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# ANALYTIC EVALUATION OF CERTAIN CHARACTERISTIC CLASSES(1)

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1. **Introduction.** We have an alternative proof of the following result of Kervaire [2]:

THEOREM. *Let $V \to M$ be a real vector bundle with fibre dimension $n \geq 4k+1$ over a compact $4k$-manifold. Suppose $V$ restricted to $M-\{x\}$ is trivial. Choose a Riemann structure for $V$ and an orthonormal frame for $V$ restricted to $M-\{x\}$. Thus the obstruction to extending the frame smoothly over $M$ is an element $\lambda$ in $\pi_{4k-1}(SO(n)) \cong Z$. Then up to sign the evaluation of the $k^{th}$ Pontrayagin class $P_k$ on $M$ is $a_k(2k-1)! \cdot \lambda$, where $a_k$ is 1 or 2 depending upon whether $k$ is even or odd.*

Like the original proof, our proof depends ultimately upon the computations of homotopy groups of the classical groups of R. Bott. Unlike the original algebraic topological proof, our proof is analytic and computational. The scheme is: first, to couch the Pontrayagin classes in analytic terms; second, to exhibit explicit generators of $\pi_{4k-1}(SO(a_k 2^{2k-1})) \cong \pi_{4k-1}(SO(n))$, $k \geq 3$; and third, to compute.

Enlightening conversations with K. Y. Lam contributed to the development of this paper.

2. **Notation.** A metric connection is a connection $\nabla$ in $V$ chosen with respect to a Riemann structure $\mathscr{G}$ such that $\nabla\mathscr{G}=0$. The Pontrayagin classes can be expressed in terms of the elementary symmetric functions of the curvature matrix $\Omega$ of any such connection. Since $V$ restricted to $M-\{x\}$ is trivial, only the highest Pontrayagin class is nontrivial. It follows that an alternative technique, using the traces of powers of $\Omega$ gives the same Pontrayagin class structure. Thus the $k^{th}$ Pontrayagin class is represented by the $4k$-form:

$$P_k = \frac{1}{(2\pi)^{2k}(2k)!} \operatorname{tr} \Omega^{2k}$$

Let $\omega$ denote the local connection forms obtained from a metric connection $\nabla$ and

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a local frame $\xi_i$; $i = 1, 2, \ldots, n$; for $V$ orthonormal with respect to $\mathscr{G}$. That is, $\omega_i^j = \mathscr{G}(\nabla\xi_i, \xi_j)$, so $\Omega_i^j = d\omega_i^j - \sum_{k=1}^{n} \omega_i^k \wedge \omega_k^j$, or simply $\Omega = d\omega - \omega^2$.

An unpublished result of A. Deicke follows.

LEMMA. *Locally we may define* $(4l-1)$*-forms* $\varphi_{4l-1}$:

$$\varphi_{4l-1} = (2l)! \sum_{i=1}^{2l} \frac{(i-1)!}{(2l+i-1)!} \sum_{\Sigma\, m_j = 2l-i} \operatorname{tr}(\omega\Omega^{m_1}\omega^2\Omega^{m_2} \cdots \omega^2\Omega^{m_i})$$

*Then* $d\varphi^{4l-1} = \operatorname{tr}\, \Omega^{2l}$.

This expresses the local triviality of the closed $4l$-form $\operatorname{tr}\, \Omega^{2l}$. The proof is straightforward but tedious.

3. **Algebraic topological considerations.** First we note that in proving the theorem we may as well assume that the fibre dimension $n$ of $V$ is at least $a_k 2^{2k-1}$, $k \geq 3$, since the Pontrayagin classes ignore the addition of trivial bundles.

Second, recall that the Hurwitz-Radon theorem provides $4k-1$ skew anti-commuting $(a_k 2^{2k-1} \times a_k 2^{2k-1})$-matrices $\{M_i\}$, $i = 2, \ldots, 4k$, with $M_i^2 = -I$ (again and hereafter $a_k = 1$ if $k$ is even and $a_k = 2$ if $k$ is odd). The quaternionic multiplication in $\mathscr{R}^4$ and the Cayley multiplication in $\mathscr{R}^8$ arise from matrices of this type. We will restrict our attention to such sets of matrices with the additional condition that the product $M_2 M_3 \cdots M_{4k}$ (necessarily $\pm I$) is $+I$. Thus we choose one of the two (up to isomorphism) irreducible Clifford modules of dimension $4k-1$. The quaternionic and Cayley multiplications are representatives of the alternative choice. The iterative scheme for construction of the matrices is given intrinsically by Zvengrowski [6].

Let $M_1 = I$ and $\{M_i\}$; $i = 2, \ldots, 4k$; be matrices of the above type. Consider the $(4k-1)$-sphere $S^{4k-1} \subset \mathscr{R}^{4k}$ as the set $\{(x_1, x_2, \ldots, x_{4k}) \mid x_1^2 + x_2^2 + \cdots + x_{4k}^2 = 1\}$. There are natural maps

$$\sigma_k : S^{4k-1} \to SO(a_k 2^{2k-1})$$

defined by

$$\sigma_k(x_1, \ldots, x_{4k}) = \sum_{i=1}^{4k} x_i M_i$$

That the map $\sigma_1$ is a generator of $\pi_3(SO(4)) \simeq Z+Z$ and leads to a generator of $\pi_3(SO(s)) \simeq Z$, $s \geq 5$, under inclusion follows from Steenrod [4, pp. 115–117]. Similarly, that the map $\sigma_2$ is a generator of $\pi_7(SO(8)) \simeq Z+Z$ and leads to a generator of $\pi_7(SO(t)) \simeq Z$, $t \geq 9$, under inclusion follows from Toda, Saito, and Yokota [5]. In higher dimensions, $\sigma_k$ is a generator of $\pi_{4k-1}(a_k 2^{2k-1}) \simeq Z$, $k \geq 3$, and this is a consequence of the Periodicity Theorem as presented by Atiyah, Bott, and Shapiro [1]. Roughly speaking, just as $S^{4k-1}$ is the intrinsic join of $S^3$ or $S^7$ with $S^7$

several times, $\sigma_k$ is the "join" in the sense of Zvengrowski of either $\sigma_1$ or $\sigma_2$ with $\sigma_2$ several times.

We will prove the theorem making use of the above explicit generators of $\pi_3(SO(s))\simeq Z, s\geq 5$; $\pi_7(SO(t))\simeq Z, t\geq 9$; and $\pi_{4k-1}(SO(a_k 2^{2k-1}))\simeq\pi_{4k-1}(SO(n))\simeq Z$, $k\geq 3$, $n\geq a_k 2^{2k-1}$.

4. **Proof of the Theorem.** The evaluation of $P_k$ is, of course, independent of the choice of connection. Thus we may choose a special metric and metric connection pair. Describe $V$ near $x$ as $\mathscr{R}^n\times\mathscr{R}^{4k}$. Then choose the metric and metric connection to be Euclidean near $x$ and arbitrary elsewhere. Let $\{x_i\}$ be the usual Euclidean frame for $\mathscr{R}^n$, so near $x$ $\nabla x_i=0$ and $\mathscr{G}(x_i, x_j)=\delta_{ij}$. Let $\{\xi_i\}$ be an orthonormal frame for $V$ restricted to $M-\{x\}$. Connection forms defined over $M-\{x\}$ are $\omega_i^j=\mathscr{G}(\nabla\xi_i, \xi_j)$. By our choice of connection, $\Omega=0$ near $x$. Let $B^{4k}$ be the unit ball in $M$ with centre at $x$, so $\partial B^{4k}=S^{4k-1}$. By Stokes' theorem, the lemma, and the local vanishing of $\Omega$,

$$\int_M P_k = \frac{1}{(2\pi)^{2k}(2k)!}\int_M \operatorname{tr}\,\Omega^{2k}$$

$$= \frac{1}{(2\pi)^{2k}(2k)!}\int_{M-B^{4k}}\operatorname{tr}\,\Omega^{2k} = \frac{1}{(2\pi)^{2k}(2k)!}\,\frac{(2k)!\,(2k-1)!}{(4k-1)!}\int_{S^{4k-1}}\operatorname{tr}\,\omega^{4k-1}$$

Now suppose the comparison of the frames $\{\xi_i\}$ and $\{x_i\}$ over $S^{4k-1}$ gives rise to the integer $\lambda\in Z\simeq\pi_{4k-1}(SO(n))$. Then the comparison map is homotopic to the composition

$$S^{4k-1}\xrightarrow{\deg\lambda}S^{4k-1}\xrightarrow{\sigma_k\oplus I}SO(n)$$

Thus $\int_M P_k$ will be $\lambda$ times the number which would be obtained from a generating frame corresponding to $\sigma_k$.

Thus it remains only to show that if the comparison map is the generating map, that is, if $\omega=(d\sigma_k\oplus 0)(\sigma_k\oplus I)^{-1}$ over $S^{4k-1}$, then the evaluation of $\int_M P_k$ is $a_k(2k-1)!$.

Therefore suppose

$$\omega = \left[\left(\sum_{i=1}^{4k} dx_i M_i\right)\oplus O\right]\left[\left(x_1M_1-\sum_{j=2}^{4k}x_jM_j\right)\oplus I\right]$$

Since we will ultimately evaluate the trace of $\omega^{4k-1}$, we disregard the $O$ block of $\omega$. Rearranging terms leads to

$$\omega = \sum_l x_l\,dx_l M_1+\sum_{i<j}(x_i\,dx_j-x_j\,dx_i)M_iM_j$$

Define 2-forms $\{a_{ij}\}$; $i, j=1,\ldots,4k$; on $S^{4k-1}$ by

$$a_{ij} = \sum_l (x_l\,dx_i-x_i\,dx_l)\wedge(x_l\,dx_j-x_j\,dx_l)$$

It is easy to check that $a_{ij}=-a_{ji}$ and $a_{ij}\wedge a_{kl}=0$ unless $i, j, k, l$ are distinct.

Furthermore,

$$\omega^2 = 2\sum_{i<j} a_{ij} M_i M_j$$

It follows that

$$\omega^{4k-2} = \sum_{1<i_2,i_3,\ldots,i_{4k-2}} a_{1i_2} \wedge a_{i_3i_4} \wedge \cdots \wedge a_{i_{4k-3}i_{4k-2}} M_{i_2} M_{i_3} \cdots M_{i_{4k-2}}$$
$$+ \sum_{1<j_2,j_3,\ldots,j_{4k-1}} a_{j_2j_3} \wedge \cdots \wedge a_{j_{4k-2}j_{4k-1}} M_{j_2} M_{j_3} \cdots M_{j_{4k-1}}$$

Finally, it may be shown that the symmetric part $(\omega^{4k-1})_{sym}$ of $\omega^{4k-1}$ is

$$(\omega^{4k-1})_{\mathrm{sym}} = \tfrac{1}{2} \sum_{i \in \mathrm{perm}\{1,\ldots,4k\}} \mathrm{sign}(i)(x_{i_1}\, dx_{i_2} - x_{i_2}\, dx_{i_1}) \wedge a_{i_3i_4} \wedge \cdots \wedge a_{i_{4k-1}i_{4k}} I$$

Now

$$\tfrac{1}{2} \sum_{i \in \mathrm{perm}\{1,\ldots,4k\}} \mathrm{sign}(i)(x_{i_1}\, dx_{i_2} - x_{i_2}\, dx_{i_1}) \wedge a_{i_3i_4} \wedge \cdots \wedge a_{i_{4k-1}i_{4k}}$$
$$= (4k-1)! \sum_{j \in cyclic\, \mathrm{perm}\{1,\ldots,4k\}} \mathrm{sign}(j) x_{j_1}\, dx_{j_2} \wedge dx_{j_3} \wedge \cdots \wedge dx_{j_{4k}} = (4k-1)!\, \Delta_{4k-1}$$

where $\Delta_{4k-1}$ is the standard orientation form induced on $S^{4k-1}$ via the standard embedding in $\mathscr{R}^{4k}$. Recall

$$\int_{S^{4k-1}} \Delta_{4k-1} = 2 \cdot \pi^{2k}$$

Since tr $\omega^{4k-1} = (4k-1)!\, a_k 2^{2k-1}\, \Delta_{4k-1}$,

$$\frac{(2k-1)!}{(2\pi)^{2k}(4k-1)!} \int_{S^{4k-1}} \mathrm{tr}\, \omega^{4k-1} = \frac{(2k-1)!}{(2\pi)^{2k}(4k-1)!} (4k-1)!\, a_k 2^{2k-1} \cdot 2 \cdot \pi^{2k} = a_k(2k-1)!$$

and the proof is complete.

5. **Remarks.** We conclude with a

COROLLARY. *Suppose $V$ has fibre dimension $4k$ and is trivial over $M-\{x\}$. Then generators of $\pi_{4k-1}(SO(4k)) \simeq Z+Z$ may be chosen so that the integers associated with the obstruction to a global frame for $V$ are:*

$$\begin{array}{lll} (\tfrac{1}{2}\chi + \{2a_k(2K-1)!\}^{-1} P_k, & \{a_k(2K-1)!\}^{-1} \rho_k) & k=1,2 \\ (\tfrac{1}{2}\chi, \ \{a_k(2K-1)!\}^{-1} P_k) & & k \geq 3 \end{array}$$

*where $\chi$ denotes the Euler characteristic of $V$.*

**Proof.** Recall that the Euler characteristic and Pontrayagin characteristic of $T(S^{4k})$ are 2 and 0. According to Steenrod [4, p. 121], the characteristic map of $S^{4k}$ generates the kernel $(\simeq Z)$ of $\pi_{4k-1}(SO(4k)) \simeq Z+Z \to \pi_{4k-1}(SO(4k+1) \simeq Z$.

*Case* $k=1$, 2. Choose as generators: $(1, 0) \sim$ characteristic map of $S^{4k}$; and $(0, 1) \sim \sigma_k$. Thus $\chi(1, 0) = 2$ and $P_k(1, 0) = 0$. Since we have chosen $M_2 M_3 \cdots M_{4k} = +I$ (not $-I$), $\sigma_k^{-1} \sim (0, -1)$ is representative of the quaternionic or Cayley multiplication. In view of the expositions of $\pi_3(SO(4))$ and $\pi_7(SO(8))$ given by Steenrod,

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and Toda, Saito, and Yokota, the image of a generator of $\pi_{4k-1}(SO(4k-1))\cong Z$ under inclusion in $\pi_{4k-1}(SO(4k))$ is $(1, 2)$ up to sign. Thus $0=\chi(1, 2)=2+2\chi(0, 1)$, so $\chi(0, 1)=-1$. The theorem provides $P_k(0, 1)=a_k(2k-1)!$.

*Case* $k\geq 3$. Consider the exact sequence

$$\pi_{4k-1}(SO(4k-1))\cong Z \xrightarrow{f} \pi_{4k-1}(SO(4k+1))\cong Z \longrightarrow \pi_{4k-1}(V_{4k+1,2})\cong Z_2$$
$$\longrightarrow \pi_{4k-2}(SO(4k-1))\cong Z_2 \longrightarrow \pi_{4k-2}(SO(4k+1))=0$$

That $\pi_{4k-2}(SO(4k+1))=0$ follows from the Periodicity Theorem, and the other homotopy groups of rotation groups may be found in Kervaire [3]. It follows that $f$ is an isomorphism. We choose generators of $\pi_{4k-1}(SO(4k))$ as follows: $(1, 0)\sim$the characteristic map of $S^{4k}$, as before; and $(0, 1)\sim$the image under inclusion of the generator of $\pi_{4k-1}(SO(4k-1))$ chosen so that $P_k(0, 1)$ is positive. Thus $\chi(1, 0)=2$, $\chi(0, 1)=0$, $P_k(1, 0)=0$, and $P_k(0, 1)=a_k(2k-1)!$.

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