — is precisely of the form $T(\alpha) = \alpha(az + b)$, where $\alpha \in \Gamma_N(\rho)$ and a and b are complex numbers, with $a \neq 0$.*

It is an immediate consequence of this theorem that the group of automorphisms of $\Gamma_N(\rho)$ is isomorphic to the group of one-to-one conformal transformations of the complex plane onto itself, leaving the point at ∞ invariant.

14. The Algebra $\Gamma_C(\rho)$

We define multiplication in $\Gamma(\rho)$ by using Hadamard composition, namely, if $\alpha = \Sigma a_n e_n \in \Gamma(\rho)$ and $\beta = \Sigma b_n e_n \in \Gamma(\rho)$, we write

$$\alpha \circ \beta = \sum a_n b_n e_n.$$

From

$$|a_n|^{1/n} n^{1/(\rho+\delta)} \to 0 \quad \text{for each } \delta > 0$$

and

$$|b_n|^{1/n} n^{1(\rho+\delta)} \to 0 \quad \text{for each } \delta > 0,$$

it follows that $\alpha \circ \beta$ is also in $\Gamma(\rho)$. $\Gamma(\rho)$ thus becomes an algebra under the multiplication defined. That the multiplication is continuous in $\Gamma(\rho, \delta)$ for each $\delta > 0$ and therefore continuous in $\Gamma(\rho)$ is shown by the inequality

$$||\alpha \circ \beta; \rho + \delta|| \leqq ||\alpha; \rho + \delta|| \; ||\beta; \rho + \delta||,$$

seen to be true by writing out the norms in full. $\Gamma(\rho)$ is therefore a topological algebra, which we denote by $\Gamma_C(\rho)$. It is a commutative algebra without any unit element. The characterisation of automorphisms in this algebra may now be stated in the form of

THEOREM 15. *The general automorphism of $\Gamma_C(\rho)$ is precisely of the form*

$$T(\alpha) = \sum a_n T(e_n) \quad \text{for all } \alpha = \sum a_n e_n \in \Gamma_C(\rho),$$

where

$$T(e_n) = e_{\theta(n)}$$

and $\theta : n \to N = \theta(n)$ *denotes a permutation of the set* I *of non-negative integers, satisfying the conditions*

$$N \log N = 0(n \log n); \; n \log n = 0(N \log N). \tag{13.1}$$

These conditions may also be combined in the form

$$|\log n - \log N| = 0(1). \tag{13.2}$$

REMARK. It follows at once from this theorem that the group of automorphisms of $\Gamma_C(\rho)$ is isomorphic to the group $G(I)$ of all permutations θ of I with the property (13.1).

PROOF OF THEOREM 15. (i) Taking T to be an automorphism we shall prove that $T(e_n) = e_N$. The formula for $T(\alpha)$ will then follow from the linearity of T.

Since T is an atomorphism, $T(e_n) \neq 0$. So $T(e_n) = \Sigma a_{np} e_p$, where, for a fixed n not all a_{nn}'s are zero. Since T preserves multiplication, $T(e_m) \circ T(e_n) = T(e_m \cdot e_n)$. This latter is equal to $T(0) = 0$ when $m = n$, and is equal to $T(e_m)$ when $m = n$. So

$$a_{mp} a_{np} = 0 \quad \text{for all } p \geqq 0 \text{ and } m \neq n$$

and

$$a_{mp}^2 = a_{mp} \text{ for all } p \geqq 0 \text{ and } m \geqq 0.$$

The second of these two equations shows that, for a fixed m,

$$a_{mp} = 0 \text{ or } 1 \text{ for all } p \geqq 0.$$

But $T(e_m)$ being an entire function, only a finite number of a_{mp}'s are equal to 1, for each fixed m. The other equation now shows that, if $a_{mp_1} = 1$ for a fixed $p = p_1$, then $a_{np_1} = 0$ for all $n \neq m$. Thus the matrix (a_{np}) has the following properties: (i) each row contains at least one 1; (ii) each row contains at most a finite number of 1's; (iii) the remaining elements in each row are 0's; and (iv) no column contains more than one 1. So we can write

$$T(e_n) = \sum_{N \in H_n} e_N,$$

where H_n, $n = 0, 1, 2, \cdots$ are nonempty disjoint finite subsets of the set I. We note the following properties about H_n:

No H_n can contain more than one integer of I. For if some H_n, say H_m, contained $m_1, m_2, \cdots m_r$, then, an element of $\Gamma_C(\rho)$ of the form

$$b_1 e_{m_1} + b_2 e_{m_2} + \cdots + b_r e_{m_r}$$

with no two of the b's equal, cannot be the transform of any element of

$\Gamma_C(\rho)$. Thus H_n is made up of a single integer, which we may call $\theta(n)$ or N. Also, every $N \in I$ must be in some H_n, for, if an integer $N' \in I$ is not in any H_n, then $T(\Gamma_C(\rho))$ cannot contain an entire function having a Taylor expansion with a nonzero term in $z^{N'}$.

Hence the correspondence $n \to N = \theta(n)$ is one-one and so defines a permutation of I. We have now only to prove (13.1).

From $T(e_n) = e_N$ and Theorem 11, we have

$$\limsup_{n\to\infty} \frac{\log ||e_N; \rho + \delta||}{n \log n} < 1/\rho \text{ for each } \delta > 0.$$

This gives, for all large n, the inequality

$$N/(\rho + \delta) \log N \leqq n/\rho \log n \text{ for each } \delta > 0.$$

Therefore,

$$N \log N = 0(n \log n).$$

The same argument applied to the inverse of T shows that

$$n \log n = 0(N \log N).$$

(ii) Defining $T(\alpha) = \Sigma a_n T(e_n)$, where $T(e_n) = e_{\theta(n)}$ with $N = \theta(n)$ satisfying (13.1), we shall prove that T is an automorphism.

That T is linear, follows from its definition. That it is continuous is a consequence of the hypothesis $N \log N = 0(n \log n)$, from which we may deduce the condition of Theorem 11. We now assert that T is one-one; for, if $T(\alpha) = T(\beta)$ for $\alpha = \Sigma a_n e_n$ and $\beta = \Sigma b_n e_n$, then, $\Sigma a_n T(e_n) = \Sigma b_n T(e_n)$, which implies $\Sigma a_n e_{\theta(n)} = \Sigma b_n e_{\theta(n)}$, so that $a_n = b_n$ for all n. This shows that $\alpha = \beta$. Further, the image of $\Gamma_C(\rho)$ by T is the whole of $\Gamma_C(\rho)$. This is because every $\alpha = \Sigma a_n e_n \in \Gamma_C(\rho)$ can be written as $\alpha = T(\alpha')$, where $\alpha' = \Sigma a_n e_{N'}$ and $N' \in I$ such that $n \log n = 0(N' \log N')$. Thus T is a linear continuous map of $\Gamma_C(\rho)$ onto $T(\Gamma_C(\rho))$ in a one-to-one manner. It follows then from a theorem of Banach [4, p. 41, Theorem 5] that the inverse of T, which is linear, is also continuous.

To complete the proof it only remains to demonstrate the preservation of multiplication. For this, we note,

$$T(\alpha) = \sum a_n T(e_n) = \sum a_n e_{\theta(n)},$$
$$T(\beta) = \sum b_n T(e_n) = \sum b_n e_{\theta(n)};$$

and

$$\begin{aligned} T(\alpha) \circ T(\beta) &= \sum a_n b_n e_{\theta(n)} \\ &= \sum a_n b_n T(e_n) \\ &= T(\sum a_n b_n e_n) = T(\alpha \circ \beta). \end{aligned}$$

So T is multiplicative and this completes the proof of the theorem.

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