Macroeconomic Dynamics (2023), **27**, pp. 1166–1177
doi:10.1017/S1365100522000086

ARTICLE

A Note on nonconvex adjustment costs in lumpy investment models: Mean versus variance

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Abstract
This paper revisits the canonical assumption of nonconvex capital adjustment costs in lumpy investment models as in Khan and Thomas [(2008) *Econometrica* 76(2), 395–436], which are assumed to follow a uniform distribution from zero to an upper bound, without distinguishing between the *mean* and the *variance* of the distribution. Unlike the usual claim that the upper bound stands for the size (represented by the *mean*) of a nonconvex cost, I show that in order to generate an empirically consistent interest elasticity of aggregate investment, both a sizable *mean* and a sizable *variance* are necessary. The *mean* governs the importance of the extensive margin in aggregate investment dynamics, while the *variance* governs how sensitive the extensive margin is to changes in the real interest rate. As a result, both the *mean* and the *variance* are quantitatively important for aggregate investment dynamics.

Keywords: Lumpy investment; Ss model; firm heterogeneity

1. Introduction

In all current lumpy investment models such as Khan and Thomas (2008), the nonconvex capital adjustment cost ξ_{jt} is uniformly distributed with support $U[0, \bar{\xi}]$ independently across firms and time. A conventional calibration of a small upper bound $\bar{\xi}$ which matches the firm-level lumpy investment moments claims that lumpy investment is irrelevant for aggregate dynamics.

However, recent literature [Fang (2020), Koby and Wolf (2020) and Winberry (2021)] reverses the claim that microeconomic investment lumpiness is inconsequential for macroeconomic analysis. As argued by Koby and Wolf (2020), whether general equilibrium feedback smooths out the effects of micro frictions is governed by the sensitivity of investment with respect to changes in the costs of and the returns to capital. These costs and returns are either mainly reflected in real interest rate changes or are isomorphic to such changes.[1]

In Khan and Thomas (2008), aggregate investment is extremely price-sensitive; the partial equilibrium interest rate semi-elasticity of firm investment is almost 500 percent. As a result, small countercyclical changes in prices are enough to smooth out the movements in investment demand caused by micro lumpiness given prices. In contrast, in Winberry (2021), Koby and Wolf (2020) and Fang (2020), the partial equilibrium interest rate semi-elasticity of firm investment is only between 5 to 8 percent. As a result, they show that lumpy investment matters for business cycle dynamics and the effectiveness of economic policies.

*I would like to thank George Alessandria, Yan Bai, Roman Merga, Narayana Kocherlakota, Christian Wolf, Joseph Zeira, and audiences at the University of Rochester for helpful comments and discussions. I am also grateful to an anonymous referee whose comments and suggestions have largely improved this note. All errors are my own.

The key argument in these papers Fang (2020), Koby and Wolf (2020) and Winberry (2021)] is that the upper bound $\bar{\xi}$ of the distribution of nonconvex capital adjustment costs should be calibrated as much larger. This upper bound $\bar{\xi}$ determines how sensitive aggregate investment is to changes in the real interest rate. The calibration of a small (large) upper bound $\bar{\xi}$ generates a large (small) interest elasticity of aggregate investment. Quasi-experimental evidence on firm-level investment responses to tax changes [Zwick and Mahon (2017) and Koby and Wolf (2020)] suggests a small interest elasticity. Consequently, the upper bound $\bar{\xi}$ should be large.

What is the economics meaning of calibrating such a large upper bound $\bar{\xi}$? In all current models, the setup implicitly assumes that the mean $\mu_\xi = \bar{\xi}/2$ and the standard deviation $\sigma_\xi = \bar{\xi}/\sqrt{12}$ are isometric $\left(\mu_\xi = \sqrt{3}\sigma_\xi\right)$. This assumption means that when calibrating $\bar{\xi}$, we jointly choose the expected size (*mean*) and the uncertainty (*variance*) of the nonconvex adjustment cost faced by the firms. ***Then, which one determines the interest elasticity of aggregate investment?***

In this paper, I answer this question through disciplining the economic meanings of the expected size (*mean*) and the uncertainty (*variance*) of the nonconvex adjustment cost. I assume the nonconvex adjustment cost follows a uniform distribution with mean and variance $\{\mu_\xi, \sigma_\xi\}$:

$$\xi_{jt} \sim U\left[\mu_\xi - \sqrt{3}\sigma_\xi, \mu_\xi + \sqrt{3}\sigma_\xi\right]. \tag{1}$$

I first compare the interest elasticity of aggregate investment over both dimensions of $\{\mu_\xi, \sigma_\xi\}$, departing from a conventionally calibrated lumpy investment model. The model generates unrealistically large interest elasticities of aggregate investment when either the *mean* or the *variance* approach zero. I find that both a sizable *mean* and a sizable *variance* are necessary to generate an empirically consistent interest elasticity of aggregate investment.

Further inspection of the mechanism shows that the *mean* and the *variance* play different roles. A decomposition of the interest elasticity between the extensive margin and the intensive margin indicates the different roles of the *mean* and the *variance*. Without a sizable *mean*, the unrealistically large interest elasticity is mainly from the unconstrained intensive margin.[2] Without a sizable *variance*, the unrealistically large interest elasticity is mainly from the oversensitive extensive margin. The underlying distribution of the extensive margin adjustment probability and intensive margin investment rate confirm these patterns.

Finally, I show the dynamic implications of the importance of having a sizable *mean* and a sizable *variance*. Without a sizable *mean*, there is neither state-dependency nor non-linearity of the state-dependency due to the micro lumpiness. The countercyclical changes in real interest rates hardly generate any state-dependency in aggregate investment. Without a sizable *variance*, the state-dependency and the non-linearity of the state-dependency are both unrealistically strong. The large interest elasticity to countercyclical changes in real interest rates smooths out the recessionary effects when the economy is in recession. Therefore, a sizable *mean* and a sizable *variance* are both quantitatively necessary for aggregate investment dynamics.

This paper is organized as follows. Section 2 presents the model and the solution method. Section 3 shows the interest elasticity of aggregate investment with respect to the *mean* and the *variance*, respectively. Section 4 further inspects the mechanism. Section 5 shows the impulse responses in alternative calibrations. Finally, Section 6 concludes.

2. The model

The economy consists of a fixed unit mass of firms $j \in [0, 1]$ which produce homogeneous output y_{jt} and a unit measure continuum of identical households who consume output and supply labor.

Technology: The production function is as follows:

$$y_{jt} = A_t z_{jt} k_{jt}^{\alpha} n_{jt}^{\nu}, \alpha + \nu < 1, \tag{2}$$

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where k_{jt} and n_{jt} indicates the idiosyncratic capital and labor employed by the firm j, and A_t is aggregate productivity. For each firm, the idiosyncratic TFP z_{jt} follows a log-normal AR(1):

$$log\left(z_{jt}\right) = -\left(1-\rho^z\right)\frac{\sigma^{z^2}}{2\left(1-\rho^{z^2}\right)} + \rho^z log\left(z_{jt-1}\right) + \epsilon_{jt}, \quad \epsilon_{jt} \sim N\left(0, \sigma^z\right). \tag{3}$$

Adjustment costs: The investment cost function includes two components: a direct cost i_{jt} and a fixed nonconvex capital adjustment cost ξ_{jt} paid in units of labor if the firm adjusts by more than a small proportion of their current capital stock ($|ak|$):

$$c\left(i_{jt}\right) = i_{jt} + \mathbf{1}_{\left(|i_{jt}|>ak_{jt}\right)} \cdot w_t \cdot \xi_{jt}, \quad \xi_{jt}. \tag{4}$$

Firm Optimization: I denote by $V^A\left(k_{jt}, z_{jt}; \Omega_t\right)$, $V^{NA}\left(k_{jt}, z_{jt}; \Omega_t\right)$, and $V\left(k_{jt}, z_{jt}; \Omega_t\right) \equiv E_{\xi_{jt}}\tilde{V}\left(k_{jt}, z_{jt}, \ \xi_{jt}; \Omega_t\right)$ the value functions of a firm with an active investment choice, without an active investment choice, and with expected draw of ξ_{jt}. The aggregate state $\Omega_t = \left(A_t, \Theta_t, \mu_t\left(k, z, \xi\right)\right)$ where Θ_t is a vector comprising the stochastic discount factor and wage at time t, and $\mu_t(k, z, \xi)$ is the distribution of firms. The value functions are as follows:

$$V^A\left(k_{jt}, z_{jt}; \Omega_t\right) = \max_{i,n}\left\{y_{jt} - w_t n_{jt} - c\left(i_{jt}\right) + \mathbb{E}\left[\Lambda_{t,t+1} V\left(k^*_{jt+1}, z_{jt+1}; \Omega_{t+1}\right)\right]\right\}, \tag{5}$$

$$V^{NA}\left(k_{jt}, z_{jt}; \Omega_t\right) = \max_{i\in[-ak,ak],n}\left\{y_{jt} - w_t n_{jt} - c\left(i_{jt}\right) + \mathbb{E}\left[\Lambda_{t,t+1} V\left(k^C_{jt+1}, z_{jt+1}; \Omega_{t+1}\right)\right]\right\}, \tag{6}$$

where the stochastic discount factor $\Lambda_{t,t+1}$ is derived from the household problem since households own all the firms. k^C_{jt+1} and k^*_{jt+1} are the constrained and non-constrained capital choices.

The firm will choose to pay the fixed cost if and only if $V^A\left(k_{jt}, z_{jt}; \Omega_t\right) - w_t\xi_{jt} > V^{NA}\left(k_{jt}, z_{jt}; \Omega_t\right)$. There is a unique threshold $\xi^*\left(k_{jt}, z_{jt}; \Omega_t\right)$ at which the firm breaks even:

$$\xi^*_t\left(k_{jt}, z_{jt}; \Omega_t\right) = \frac{V^A\left(k_{jt}, z_{jt}; \Omega_t\right) - V^{NA}\left(k_{jt}, z_{jt}; \Omega_t\right)}{w_t}. \tag{7}$$

If a firm draws a fixed cost ξ_{jt} below $\xi^*(k_{jt}, z_{jt}; \Omega_t)$ (which I denote as ξ^* *for short*), the firm pays the fixed cost and then actively adjusts its capital, otherwise it does not. The value function is:

$$V\left(k_{jt}, z_{jt}; \Omega_t\right) = -\frac{w_t\left(\xi^* + \underline{\xi}\right)}{2} + \frac{\xi^* - \underline{\xi}}{2\sqrt{3}\sigma_\xi} V^A\left(k_{jt}, z_{jt}; \Omega_t\right) + \left(1 - \frac{\xi^* - \underline{\xi}}{2\sqrt{3}\sigma_\xi}\right) V^{NA}\left(k_{jt}, z_{jt}; \Omega_t\right), \tag{8}$$

where $\underline{\xi} = \mu_\xi - \sqrt{3}\sigma_\xi$ is the lower bound of the fixed cost. The firm expects to pay the fixed cost when drawing ξ_{jt} lower than $\xi^*(k_{jt}, z_{jt}; \Omega_t)$. With probability $\frac{\xi^*-\underline{\xi}}{2\sqrt{3}\sigma_\xi}$, the firm chooses to actively invest, otherwise it stays inactive. Therefore, the capital stock evolves by the law of motion:

$$k_{jt+1} = \begin{cases} (1-\delta)\, k_{jt} + i^*_{jt} & \xi_{jt} < \xi^*\left(k_{jt}, z_{jt}; \Omega_t\right) \\ (1-\delta)\, k_{jt} + i^C_{jt} & otherwise \end{cases}. \tag{9}$$

Household optimization: Households' expected utility is as follows:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\eta}}{1-\eta} - \theta N_t\right),$$

subject to the budget constraint: $C_t + \frac{1}{R_t}B_t \leq B_{t-1} + w_t N_t + \Pi^F_t$. Here β is the discount factor of households, θ is the disutility of working, R_t is the real interest rate, B_t is one period bonds, w_t is

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the nominal wage, and Π_t^F is the nominal profits from all the firms. The first order conditions of consumption, labor, and bonds deliver:

$$w_t = -\frac{U_n(C_t, N_t)}{U_c(C_t, N_t)} = \theta C_t^{\eta}, \tag{10}$$

$$\Lambda_{t,t+1} = \frac{1}{R_t} = \beta \frac{U_c(C_{t+1}, N_{t+1})}{U_c(C_t, N_t)} = \beta \left(\frac{C_t}{C_{t+1}} \right)^{\eta}. \tag{11}$$

Equilibrium Definition:

Definition 1. *A Recursive Equilibrium for this economy is defined by a set of value functions and policy functions* $\{V(k, z; \Omega), V^A(k, z; \Omega), V^{NA}(k, z; \Omega), \xi^*(k, z;\Omega), k^*(k, z; \Omega), k^C(k, z; \Omega)\}$, *a set of quantity functions* $\{C(\Omega), N(\Omega), Y(\Omega), K(\Omega)\}$, *a set of price functions* $\{w(\Omega), \Lambda(\Omega), R(\Omega)\}$, *and a distribution* $\mu'(\Omega)$ *that solves the firms' and households' problems and satisfies market clearing such that:*

(i) Taking the price functions as given, the policy functions solve firms' optimization.

(ii) Taking the price functions as given, the quantity functions solve households' optimization.

(iii) Goods market clears: $Y(\Omega) = C(\Omega) + I(\Omega) + \Theta_k(\Omega)$, *where* $\Theta_k(\Omega)$ *is the total adjustment cost.*

Solution method: I follow the sequence space solution strategy as in Boppart et al. (2018) to solve the model which involves two parts. First, I solve the *Stationary Equilibrium* at the steady-state, which delivers all the steady-state equilibrium objects and provides the cross-sectional moments for the calibration. Second, I solve the *Transitional Equilibrium* starting from the *Stationary Equilibrium* and transit back to the same *Stationary Equilibrium*. The *Transitional Equilibrium* then provides the dynamic moments for the calibration and the impulse response functions. Details of the solution method are presented in the appendix.

3. Mean, variance, and the interest elasticity of investment

Benchmark calibration: I calibrate the benchmark model with mean and variance bundled (*henceforth, bundled model*) as in Khan and Thomas (2008) $\Big($the uniform distribution has support from 0 to an upper bound: $\xi_{jt} \sim U\left[0, \bar{\xi}\right]$, so $\mu_\xi = \sqrt{3}\sigma_\xi = \bar{\xi}/2\Big)$ to hit the target investment moments. For fixed parameters, I choose the discount factor $\beta = 0.99$ to match an annual interest rate of 4%., elasticity of intertemporal substitution $\eta = 1$ for log utility, leisure preference $\theta = 2$ to match a one-third working time share, capital exponent $\alpha = 0.25$ and the labor exponent $\nu = 0.60$ to match a labor share of two-thirds and decreasing returns to scale of 85%, quarterly capital depreciation $\delta = 0.026$, free capital adjustment region $a = 0.001$, and persistence of idiosyncratic TFP shock $\rho^z = 0.95$. For fitted parameters, I choose $\sigma^z = 0.05$ and $\bar{\xi} = 0.6$ to match the average investment rate (10.5%), the standard deviation of investment rates (0.13), the spike rate[3] (17%), and the partial equilibrium interest elasticity of aggregate investment (−5), reflecting the empirical moments as measured in Zwick and Mahon (2017) and Koby and Wolf (2020).[4]

How to measure the interest elasticity? The partial equilibrium interest elasticity of aggregate investment is defined by how aggregate investment, as yielded by the collective decisions of all heterogeneous firms, responds to an unexpected real interest rate shock[5]. For instance, −5 means when firms face an unexpected real interest rate cut of 1%, partial equilibrium aggregate investment increases by 5%. Quasi-experimental evidence in Zwick and Mahon (2017) and Koby and Wolf (2020) suggests this interest-elasticity should be about −5.

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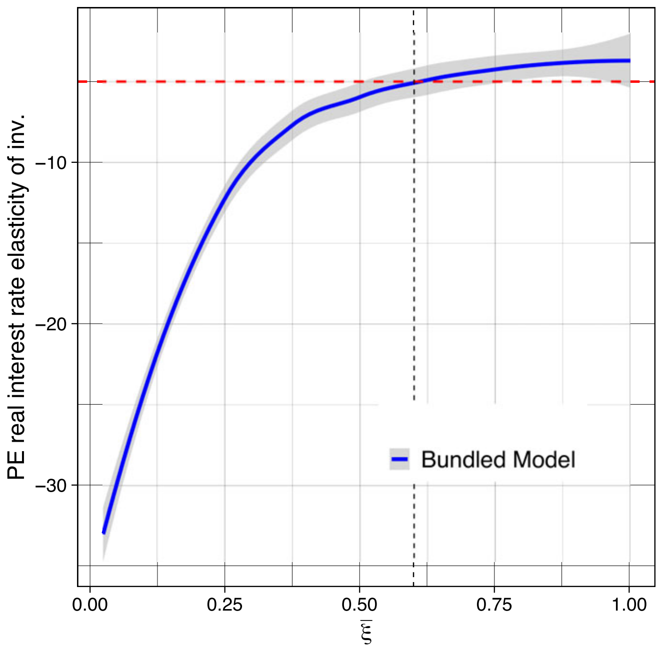

Figure 1. PE interest elasticity over $\bar{\xi}$.
Note: In the benchmark model, the uniform distribution starts from 0 to an upper bound: $\xi_{jt} \sim U[0, \bar{\xi}]$. It bundles the mean and the variance by $\bar{\xi}$: $\mu_\xi = \sqrt{3}\sigma_\xi = \bar{\xi}/2$. Therefore, increasing $\bar{\xi}$ increases both μ_ξ and σ_ξ simultaneously.

My exact numerical exercise in Sections 3 and 4 is to have the economy start at steady-state and hit by a one-time unexpected drop in real interest rate in the first period to generate the real interest rate series $\{r_t\}_{t=0}^T = \{r^*, r^* + \Delta r, r^*, \ldots, r^*\}$. I then feed the stochastic discount factor series $\{\Lambda_t\}_{t=0}^T = \left\{\frac{1}{1+r^*}, \frac{1}{1+r^*+\Delta r}, \ldots, \frac{1}{1+r^*}\right\}$ and the steady-state wage series $\{w_t\}_{t=0}^T = \{w^*\}$ into the partial equilibrium and solve for the aggregate investment series $\{I_t\}_{t=0}^T$. The partial equilibrium interest elasticity is then calculated as $\partial log I_t / \partial r_t$ at time $t = 1$. More specifically, I choose $\Delta r = -0.25\%$, therefore, a $\partial log I_1 / \partial r_1 = -5$ means such a one-time unexpected drop in the real interest rate boosts aggregate investment by 1.25%.

Upper bound $\bar{\xi}$ and interest elasticity: In Figure 1, I plot the model's interest elasticity as the choice of $\bar{\xi}$ varies from 0.025 to 1. First, the interest elasticity of aggregate investment is very sensitive to changes in the upper bound of the nonconvex adjustment costs.[6] Second, a relatively large value of $\bar{\xi} = 0.6$ gives an interest elasticity of -5. Conversely, if $\bar{\xi}$ is smaller, the aggregate investment will be oversensitive to interest rate changes. In Khan and Thomas (2008), $\bar{\xi} = 0.0083/4$, which will deliver an interest elasticity around -500. This will imply that lumpy investment is irrelevant for aggregate dynamics as in Khan and Thomas (2008). However, their irrelevance result is not consistent with the joint dynamics of aggregate investment and the real interest rate over the business cycle as shown in Winberry (2021).

Mean, variance, and interest elasticity: Now I depart from the benchmark calibration of $\bar{\xi} = 0.6$. Instead, I study two alternative groups of calibrations: One, fixing the variance $\sigma_\xi^* = \bar{\xi}/\sqrt{12} = 0.6/\sqrt{12}$ and varying the mean μ_ξ from 0 to $2\mu_\xi^* = \bar{\xi}$ to show how the interest-elasticity changes and two, fixing the mean $\mu_\xi^* = \bar{\xi}/2 = 0.6/2$ and varying the variance σ_ξ from 0 to $2\sigma_\xi^* = \bar{\xi}/\sqrt{3}$, to show how the interest elasticity changes. I use an identical quasi-experimental real interest rate shock as the one in the bundled model experiment above.

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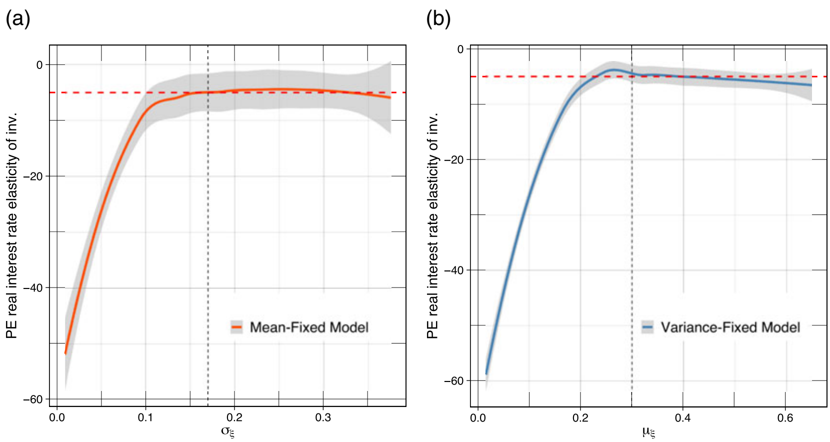

Figure 2. PE interest elasticity over μ_ξ and σ_ξ.
Note: The variance-fixed model fixes the variance by choosing $\sigma_\xi^* = \bar{\xi}/\sqrt{12}$ and the mean-fixed model fixes the mean by choosing $\mu_\xi^* = \bar{\xi}/2$. The two models are identical along both vertical dotted lines when $\mu_\xi = 0.3$ and $\sigma_\xi \simeq 0.17$.

The findings plotted in Figure 2 are very interesting. First, the interest elasticity is not solely determined by the expected size (*mean*) of the nonconvex cost. Unlike common claims that the interest elasticity is controlled by the expected size of the nonconvex cost, the uncertainty (*variance*) plays a role. In panel (a), even though the *mean* is set to be relatively large, when the *variance* approaches zero, the interest elasticity is massive. Given the fixed *mean*, the model hits the targeted interest elasticity when the *variance* is equal or larger to that of the bundled model. Second, the interest elasticity is not solely determined by the uncertainty (*variance*) of the nonconvex cost. In panel (b), even though the *variance* is fixed to be relatively large when the *mean* approaches zero, the interest elasticity is again massive. Given the fixed *variance*, the model hits the targeted interest elasticity when the *mean* is equal or larger to that of the bundled model.

What is the mechanism behind choosing the *mean* and the *variance*, respectively?

4. The mechanism

To further inspect the mechanism behind the differences between the *mean* and the *variance*, I demonstrate results from three models with three alternative calibrations: (1) the carefully calibrated bundled model (*Bundled*); (2) a mean-fixed model $\left(\mu_\xi^* = \bar{\xi}/\sqrt{12}\right)$ with zero variance $\left(Zero\text{-}\sigma_\xi\right)$; and (3) a variance-fixed model $\left(\sigma_\xi^* = \bar{\xi}/2\right)$ with zero mean $\left(Zero\text{-}\mu_\xi\right)$.

A decomposition of the interest elasticity: I first show the decomposition of the interest elasticity in all three models in terms of both extensive margin and intensive margin investment in Table 1 following the equation below:

$$\frac{d\sum I_j}{dr} = \frac{d\sum_{EM} I_j}{dr} + \frac{d\sum_{IM} I_j}{dr}, \tag{12}$$

where $d\sum I_j$ is aggregate investment, $d\sum_{EM} I_j$ is aggregate extensive margin investment, and $d\sum_{IM} I_j$ is aggregate intensive margin investment. The carefully calibrated *Bundled* model has an interest elasticity of −5.1, 96% of the investment response is from the extensive margin, and 4% is from the intensive margin. The *Zero*-σ_ξ model has an interest elasticity of about −600, which is almost entirely from the extensive margin. The *Zero*-μ_ξ model has an interest elasticity of −80, but which is mainly from the intensive margin.

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Table 1. A decomposition of the interest elasticity

	Bundled			Zero-σ_ξ			Zero-μ_ξ		
Decomp.	Total	EM	IM	Total	EM	IM	Total	EM	IM
Elasticity	−5.1	−4.9	−0.2	−601.8	−601.7	−0.1	−80.0	−4.4	−75.6
Percentage	—	96%	4%	—	100%	0%	—	5.5%	94.5%

Note: The *Bundled* model has the calibration that matches the micro investment moments. The Zero-σ_ξ deviates by setting σ_ξ to zero while all other parameters are unchanged. The Zero-μ_ξ deviates by setting μ_ξ to zero while all other parameters are unchanged. EM stands for the extensive margin and IM stands for the intensive margin.

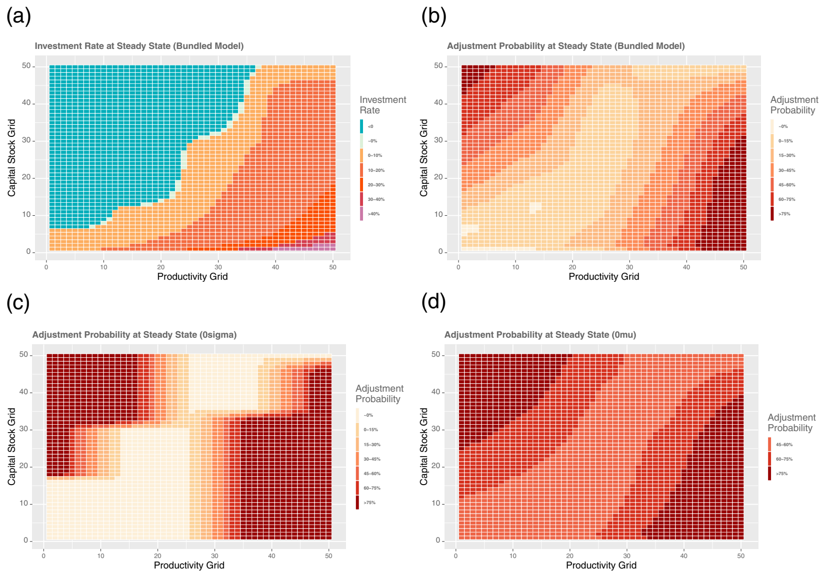

Figure 3. Distributions of the extensive margin and the intensive margin.
Note: This figure shows the distribution of firms' investment decisions at both the extensive margin and intensive margin conditional on their productivity and capital stock. Since the intensive margin distributions are not much changed across models, I only plot these for the *Bundled* model. We could decompose firms' investment decisions in two steps. Take the *Bundled* model for example; for firms at Productivity Grid 40 and Capital Grid 30, between 15% and 30% of these firms, according to the extensive margin rules in panel (b), would invest positively by 10% to 20%, according to the intensive margin rules in panel (a), and for firms at Productivity Grid 10 and Capital Grid 35, between 45% and 60% of these firms, according to panel (b), would disinvest, according to panel (a). Aggregate investment of the economy is, therefore, an integration of the extensive margin multiplying the intensive margin over the entire distribution.

This decomposition shows that the *mean* and the *variance* play different roles. Without a sizable *mean*, the response of the aggregate investment to the interest rate is mainly from the intensive margin. The intensive margin is much too sensitive to real interest rate changes, which delivers a falsely large interest elasticity of aggregate investment. Without a sizable *variance*, the response of aggregate investment to the interest rate is mainly from the extensive margin. The extensive margin is extremely sensitive to changes in real interest rates. A firm either chooses to pay μ_ξ^* and invest a lot or stay inactive when the real interest rate changes. Firms on the extensive margin choose to "all-in" which creates the unrealistically large interest elasticity.

Distributions of the extensive margin and the intensive margin: In Figure 3, I plot the interpolated distributions of the extensive margin (adjustment probability) and the intensive margin (investment rate conditional on adjustment) at the steady states using two-dimensional

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interpolation with respect to productivity and capital stock. Since the intensive margin distributions are not much changed across models, I only plot these for the *Bundled* model. Warmer and darker colors indicate higher investment rates and higher adjustment probabilities.

In panel (a) for the *Bundled* model, we see that higher productivity and lower capital firms invest more at the steady state. These firm will also invest more in response to the opportunity presented by the interest rate shock. In contrast, lower productivity and higher capital firms disinvest at the steady state. In panel (b), we see that the extensive margin distribution is layered from 0% probability of adjustment along the diagonal to higher probabilities away from the diagonal where productivity-capital mismatches are more severe. For the highest (lowest) productivity and lowest (highest) capital firms, the adjustment probabilities are larger than 75%. Conditional on draws of the nonconvex adjustment costs, a proportion of the high productivity and low capital firms to the right of the diagonal would then invest positively and the low productivity and high capital firms to the left of the diagonal would than disinvest, both according to the intensive margin rules in panel (a).

However, for the *Zero-*σ_ξ model and the *Zero-*μ_ξ model, the extensive margin distributions are entirely different. The extensive margin in the *Zero-*σ_ξ model shows a vertical sorting pattern that is sharply moving from a 0% probability of adjusting to almost a 100% possibility of adjusting. Even slight changes in interest rate would easily cause massive movements in the boundaries, boosting firms at the margins to dramatically change from 0% to >75% or vice versa. As a result, the extensive margin is extremely interest rate sensitive. In contrast, the extensive margin adjustment probability in the *Zero-*μ_ξ model is always higher than 45% and has much smaller variations. Changes in the interest rate barely cause movements in the boundaries - the extensive margin is not that sensitive to interest rate changes.

5. Implications of aggregate dynamics

To demonstrate the dynamics implications of the the mechanism behind the differences between the *mean* and the *variance*, I show the impulse responses of aggregate investment to aggregate TFP shocks in the same three alternative calibrations: (1) the carefully calibrated bundled model (*Bundled*); (2) the mean-fixed model $\left(\mu_\xi^* = \bar{\xi}/\sqrt{12}\right)$ with zero variance (*Zero-*σ_ξ); and (3) the variance-fixed model $\left(\sigma_\xi^* = \bar{\xi}/2\right)$ with zero mean (*Zero-*μ_ξ).

My exact numerical exercises start the economy at steady-state, then impose an unexpected aggregate productivity shock with persistence $\rho = 0.8$: $\{A_t\}_{t=0}^{T} = \{A^*, A^* + a, A^* + \rho a, A^* + \rho^2 a, \ldots, A^*\}$. I feed the aggregate productivity series into the general equilibrium and solve for the aggregate investment series $\{I_t\}_{t=0}^{T}$. To demonstrate the state-dependency and the nonlinearity of the state-dependency due to the micro lumpiness, I solve for four scenarios $\{a\} = \{+5\%, -5\%, +10\%, -10\%\}$ as representations for {*Small Boom*, *Small Recession*, *Large Boom*, *Large Recession*} for all three models, respectively. I then calculate the impulse responses relative to the steady-state in absolute value in percentages $\left\{100\% \times \left|\frac{I_t - I^*}{I*}\right|\right\}_{t=1}^{T}$.

The impulse responses are plotted in Figure 4. In panel (a) for the *Bundled* model, we first observe strong state-dependency: compared to a small recession, aggregate investment responds by 6% more (21.6% relative to 20.4%) in a small boom. Second, the state-dependency is non-linear due to the size of the TFP shock: compared to a large recession, aggregate investment responds by 11% more (43.9% relative to 39.5%) in a large boom. These results show how lumpy investment is consequential for macroeconomic analysis.

However, this is not the case for either the *Zero-*σ_ξ model or the *Zero-*μ_ξ model. In panel (b) for the *Zero-*σ_ξ model, the state-dependency and the non-linearity of the state-dependency are both

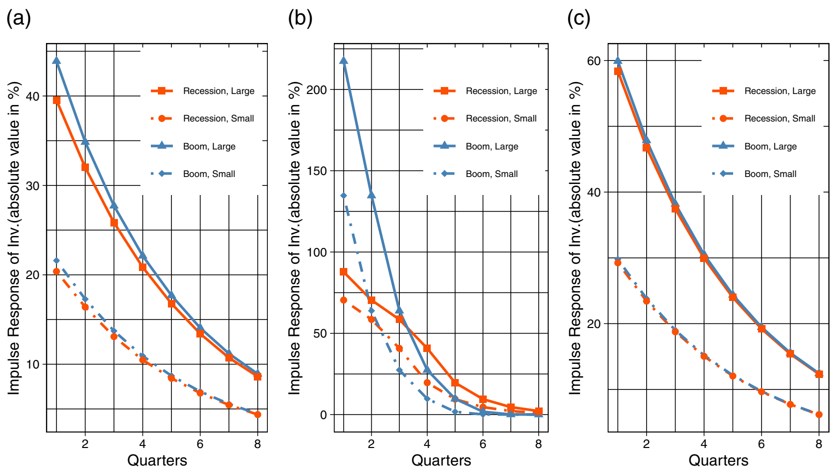

Figure 4. GE impulse responses to TFP shocks.
Note: The economy starts at steady-state $t = 0$ is hit by an unexpected aggregate productivity shock with persistence $\rho = 0.8$: $\{A_t\}_{t=0}^{T} = \left\{A^*, A^* + a, A^* + \rho a, A^* + \rho^2 a, \ldots, A^*\right\}$. Scenarios {*Small Boom*, *Small Recession*, *Large Boom*, *Large Recession*} for all three models have a corresponding TFP shocks $\{a\} = \{+5\%, -5\%, +10\%, -10\%\}$, respectively. The impulse responses relative to the steady-state are absolute value in percentages $\left\{100\% \times \left|\frac{I_t - I^*}{I*}\right|\right\}_{t=1}^{T}$.

unusually strong as well as are the magnitudes of the impulse responses. Aggregate investment responds by 83% more(134% relative to 73%) and by 153% more (220% relative to 87%) in a small/larger boom relative to a small/large recession, respectively. This is all because the extensive margin is extremely sensitive to the countercyclical changes in real interest rates which smooth out the recessionary effects when the economy is in recessions. In panel (c) for the *Zero-*μ_ξ model, in contrast, there is almost no state-dependency or non-linearity of the state-dependency. Since the response of aggregate investment to the interest rate is mainly from the intensive margin, the countercyclical changes in real interest rates hardly generate any state-dependency in aggregate investment.

6. Concluding remarks

Nonconvex capital adjustment costs play an essential role in generating data-consistent lumpy investment behaviors. The literature usually assumes a uniform distribution for the nonconvex adjustment cost with support from 0 to an upper bound, which does not distinguish the separate roles played by the *mean* and the *variance* of the distribution. In this paper, I show that both a sizable *mean* and a sizable *variance* are necessary for lumpy investment models to generate an empirically consistent interest elasticity of aggregate investment. The *mean* governs the degree to which the extensive margin accounts for aggregate investment dynamics. In contrast, the *variance* controls how sensitive the extensive margin is to interest rate changes. Therefore, both are quantitatively necessary in a reasonably calibrated lumpy investment model.

There are two potential directions of future research. First, more realistic estimations of the *mean* and the *variance* using microdata on firm-level investment to better represent the expected size and the uncertainty of the nonconvex capital adjustment cost faced by firms. Second, the separate roles of the *mean* and the *variance* potentially matter for other dynamic models with nonconvex adjustment costs such as firm entry and exit, worker hiring and firing, trade entry and exit, inventory dynamics, and many others.

Acknowledgements. This work is generously supported by a grant from the Swiss National Science Foundation under project ID "New methods for asset pricing with frictions".

Notes

1 Consider the dynamics of aggregate investment model, general equilibrium feedbacks and monetary policy are directly reflected in real interest rate changes and investment stimulus policies are isomorphic to such changes through the marginal prices of capital.
2 This unconstrained intensive margin is usually constrained by a quadratic adjustment cost in recent literature. To keep this note dedicated to disciplining the mean and the variance, I leave out the quadratic adjustment cost.
3 Spike rate is defined as the proportion of investment rate larger than 20% in a quarter.
4 To make the results more intuitive, I only include the nonconvex fixed cost and did not include the quadratic adjustment cost which usually serves to constrain extreme investment behaviors. As a result, the model cannot exactly match all the micro-investment moments as in Zwick and Mahon (2017).
5 By partial equilibrium, I assume the firms do not take consideration of wage changes as a feedback loop from household decisions. This is consistent with the reduced form estimation from the partial equilibrium perspective.
6 In contrast, I show the PE wage elasticity of aggregate investment over $\bar{\xi}$ in the appendix. Since changes in wage are not directly (but indirect) changes in the costs of and the returns to capital. the wage elasticity of aggregate investment is not sensitive to $\bar{\xi}$ at all.
7 By partial equilibrium, I assume the firms not taking consideration of real interest rate changes as a feedback loop from household decisions. This is consistent with the reduced form estimation from the partial equilibrium perspective.

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Appendix A. Details of the solution methods

Part I: Solving the stationary equilibrium

I first assume the economy at steady-state. This part is very similar as solving an Aiyagari model. The only difference is firms own capital which is subject to adjustment costs. I search for equilibrium wage to clear the labor market. The algorithm is as following:

Step. 1. Guess an equilibrium wage;

Step. 2. Solve the firm's problem using Value Function Iteration;

Step. 3. Calculate aggregate variables from the firm distribution using Young (2010);.

Step. 4. Update the wage with a given weight and return to Step 2 until convergence.

After the convergence, I have the stationary equilibrium aggregate prices $\Omega^* = \{\Lambda^* = \beta, w^* = w^*\}$, aggregate quantities $\{C^*(\Omega^*), N^*(\Omega^*), Y^*(\Omega^*), K^*(\Omega^*)\}$, firm value functions $\{V^*(k, z; \Omega^*)$, $V^{A^*}(k, z; \Omega^*)$, $V^{NA^*}(k, z; \Omega^*)$, policy functions $\xi^{**}(k, z; \Omega^*)$, $k'^*(k, z; \Omega^*)$, $l'^*(k, z; \Omega^*)\}$, and distribution $\mu(k, z; \Omega^*)$ at the stationary equilibrium state.

Part II: Solving the transitional equilibrium

With the stationary equilibrium solutions in hand, I now move to the solution of the transitional equilibrium using a shooting algorithm. The key assumption here is that after a sufficiently

https://doi.org/10.1017/S1365100522000086 Published online by Cambridge University Press

long enough time, the economy will always converge back to its initial stationary equilibrium after any temporary and unexpected (MIT) shocks. The following steps outline the shooting algorithm:

Step. 1. Fix a sufficient long transition period t = 1 to t = T (say 200);

Step. 2. Guess or given a sequence of aggregate price $\{w_t, \Lambda_t\}$ of length T such that the initial prices $\{w_1 = w^*, \Lambda_1 = \Lambda^*\}$ (just simply assuming all the prices stay at steady state works well) and terminal prices $\{w_T = w^*, \Lambda_T = \Lambda^*\}$. Provide a predetermined shock process of interest, that is, $\{A_t\}$. This implies a time series for the aggregate state $\{\Omega_t\}_{t=1}^{T}$. The aggregate state is just time t.

Step. 3. I know that at time T, the economy is back to its steady state. I have the steady state value function $V\left(k, z; \Omega_T\right) = V^*\left(k, z; \Omega^*\right)$ in hand for time T. I solve for the firms' problem by ***backward induction*** given $V\left(k, z; \Omega_T\right)$ and $\{w_{T-1}, \Lambda_{T-1}\}$. This yields the firm value function $V\left(k, z; \Omega_{T-1}\right)$ and associated policy functions for capital $k'\left(k, z; \Omega_{T-1}\right)$ and labor $l\left(k, z; \Omega_{T-1}\right)$. By iterating backward, I solve the whole series of both policy functions $\left\{k'(k, z; \Omega_t)\right\}_{t=1}^{T}$ and $\left\{l'(k, z; \Omega_t)\right\}_{t=1}^{T}$.

Step. 4. Given the policy functions and the steady state distribution as the initial distribution $\mu\left(k, z; \Omega_1\right) = \mu\left(k, z; \Omega^*\right)$, I use ***forward simulation*** with the non-stochastic simulation in Young (2010) to recover the whole path $\left\{\mu(k, z; \Omega_t)\right\}_{t=1}^{T}$.

Step. 5. Using the distribution $\left\{\mu(k, z)\right\}_{1}^{T}$, I obtain all the ***aggregate quantities***: aggregate output $\{Y\}_{t=1}^{T}$, aggregate investment $\{I\}_{t=1}^{T}$, aggregate labor demand $\{N\}_{t=1}^{T}$, and aggregate capital adjustment costs $\{\Theta_k\}_{t=1}^{T}$, we could calculate aggregate adjustment costs $\{\Theta_p\}_{t=1}^{T}$. I then use the goods market clearing condition to calculate aggregate consumption $\{C\}_{t=1}^{T}$. I then calculate the *Excessive Demand* $\{\Delta C\}_{t=1}^{T}$ by taking the differences between currently iterated $\{C\}_{t=1}^{T}$ and the previous iteration $\{C_{old}\}_{t=1}^{T}$.

Step. 6. Given all the aggregate quantities in the previous step and the *Excessive Demand* $\{\Delta C\}_{t=1}^{T}$, I update all the ***aggregate prices***. I update all equilibrium prices with a line search: $X_t^{new} = speed \cdot f_X\left(\{\Delta C\}_{t=1}^{T}\right) + (1 - speed) \cdot X_t^{old}$.

Partial equilibrium: Step 1-5 for given sequences of shocks, that is, real interest rate shock which changes the stochastic discount factor $\left\{w_t = w^*, \Lambda_t = \frac{1}{1+r_t}\right\}_{t=1}^{T}$ for given $\{r_t\}_{t=1}^{T}$ series.

General equilibrium: Step 1-6 for given sequences of shocks, that is, aggregate productivity shock for given $\{A_t\}_{t=1}^{T}$ series. Repeat Steps 2-6 until X_t^{new} and X_t^{old} are close enough. Updating all prices in all periods simultaneously reduces the computation burden dramatically.

In all the experiments, I set T = 200, and a step size of 0.1 (only for the $Zero - \sigma_\xi$ model, I choose a step size of 0.0001) to ensure convergence, with the necessary distance between X_t^{new} and X_t^{old} smaller than 1e-7. I also tested with various choices of T from 50 to 400 to ensure that the choice of T = 200 does not affect the accuracy of the solution.

B. Upper bound $\bar{\xi}$ and wage elasticity of investment

In contrast, I show the PE wage elasticity of aggregate investment is not sensitive to $\bar{\xi}$. Since changes in wage is not directly (but indirect) changes in the costs of and the returns to capital. the wage elasticity of aggregate investment is not sensitive to $\bar{\xi}$ at all.

How to measure the wage elasticity? The partial equilibrium wage elasticity of aggregate investment is defined by how aggregate investment, as yielded by the collective decisions of all

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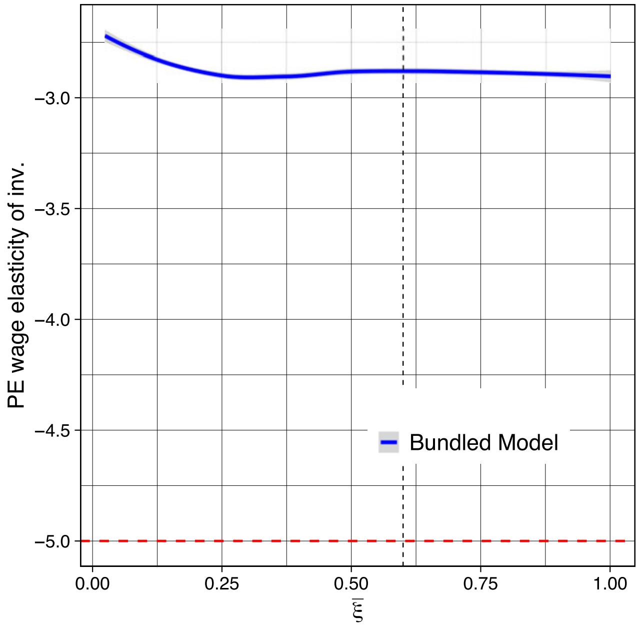

Figure 5. PE wage elasticity over $\bar{\xi}$.
Note: In the benchmark model, the uniform distribution starts from 0 to an upper bound: $\xi_{jt} \sim U[0, \bar{\xi}]$. The red dash line is the PE interest rate elasticity of investment in the data which is about −5.

heterogeneous firms, responds to an unexpected wage shock[7]. For instance, -5 means when firms face an unexpected wage cut of 1%, the partial equilibrium aggregate investment increases by 5%.

The exact numerical exercise in this subsection is to have the economy start at steady-state, hit by a one-time unexpected drop in wage at the first period to generate the wage series $\{w_t\}_{t=0}^{T} = \{w^*, w^* + \Delta w, \ldots, w^*\}$. I then feed the steady-state stochastic discount factor series $\{\Lambda_t\}_{t=0}^{T} = \left\{\frac{1}{1+r^*}\right\}$ and the wage series $\{w_t\}_{t=0}^{T}$ into the partial equilibrium transaction and solve for the aggregate investment series $\{I_t\}_{t=0}^{T}$. The partial equilibrium wage elasticity is then calculated as $\partial logI_t/\partial r_t$ at time $t = 1$. More specifically, I choose $\Delta w = -1\% * w^*$, therefore, a $\partial logI_1/\partial w_1 = -5$ means such a one-time 1% unexpected drop in wage boosts 5% aggregate investment increment.

In Figure 5, I plot the model's wage elasticity against the choice of $\bar{\xi}$ from 0.025 to 1. From the figure we could first tell that wage elasticity of aggregate investment is not sensitive to changes in the upper bound of the nonconvex adjustment costs. Second, aggregate investment is not as sensitive to changes in wage than to changes in real interest rate.

Cite this article: Fang M (2023). "A Note on nonconvex adjustment costs in lumpy investment models: Mean versus variance." *Macroeconomic Dynamics* **27**, 1166–1177. https://doi.org/10.1017/S1365100522000086