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ERGODICITY AND STABILITY OF ORBITS OF UNBOUNDED SEMIGROUP REPRESENTATIONS

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Abstract

We develop a theory of ergodicity for unbounded functions $\phi : J \to X$, where J is a subsemigroup of a locally compact abelian group G and X is a Banach space. It is assumed that ϕ is continuous and dominated by a weight w defined on G. In particular, we establish total ergodicity for the orbits of an (unbounded) strongly continuous representation $T : G \to L(X)$ whose dual representation has no unitary point spectrum. Under additional conditions stability of the orbits follows. To study spectra of functions, we use Beurling algebras $L^1_w(G)$ and obtain new characterizations of their maximal primary ideals, when w is non-quasianalytic, and of their minimal primary ideals, when w has polynomial growth. It follows that, relative to certain translation invariant function classes $\mathscr{F}$, the reduced Beurling spectrum of ϕ is empty if and only if $\phi \in \mathscr{F}$. For the zero class, this is Wiener's tauberian theorem.

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1. Introduction

Throughout this paper G denotes a locally compact abelian topological group with a fixed Haar measure μ and dual group $\hat{G}$. We use additive notation for G and multiplicative for $\hat{G}$. The *Fourier transform* of a function $f \in L^1(G)$ is then defined by $\hat{f}(\gamma) = \int_G \gamma(-t) f(t)\, d\mu(t)$ for $\gamma \in \hat{G}$.

By J we denote a closed sub-semigroup of G with non-empty interior such that $G = J - J$ and by X a complex Banach space. For a function $\phi : J \to X$, its *translate* ϕ_h and *difference* $\Delta_h\phi$ by $h \in J$ are given by $\phi_h(t) = \phi(t+h)$ and $\Delta_h\phi = \phi_h - \phi$. If $h = (h_1, \ldots, h_n) \in J^n$, then $\Delta_h\phi = \Delta_{h_n}(\Delta_{h_{n-1}} \cdots (\Delta_{h_1}\phi)\cdots)$, $n \in \mathbb{N}$; if $h_j = t$, for all $1 \le j \le n$, we write $\Delta_t^n \phi$ instead of $\Delta_h \phi$. Finally, $|\phi|$ will stand for the function defined by $|\phi|(t) = \|\phi(t)\|$ for $t \in J$.

Weights are functions $w : G \to \mathbb{R}$ which, unless otherwise stated, are assumed to satisfy the following conditions:

(1.1) w is continuous, $w(t) \ge 1$ and $w(s+t) \le w(s)w(t)$ for all $s, t \in G$;

(1.2) $w(-t) = w(t)$ for every $t \in G$;

(1.3) $$\sum_{n=1}^{\infty} \frac{1}{n^2} \log w(nt) < \infty \quad \text{for every } t \in G;$$

(1.4) $$\frac{\Delta_s w}{w} \in C_0(G) \quad \text{for every } s \in G;$$

(1.5) $$\sup_{t \in G} \frac{|\Delta_s w(t)|}{w(t)} \to 0 \quad \text{as } s \to 0 \text{ in } G.$$

The symmetry condition (1.2) is only used to simplify the exposition. Without it, the definition of the Beurling spectrum is modified as in [8, (1.9)]. Moreover, if w satisfies all these conditions except (1.2) then $w(t) + w(-t)$ satisfies all of them. Condition (1.3) is the Beurling-Domar condition (see [14])and a weight satisfying (1.3) is called *non-quasianalytic*. In the case that w is bounded we will assume $w = 1$, as this will cause no loss of generality. For certain results, as we shall see, condition (1.4) may be weakened. We can also pass to equivalent weights. Functions $w, w_1 : G \to \mathbb{R}$ are *equivalent* if $c_1 w(t) \le w_1(t) \le c_2 w(t)$ for some $c_1, c_2 > 0$ and all $t \in G$. The function $w(t) = (1 + |\sin t|)(1 + |t|)$ on $\mathbb{R}$ does not satisfy (1.4), but is equivalent to $w_1(t) = 1 + |t|$ which does satisfy (1.4).

Frequently we will also assume the existence of $N \in \mathbb{Z}_+$ such that

(1.6) $$\lim_{|m| \to \infty} \frac{w(mt)}{1 + |m|^{N+1}} = 0 \quad \text{for all } t \in G; \text{ and}$$

(1.7) $$\inf_{m \in \mathbb{Z}} \frac{w(mt)}{|m|^N} > 0 \quad \text{for some } t \in G.$$

We will say that a weight w has *polynomial growth* of order $N \in \mathbb{Z}_+$ if it satisfies (1.6)–(1.7). The Beurling-Domar condition (1.3) follows from (1.6).

A function $\phi : J \to X$ is called *w-bounded* if ϕ/w is bounded. The space $BC_w(J, X)$ of all continuous w-bounded functions $\phi : J \to X$ is a Banach space with norm $|\phi\|_{w,\infty} = \sup_{t \in J}(\|\phi(t)\|/w(t))$. For this space and others, we will omit the subscript w when $w = 1$.

Following [31, page 142], we say that a function $\phi : J \to X$ is *w-uniformly continuous* if for each $\varepsilon > 0$ there is a neighbourhood U of 0 in G such that $\|\phi(s) - \phi(t)\| < \varepsilon w(t)$ for all $t \in J$ and $s \in (t + U) \cap J$. The closed subspace of $BC_w(J, X)$ consisting of all w-uniformly continuous functions is denoted $BUC_w(J, X)$; the closed subspace of $BC_w(J, X)$ consisting of functions ϕ for which $\phi/w \in C_0(J, X)$ is denoted $C_{w,0}(J, X)$.

Condition (1.5) is equivalent to $w \in BUC_w(G, \mathbb{C})$. Also, if w satisfies (1.1) and (1.2), then $|\Delta_h w(t)/w(t)| \leq w(h) - 1$ for all $h, t \in G$ and so (1.5) holds if $w(0) = 1$. Moreover, $\Delta_h(\phi/w) = \Delta_h\phi/w - (\phi/w)_h(\Delta_h w/w)$ and therefore from (1.5) we conclude

(1.8) $\qquad \phi \in BUC_w(J, X) \Leftrightarrow \phi/w$ is uniformly continuous and bounded.

Furthermore, $\|\phi_{t+h} - \phi_t\|_{w,\infty} \leq w(t)\|\phi_h - \phi\|_{w,\infty}$ and so

(1.9) $\qquad \phi \in BUC_w(J, X) \Rightarrow t \mapsto \phi_t : J \to BUC_w(J, X)$ is continuous.

EXAMPLE 1.1. The function $w(t) = c(1 + |t|)^N \exp(1 + |t|)^p$ on $\mathbb{R}^d$ or $\mathbb{Z}^d$ satisfies (1.1)–(1.5) whenever $c \geq 1/e$, $N \geq 0$ and $0 \leq p < 1$. If also $p = 0$ then w has polynomial growth of order N.

The *Beurling algebra* $L^1_w(G) = \{f \in L^1(G) : wf \in L^1(G)\}$ is a subalgebra of the convolution algebra $L^1(G)$ and a Banach algebra under the norm

$$\|f\|_{w,1} = \int_G |f(t)|w(t)\, d\mu(t)$$

(see [31, page 83]). The *co-spectrum* of a closed ideal I of $L^1_w(G)$, is defined by

$$\operatorname{cosp}(I) = \{\gamma \in \hat{G} : \hat{f}(\gamma) = 0 \ \text{ for all } f \in I\}.$$

In this paper we introduce a new method for studying the asymptotic behaviour of strongly continuous representations $T : J \to X$. In particular, the results are applied to unbounded solutions of the Cauchy problem on the half-line $\mathbb{R}_+$. There are three major ingredients of this method. Firstly we introduce the notion of w-ergodicity for unbounded functions. For weights satisfying (1.4) many results for bounded ergodic functions have analogues for w-ergodic functions (see Section 2). Note that while the spaces $BC_w(J, X)$ and $L^1_w(G)$ are unchanged if w is replaced by an equivalent weight, this is not the case for spaces of w-ergodic functions. Secondly we introduce the reduced Beurling spectrum of unbounded functions ϕ relative to certain function classes $\mathcal{F}$. This spectrum is used to determine membership of $\mathcal{F}$. As a consequence, we reduce the study of the asymptotic behaviour of ϕ relative to $\mathcal{F}$ to that of ϕ/w relative to $\mathcal{F}/w$. Thirdly we employ the method used by the first author in [6] to unify the study of homogeneous and inhomogeneous equations for the Cauchy problem on the half-line.

The structure of this paper is as follows. In Section 2 we study some translation invariant closed subspaces $\mathcal{F}$ of $BC_w(J, X)$ that will be used in the applications.

These spaces have an additional property that we call BUC_w-invariance. Our main examples are the spaces $E_w(J, X)$ of w-ergodic functions. Though other authors use different characterizations of ergodicity, usually for bounded functions (see [7]), we use that of Maak [25, 26] because of its simplicity and wide applicability. See also [20, 21] and references therein. (We thank Hans Günzler for pointing out that Maak [25] preceded Isekii [20] cited in our paper [7]). In particular, we obtain conditions on a subspace $\mathscr{F}$ of $BC_w(J, X)$ under which a w-ergodic function belongs to $\mathscr{F}$ whenever its differences belong to $\mathscr{F}$. Important examples of w-ergodic functions are certain orbits $T(\cdot)x$ of strongly continuous representations $T : J \to X$ (see Theorem 2.6, Theorem 2.7) and $\phi * f$ whenever $\phi \in BUC_w(G, X)$, $f \in L^1_w(G)$ and $\hat{f}(1) = 0$ (see Corollary 3.2).

Beurling algebras play an important role in harmonic analysis. In particular a knowledge of their ideal structure is useful in applications as we shall demonstrate. However, the identification of the primary ideals of a general Beurling algebra is a difficult problem. If w is non-quasianalytic, then $L^1_w(G)$ is a Wiener algebra (see [31, page 132]). Moreover, its maximal ideals are the sets $I_w(\gamma) = \{f \in L^1_w(G) : \hat{f}(\gamma) = 0\}$ where $\gamma \in \hat{G}$, and its primary ideals are those whose co-spectrum is a singleton. By Wiener's tauberian theorem, all (closed) primary ideals in $L^1(G)$ are maximal (see [32, 7.2.5, 7.2.6]). This is not the case for general $L^1_w(G)$. For example, if $G = \mathbb{R}$, then

$$I_k = \left\{ f \in L^1_w(\mathbb{R}) : \int t^j f(t)\, dt = 0 \text{ for } 0 \le j \le k \right\}$$

defines a chain of primary ideals (see Gurarii [17]). Moreover, for a weight of polynomial growth N, the primary ideals of $L^1_w(\mathbb{Z})$ are the sets $I_k = \{f \in L^1_w(\mathbb{Z}) : \hat{f}^{(j)}(1) = 0 \text{ for } 0 \le j \le k\}$, where $0 \le k \le N$ (see [8, Theorem 3.1]). In Section 3 we obtain two characterizations of the minimal primary ideals of $L^1_w(G)$ when w has polynomial growth—one in terms of differences and one in terms of w-spectral synthesis (see Theorem 3.6 and Corollary 3.7). This is achieved using polynomials $p : G \to X$, a study of which was commenced in [8, Theorem 2.4]. In particular, for weights of polynomial growth, polynomials are the w-bounded functions with Beurling spectrum $\{1\}$. Moreover, functions in $BC_w(G, X)$ with finite Beurling spectra are sums of products of characters and polynomials. We also characterize the maximal ideals in terms of differences when w is non-quasianalytic (see Theorem 3.1).

In Section 4 we define the spectrum $\mathrm{sp}_{\mathscr{F}}(\phi)$ relative to the class $\mathscr{F} \subseteq BC_w(J, X)$ of a function $\phi \in BC_w(G, X)$. We prove (Theorem 4.3) a generalization of Wiener's tauberian theorem, characterizing functions for which $\mathrm{sp}_{\mathscr{F}}(\phi) = \emptyset$ as those for which $\phi|_J \in \mathscr{F}$. In turn, this is used to characterize functions for which $\mathrm{sp}_{\mathscr{F}}(\phi)$ is finite. We also generalize a tauberian theorem of Loomis (Theorem 4.7) for the case that $\mathrm{sp}_{\mathscr{F}}(\phi)$ is residual. An application to convolution operators appears in Section 5 (see Theorem 5.1 and its corollaries). In particular, we obtain tauberian theorems of the form $(k * \phi)|_J \in \mathscr{F}$ implies $\phi|_J \in \mathscr{F}$. Finally, we prove stability theorems for unbounded solutions of the Cauchy problem (Theorem 5.6) and, more generally, for the orbits of strongly continuous semigroup representations (Theorem 5.7).

2. Some function classes

We begin by defining a notion of ergodicity that applies to unbounded functions. This ergodicity differs from both that of Maak [25, 26] and that of Basit and Günzler [13, 12]. If J is $\mathbb{R}$ or $\mathbb{R}_+$, a function $\phi \in L^1_{loc}(J, X)$ is sometimes called *uniform-ergodic* with mean $x \in X$ if $\lim_{T\to\infty}(1/T)\int_0^T \phi(s+t)\,ds = x$ uniformly in t. For example, in [2, 3, 5] uniform-ergodicity is used to prove tauberian theorems for functions in $BUC(J, X)$, whereas in [12] it is used for a similar purpose for certain unbounded functions and distributions. The definition of uniform-ergodicity extends readily to functions on semigroups J which possess a Følner net. See for example [7]. However, Maak [25, 26] introduced a notion of ergodicty that applies for functions on general semigroups (see also [20, 21]).

Thus a function $\phi : J \to X$ is *Maak-ergodic* with mean $M(\phi) = x \in X$ if for each $\varepsilon > 0$ there is a finite subset $F \subseteq J$ with $\left\|(1/|F|)\sum_{t\in F}(\phi_t - x)\right\| < \varepsilon$.

We denote by $E(J, X)$ (respectively $E_0(J, X)$) the closed subspace of Maak-ergodic (respectively Maak-ergodic with mean 0) bounded continuous functions $\phi : J \to X$. (Note the difference with our notation in [7, Section 2]; there $E(J, X)$ stands for the set of all bounded Maak-ergodic functions $\phi : J \to X$).

It is proved in [7, Corollary 5.2] that for certain semigroups J, a function $\phi \in BUC(J, X)$ is uniform-ergodic if and only if it is Maak-ergodic with the same mean.

Also, the space of Maak-ergodic functions $E(J, X)$ is closely related to the difference space $\mathscr{D}(BUC(J, X))$, the span of the set of all differences

$$\Delta_h\phi, \quad \phi \in BUC(J, X),\ h \in J,$$

studied by Nillsen [27, pages 1 and 10] for the case $J = G$, $X = \mathbb{C}$. As in [7, Corollary 5.2], it can be shown that

$$E_0(J, X) \cap BUC(J, X) = \overline{\mathscr{D}(BUC(J, X))}.$$

To apply ergodic theory more generally, we introduce a new class $E_w(J, X)$, (respectively $E_{w,0}(J, X)$), the closed subspaces of $BC_w(J, X)$ consisting of functions ϕ for which ϕ/w is Maak-ergodic (respectively Maak-ergodic with mean 0). Such functions we shall refer to as *w-ergodic*. In particular, for non-zero real s, the function $\phi(t) = t\,e^{ist}$ is neither uniform-ergodic nor Maak-ergodic on $\mathbb{R}$, but if $w(t) = 1 + |t|$ then ϕ is w-ergodic and $M(\phi/w) = 0$.

Note that $\phi(t) = t \sin t^2$ is uniform-ergodic on $\mathbb{R}$ but not Maak-ergodic since

$$\frac{1}{m}\sum_{i=1}^{m}(t+t_i)\sin(t+t_i)^2 = \frac{1}{m}\sum_{i=1}^{m} t\sin(t+t_i)^2 + \frac{1}{m}\sum_{i=1}^{m} t_i \sin(t+t_i)^2$$

is not bounded for all finite collections $t_1, \ldots, t_m \in \mathbb{R}$. However, for general $\phi \in L^1_{\mathrm{loc}}(\mathbb{R}, X)$, if ϕ is uniform-ergodic, then $M_1\phi(t) = \int_0^1 \phi(t+s)ds$ is bounded and uniform-ergodic with the same mean (see [12, Proposition 7.1]). Therefore $M_1^2\phi \in BUC(\mathbb{R}, X)$. It follows that if $\phi : \mathbb{R} \to X$ is uniform-ergodic then $M_1^2\phi$ is Maak-ergodic with the same mean.

The following proposition gives some useful properties of w-ergodicity and the theorem provides a simple but important application of the concept.

PROPOSITION 2.1.

(a) *If* $\phi \in E_w(G, X)$, *then* $\phi|_J \in E_w(J, X)$ *and* $M((\phi/w)|_J) = M(\phi/w)$.
(b) *If* G *is not compact, then* $C_{w,0}(J, X) \subseteq E_{w,0}(J, X)$.
(c) *If* $\phi \in BC_w(J, X)$ *then* $\Delta_t\phi \in E_{w,0}(J, X)$ *for all* $t \in J$.

PROOF. (a) Let $x = M(\phi/w)$. Given $\varepsilon > 0$ there is a finite set $F = \{t_1, \ldots, t_m\} \subseteq G$ such that $\left\|(1/m)\sum_{j=1}^{m}(\phi/w)(t_j+t) - x\right\| < \varepsilon$ for all $t \in G$. Choose $u_j, v_j \in J$ such that $t_j = u_j - v_j$. Let $v = v_1 + \cdots + v_m$ and set $s_j = t_j + v$. So $s_j \in J$ and $\left\|(1/m)\sum_{j=1}^{m}(\phi/w)(s_j+t) - x\right\| < \varepsilon$ for all $t \in J$.

(b) Since $G = J - J$, J is not compact. Let $\phi \in C_{w,0}(J, X)$. Given $\varepsilon > 0$, choose $m \in \mathbb{N}$ such that $\|\phi(t)\| < (m\varepsilon/2)w(t)$ for all $t \in J$, and a compact subset K of J such that $\|\phi(t)\| < (\varepsilon/2)w(t)$ for all $t \notin K$. Take any $t_1 \in J$ and for $2 \le j \le m$ choose $t_j \in J$ inductively such that $t_j \notin \bigcup_{i=1}^{j-1}(t_i + K - K)$. Then for any $t \in J$, $t + t_j \in K$ for at most one j and so

$$\left\|\frac{1}{m}\sum_{j=1}^{m}\frac{\phi(t_j+t)}{w(t_j+t)}\right\| \le \frac{1}{m}\left(\frac{m\varepsilon}{2} + \frac{(m-1)\varepsilon}{2}\right) < \varepsilon.$$

This shows that $\phi/w \in E_{w,0}(J, X)$.

(c) First note that $\Delta_t\phi/w = \Delta_t(\phi/w) + (\phi/w)_t(\Delta_t w/w)$. Given $\varepsilon > 0$, choose $m \in \mathbb{N}$ such that $\|\phi(t)\| < (m\varepsilon/2)w(t)$ for all $t \in J$. Since

$$(\Delta_t(\phi/w))_s = \Delta_{t+s}(\phi/w) - \Delta_s(\phi/w),$$

$\left\|\frac{1}{m}\sum_{j=1}^{m}\Delta_t(\phi/w)(jt+u)\right\| < \varepsilon$ for all $t, u \in J$, showing $\Delta_t(\phi/w) \in E_{w,0}(J, X)$. By (1.4), $(\phi/w)_t(\Delta_t w/w) \in C_0(J, X)$. If G is not compact, then $(\phi/w)_t(\Delta_t w/w) \in E_{w,0}(J, X)$ by part (b). If G is compact, then $w = 1$ so $(\phi/w)_t(\Delta_t w/w) = 0$. $\square$

THEOREM 2.2. *Let $\mathscr{F}$ be any translation invariant closed subspace of $BC_w(J, X)$. If $\phi \in E_w(J, X)$ and $\Delta_t\phi \in \mathscr{F}$ for each $t \in J^n$ and some $n \in \mathbb{N}$, then $\phi - M(\phi/w)w \in \mathscr{F} + C_{w,0}(J, X)$. If also $w = 1$, then $\phi - M(\phi) \in \mathscr{F}$.*

PROOF. Assume firstly that $n = 1$. For any finite subset $F \subseteq J$, we have

$$\phi - M\left(\frac{\phi}{w}\right) w = w\left[\frac{1}{|F|}\sum_{t\in F}\frac{\phi}{w}\bigg|_t - M\left(\frac{\phi}{w}\right)\right] - \frac{1}{|F|}\sum_{t\in F}\Delta_t\phi + \frac{1}{|F|}\sum_{t\in F}\frac{\phi}{w}\bigg|_t \Delta_t w.$$

The first term on the right may be made arbitrarily small in norm by suitable choice of F. The second term is in $\mathscr{F}$ by assumption and the third term is in $C_{w,0}(J, X)$ by (1.4). If $w = 1$, then $\Delta_t w = 0$. The result for general n now follows. □

We say that a subspace $\mathscr{F}$ of $BC_w(J, X)$ is *BUC_w-invariant* whenever it satisfies

(2.1) if $\phi \in BUC_w(G, X)$ and $\phi|_J \in \mathscr{F}$ then $\phi_t|_J \in \mathscr{F}$ for all $t \in G$.

Other conditions that we will sometimes use are

(2.2) $\mathscr{F}$ is closed under multiplication by characters;

(2.3) if w is unbounded, $\mathscr{F} \supseteq C_{w,0}(J, X)$.

A closed linear subspace $\mathscr{F}$ of $BUC_w(J, X)$ satisfying (2.1)–(2.3) will be called a *Λ_w-class*.

REMARKS 2.3. (a) It is easy to see that if $\phi \in E_{w,0}(J, \mathbb{C})$, $\phi \geq 0$ and $\psi \in BC(J, X)$, then $\phi\psi \in E_{w,0}(J, X)$. Hence, Proposition 2.1 (c) and Theorem 2.2 remain valid with $C_{w,0}(J, X)$ replaced by $E_{w,0}(J, X)$ if, instead of (1.4), w satisfies the weaker condition

(2.4) $|\Delta_s w| \in E_{w,0}(J, \mathbb{C})$ for every $s \in J$.

(b) The spaces $E_w(J, X)$ and $E_{w,0}(J, X)$ are BUC_w-invariant. Indeed, let $\phi \in BUC_w(G, X)$ with $\phi|_J \in E_w(J, X)$. If $t \in G$, then $\phi_t = \Delta_t\phi + \phi$ and so by Proposition 2.1 (c), $\phi_t|_J \in E_w(J, X)$ and $M(\phi_t/w) = M(\phi/w)$.

(c) The partial ordering $\leq$, defined by $s \leq t$ whenever $t - s \in J \cup \{0\}$, makes J a directed set. We will use this order to define limits here and below. In particular, we may define

$$\mathscr{F}_{w,0}(J, X) = \left\{\phi \in BUC_w(J, X) : \lim_{t\in J}\left\|\frac{\phi(t)}{w(t)}\right\| = 0\right\}.$$

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Using $G = J - J$, it is easy to check that $\mathscr{F}_{w,0}(J, X)$ is BUC_w-invariant. Moreover, if $G = \mathbb{R}$ or $\mathbb{Z}$ and $J = \mathbb{R}_+$ or $\mathbb{Z}_+$, then $\mathscr{F}_{w,0}(J, X) = C_{w,0}(J, X)$ but in general this is not the case. For example, $\mathscr{F}_{w,0}(G, X) = \{0\}$. However, $\mathscr{F}_{w,0}(J, X) \supseteq C_{w,0}(J, X)$ if J satisfies the following condition:

(2.5) for every compact subset K of J there exists $t \in J$ with $K \cap (t + J) = \emptyset$.

(d) For some semigroups J we have $\mathscr{F} \supset C_{w,0}(J, X)$ for every BUC_w-invariant closed subspace $\mathscr{F}$ of $BC_w(J, X)$. For example, this is the case if $C_0(J, X) = C_0(G, X)|_J$ and J satisfies

(2.6) for every compact subset K of G there exists $t \in G$ with $(t + K) \cap J = \emptyset$.

Indeed, any $\xi \in C_{w,0}(J, X)$ can be extended to a function $\tilde{\xi} \in C_{w,0}(G, X)$. Since G is normal (see [18, page 76]) $\tilde{\xi}$ can be approximated by a function $\psi \in C_{w,0}(G, X)$ with compact support K say. Choose $t \in G$ such that $(t + K) \cap J = \emptyset$ and set $\phi = \psi_{-t} \in BUC_w(G, X)$. Then $\phi|_J = 0 \in \mathscr{F}$ so $\psi|_J = \phi_t|_J \in \mathscr{F}$. As $\mathscr{F}$ is closed, $\xi \in \mathscr{F}$.

(e) Condition (2.6) holds if $G = \mathbb{R}^d$ and $J = (\mathbb{R}_+)^d$. In fact, it holds whenever J and the interior of $-J$ are disjoint. Indeed, let $s \in J^\circ$, the interior of J, and choose an open neighbourhood U of 0 in G such that $-s + U \subseteq -J^\circ$. Given a compact subset K of G, choose a finite covering $\{c_j + U : 1 \le j \le n\}$ of K. Now $c_j = a_j - b_j$ for some $a_j, b_j \in J$. Setting $a = a_1 + \cdots + a_n$ and $t = -a - s$ we find $t + K \subseteq \bigcup_{j=1}^{n}(-a + a_j - b_j - s + U) \subseteq -J^\circ$. Hence $(t + K) \cap J = \emptyset$.

(f) Translation invariant subspaces of $BC_w(G, X)$ are BUC_w-invariant. In particular, $C_{w,0}(G, X)$ is BUC_w-invariant as is the class 0_G consisting of just the zero function from G to X.

(g) A class $\mathscr{F}$ is a BUC_w-invariant subspace of $BUC_w(J, X)$ containing $C_{w,0}(J, X)$ if and only if $\mathscr{F}/w$ is a BUC-invariant subspace of $BUC(J, X)$ containing $C_0(J, X)$. Indeed, if $\phi \in BUC_w(J, X)$ and $t \in J$, then $(\phi_t/w) - (\phi/w)_t = (\phi/w)_t(\Delta_t w/w) \in C_0(J, X)$ by (1.4). The claim follows.

(h) The spaces $C_{w,0}(G, X)$ and $\mathscr{F}_{w,0}(J, X)$ are Λ_w-classes. By remark (a), the subspace of $E_{w,0}(J, X)$ defined by

$$AE_{w,0}(J, X) = \{\phi \in BUC_w(J, X) : |\phi| \in E_{w,0}(J, \mathbb{C})\}$$

is also a Λ_w-class. Moreover, this class is closed under multiplication by functions from $BUC(J, \mathbb{C})$.

Many other examples for the case $w = 1$ are given in [5]. These include almost periodic, almost automorphic and absolutely recurrent functions. Further examples for other weights will be discussed in a subsequent paper.

PROPOSITION 2.4. *Let $\mathcal{F}$ be any BUC_w-invariant closed subspace of $BC_w(J, X)$. If $\phi \in BUC_w(G, X)$, $f \in L^1_w(G)$ and $\phi|_J \in \mathcal{F}$, then $(\phi * f)|_J \in \mathcal{F}$.*

PROOF. We may assume $f \in C_c(G)$, since this space is dense in $L^1_w(G)$ (see [31, page 83]). Now $(\phi * f)(t) = \int_K \phi_{-s}(t) f(s) d\mu(s)$ where K is the support of f and $t \in G$. By (1.9), the function $s \to \phi_s|_J : G \to \mathcal{F}$ is continuous and so the function $F(s) = \phi_{-s}|_J f(s)$ is strongly measurable. This implies that $|F|$ is integrable and hence the integral $\int_K \phi_{-s}|_J f(s)\, d\mu(s)$ is a convergent Haar-Bochner integral, by Bochner's theorem [34, page 133], and so belongs to $\mathcal{F}$. As evaluation at $t \in J$ is continuous on $\mathcal{F}$ we conclude that $(\phi * f)|_J \in \mathcal{F}$. □

Since

$$\frac{\phi}{w} * f(t) - \left(\frac{\phi * f}{w}\right)(t) = -\int_G \frac{\phi}{w}\bigg|_{-s}(t)\frac{\Delta_{-s}w(t)}{w(t)} f(s)\, d\mu(s),$$

a proof similar to the last gives

$$\frac{\phi}{w} * f - \frac{\phi * f}{w} \in C_0(G, X), \quad \text{respectively } AE_0(G, X), \tag{2.7}$$

for any $\phi \in BUC_w(G, X)$, $f \in L^1_w(G)$ and w satisfying (1.1) and (1.4), respectively (2.4).

COROLLARY 2.5. *If $\phi \in BUC_w(G, X)$, $f \in L^1_w(G)$ and $\phi|_J$ is w-ergodic, then $(\phi * f)|_J$ is w-ergodic and $M(((\phi * f)/w)|_J) = M((\phi/w)|_J)\hat{f}(1)$.*

PROOF. By Proposition 2.4, $(\phi * f)|_J$ is w-ergodic. So, by (2.7), $((\phi/w) * f)|_J$ is Maak-ergodic and $M(((\phi/w) * f)|_J) = M((\phi * f)/w)|_J$. But $((\phi/w) * f)|_J - M((\phi/w)|_J)\hat{f}(1) = (((\phi/w) - M((\phi/w)|_J)) * f)|_J \in E_0(J, X)$, again by Proposition 2.4. The corollary follows from (2.7). □

The next two theorems provide important examples of ergodic functions to be used in Section 5. Whether or not w is a weight, we say $\phi : J \to X$ is w-ergodic if ϕ/w is uniform-ergodic and totally w-ergodic if $\gamma\phi$ is w-ergodic for all $\gamma \in \hat{G}$. Moreover, a representation $T : J \to L(X)$ is dominated by w if $\|T(t)\| \le cw(t)$ for all $t \in J$ and some $c > 0$. The *unitary point spectrum* of T is given by $\sigma_{up}(T) = \{\gamma \in \hat{G} : T(t)x = \gamma(t)x$ for some $x \neq 0$ and all $t \in J\}$ and the *dual representation* $T^* : J \to L(X^*)$ by $\langle T^*(t)x^*, x\rangle = \langle x^*, T(t)x\rangle$ for $x^* \in X^*, x \in X$. The *dual* of a (densely defined) operator $A : X \to X$ is denoted by $A^* : X^* \to X^*$ and $\sigma_p(A^*)$ is its *point spectrum*.

THEOREM 2.6. *Let $w : J \to [1, \infty)$ be a continuous function satisfying $\Delta_t w/w \in C_0(J)$ for all $t \in J$. Let $T : J \to L(X)$ be a strongly continuous representation dominated by w.*

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(a) *If* $1 \notin \sigma_{up}(T^*)$, *then each orbit* $T(\cdot)x$ *is* w*-ergodic with* $M((1/w)T(\cdot)x) = 0$.

(b) *If* $\sigma_{up}(T^*)$ *is empty, then each orbit* $T(\cdot)x$ *is totally* w*-ergodic and*

$$M((\gamma/w)T(\cdot)x) = 0$$

for all $\gamma \in \hat{G}$ *and* $x \in X$.

PROOF. Note that $\langle T(h)x - x, x^* \rangle = \langle x, T^*(h)x^* - x^* \rangle$ for all $h \in J, x \in X$ and $x^* \in X^*$. It follows that $1 \notin \sigma_{up}(T^*)$ if and only if $\operatorname{span}\{(T(h)x - x : h \in J, x \in X\}$ is dense in X. But if $y = T(h)x - x$, then $T(\cdot)y = \Delta_h T(\cdot)x$ which, by the proof of Proposition 2.1 (c), is w-ergodic with $M((1/w)T(\cdot)y) = 0$. Since the span of such y is dense in X, (a) is proved and (b) then follows. □

THEOREM 2.7. *Let* $w : \mathbb{R}_+ \to [1, \infty)$ *be a differentiable function with* $w'/w \in AE_0(\mathbb{R}_+, \mathbb{C})$. *Let* A *be the generator of a* C_0*-semigroup of operators* $T(t)$, $t \geq 0$ *on* X *which is dominated by* w.

(a) *If* $\phi/w \in BC(\mathbb{R}_+, X)$ *and* $\phi' \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$, *then* ϕ'/w *is uniformly ergodic with* $M(\phi'/w) = 0$.

(b) *If* $x \in \operatorname{range}(A)$, *then the orbit* $T(\cdot)x$ *is* w*-ergodic with* $M((1/w)T(\cdot)x) = 0$.

(c) *If* $\sigma_p(A^*) \cap i\mathbb{R}$ *is empty, then each orbit* $T(\cdot)x$ *is totally* w*-ergodic and* $M((\gamma/w)T(\cdot)x) = 0$ *for all* $\gamma \in \hat{\mathbb{R}}$ *and* $x \in X$.

PROOF. (a) For each $T > 0$ and $t \geq 0$,

$$\frac{1}{T}\int_0^T \frac{\phi'(t+s)}{w(t+s)}\,ds = \frac{1}{T}\left[\frac{\phi(t+s)}{w(t+s)}\right]_0^\infty + \frac{1}{T}\int_0^T \frac{\phi(t+s)}{w(t+s)}\,\frac{w'(t+s)}{w(t+s)}\,ds.$$

But $(\phi/w)(w'/w) \in AE_0(\mathbb{R}_+, X)$ and hence

$$\lim_{t\to\infty} \frac{1}{T}\int_0^T \frac{\phi'(t+s)}{w(t+s)}\,ds = 0 \quad \text{uniformly in } t.$$

(b) If $x = Ay$ set $\phi(t) = T(t)y$. Then $T(t)x = \phi'(t)$ and the claim follows from (a).

(c) If $\gamma_s(t) = e^{ist}$, then $S(t) = \gamma_s^{-1}(t)T(t)$ defines a C_0-semigroup with generator $A - is$. By (b), $S(\cdot)x$ is w-ergodic with mean 0 for each $x \in \operatorname{range}(A - is)$. Since $\sigma_p(A^*) \cap i\mathbb{R}$ is empty, $\operatorname{range}(A - is)$ is dense for all $s \in \mathbb{R}$ and the claim follows. □

REMARK 2.8. Functions w satisfying the conditions of Theorems 2.6–2.7 arise very naturally. For example, $w = 1$. More generally, if w is a weight on $\mathbb{R}$, then $\tilde{w}(t) = \int_0^1 w(t+s)\,ds$ is differentiable for $t > 0$ and $\tilde{w}'(t) = w(t+1) - w(t)$. Moreover, by the Mean Value Theorem, $\tilde{w}(t) = w(t + \theta(t))$ for some $0 \leq \theta(t) \leq 1$

and so $(1/c)w(t) \leq \tilde{w}(t) \leq cw(t)$ for all $t \geq 0$, where $c = \max\{w(s) : 0 \leq s \leq 1\}$. Also

$$\frac{\tilde{w}}{w} - 1 = \int_0^1 \frac{\Delta_s w}{w}\, ds \quad \text{and} \quad \frac{\tilde{w}'}{\tilde{w}} = \frac{\Delta_1 w}{\tilde{w}}.$$

So by (1.4), $(\tilde{w}/w) - 1$, $\tilde{w}'/\tilde{w} \in C_0(\mathbb{R}_+)$. If $\phi \in BC_w(\mathbb{R}_+, X)$, then $(\phi/w) - (\phi/\tilde{w}) = (\phi/w)((\tilde{w}/w) - 1)(w/\tilde{w}) \in C_0(\mathbb{R}_+, X)$. Hence ϕ is w-uniformly continuous if and only if $(\phi/\tilde{w})$ is uniformly continuous and ϕ is w-ergodic if and only if ϕ is $\tilde{w}$-ergodic. Finally, $\Delta_h \tilde{w}(t) = \int_0^1 \Delta_h w(t+s)\, ds$ and so from (1.4) and (1.5) we conclude that $(\Delta_h \tilde{w}/\tilde{w}) \in C_0(\mathbb{R}_+, \mathbb{C})$ and $\sup_{t \in \mathbb{R}_+}(|\Delta_h \tilde{w}(t)|/\tilde{w}(t)) \to 0$ as $h \to 0$ in $\mathbb{R}_+$.

3. Maximal and minimal ideals

For $m \in \mathbb{N}$, $t = (t_1, \dots, t_m) \in G^m$ and $f \in L^1_w(G)$ write $\Delta_t f = \Delta_{t_1} \cdots \Delta_{t_m} f$. Then for each $\gamma \in \hat{G}$ let $J^m_w(\gamma)$ denote the closed span of $\{\gamma \Delta_t f : f \in L^1_w(G), t \in G^m\}$. Since $(\gamma \Delta_t f) * g = \gamma \Delta_t (f * \gamma^{-1} g)$ we have a chain of closed ideals $J^1_w(\gamma) \supseteq J^2_w(\gamma) \supseteq \cdots$. Moreover, if $g = \gamma \Delta_{t_1} \cdots \Delta_{t_m} f$ then $\hat{g}(\tau\gamma) = (\tau(t_1) - 1) \cdots (\tau(t_m) - 1)\hat{f}(\tau)$ which is 0 for all such f and t if and only if $\tau = 1$. Hence each of the ideals $J^m_w(\gamma)$ is primary with co-spectrum $\{\gamma\}$. Recall (see [17, page 33]) that $I_w(\gamma)$ is the maximal ideal in $L^1_w(G)$ with co-spectrum $\{\gamma\}$. The following theorem gives another characterization of these maximal ideals.

THEOREM 3.1. *For each $\gamma \in \hat{G}$, $I_w(\gamma) = J^1_w(\gamma)$.*

PROOF.

$$\begin{aligned} J^1_w(\gamma)^\perp &= \{\phi \in L^\infty_w(G) : \phi * (\gamma \Delta_t f) = 0 \ \text{ for all } t \in G \text{ and } f \in L^1_w(G)\} \\ &= \{\phi \in L^\infty_w(G) : \Delta_t(\gamma^{-1}\phi) = 0 \ \text{ for all } t \in G\} \\ &= \{\phi \in L^\infty_w(G) : \gamma^{-1}\phi \ \text{ is constant}\} = \gamma\mathbb{C}. \end{aligned}$$

Hence

$$\begin{aligned} J^1_w(\gamma)^{\perp\perp} &= \{f \in L^1_w(G) : \phi * f = 0 \ \text{ for all } \ \phi \in J^1_w(\gamma)^\perp\} \\ &= \{f \in L^1_w(G) : \gamma * f = 0\} = \{f \in L^1_w(G) : \hat{f}(\gamma) = 0\} = I_w(\gamma). \end{aligned}$$

But $I^{\perp\perp} = I$ for any closed ideal in $L^1_w(G)$ and so the theorem follows. □

As an initial application of Theorem 3.1 we prove an ergodicity result.

COROLLARY 3.2. *If $f \in I_w(\gamma)$ for some $\gamma \in \hat{G}$ and $\phi \in BUC_w(G, X)$, then $\gamma^{-1}(\phi * f) \in E_{w,0}(G, X)$.*

PROOF. Let $h = \gamma\Delta_t g$, where $t \in G$ and $g \in L^1_w(G)$. By Proposition 2.1 (c), $\gamma^{-1}(\phi * h) = \Delta_t(\gamma^{-1}\phi * g) \in E_{w,0}(G, X)$. Since f is in the closed linear span of such functions h and $E_{w,0}(G, X)$ is complete, the result follows. $\square$

Following [8, (2.1), (2.2)] we say that a function $p \in C(G, X)$ is a *polynomial* if $\Delta_t^{n+1} p = 0$ for some $n \in \mathbb{N}$ and all $t \in J$. Equivalently (see [14]), $p(s + mt)$ is a polynomial in $m \in \mathbb{Z}_+$ of degree at most n for all $s, t \in J$. Since Δ_t is a continuous mapping on $BC_w(G, X)$, the polynomials in $BC_w(J, X)$ form a closed subspace which we denote by $P_w(J, X)$. The following result was proved in [8, Theorem 3.4] under a slightly stronger assumption than (1.6) and with $X = \mathbb{C}$. The same proof is valid under the present assumptions. See also [30, Proposition 0.5] for the case $G = \mathbb{R}$.

THEOREM 3.3. *Suppose w has polynomial growth and $\phi \in BC_w(G, X)$. Then* $\mathrm{sp}_w(\phi) = \{\gamma_1, \ldots, \gamma_n\}$ *if and only if* $\phi = \sum_{j=1}^n \gamma_j p_j$ *for some non-zero* $p_j \in P_w(G, X)$.

COROLLARY 3.4. *Suppose w has polynomial growth of order N and I is a closed ideal of $L^1_w(G)$ with* $\mathrm{cosp}(I) = \{1\}$. *Then $\Delta_t g \in I$ for all $g \in L^1_w(G)$ and $t \in G^{N+1}$.*

PROOF. Consider the annihilator $I^\perp = \{\phi \in L^\infty_w(G) : \phi * f = 0 \text{ for all } f \in I\}$, a closed translation invariant subspace of $L^\infty_w(G)$. If $\phi \in I^\perp$ and $I_w(\phi) = \{f \in L^1_w(G) : \phi * f = 0\}$ then $I_w(\phi) \supseteq I$. This implies that $\mathrm{cosp}(I_w(\phi)) \subseteq \mathrm{cosp}(I) = \{1\}$. By Theorem 3.3, $\phi \in P_w(G, \mathbb{C})$ and so $\Delta_t \phi = 0$ for all $t \in G^{N+1}$. If $g \in L^1_w(G)$ then $\phi * \Delta_t g = \Delta_t \phi * g = 0$, showing $\Delta_t g \in I^{\perp\perp}$. Since $I^{\perp\perp} = I$ the theorem is proved. $\square$

Finally, we establish relationships between spectral synthesis and minimal primary ideals. For $\gamma \in \hat{G}$, let $S_w(\gamma)$ denote the closure of the set of $f \in L^1_w(G)$ for which $\hat{f}$ is 0 on a neighbourhood of γ. Functions in $S_w(\gamma)$ are said to be of *w-spectral synthesis* with respect to $\{\gamma\}$.

LEMMA 3.5. *For each $f \in L^1_w(G)$ the function $t \mapsto f_t : G \to L^1_w(G)$ is w-uniformly continuous.*

PROOF. Let V be a compact neighbourhood of 0 and set $c_1 = \sup_{t\in V} w(t)$. Given $\varepsilon > 0$ choose $g \in C(G, \mathbb{C})$ with compact support K such that $\|g - f\| < \varepsilon/3c_1$. Set $c_2 = (1 + c_1)\int_K w(t)\,d\mu(t)$. As g is uniformly continuous there is a compact neighbourhood U of 0 in G such that $U \subseteq V$ and $|g(t) - g_h(t)| < \varepsilon/3c_2$ for all $h \in U$ and $t \in G$. Hence for each $h \in U$,

$$\|g - g_h\| = \int_{K\cup(K-h)} |g(t) - g_h(t)|\,w(t)\,d\mu(t)$$

$$\leq \int_K \frac{\varepsilon}{3c_2} w(t)\, d\mu(t) + \int_K \frac{\varepsilon}{3c_2} w(t-h)\, d\mu(t) < \frac{\varepsilon}{3}.$$

So, for $t \in G$ and $h \in U$ we have

$$\begin{aligned}\|f_t - f_{t+h}\| &\leq w(t)\|f - f_h\| < w(t)(\|f - g\| + \|g - g_h\| + \|g_h - f_h\|) \\ &< w(t)\left(\frac{\varepsilon}{3c_1} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3c_1} w(h)\right) < \varepsilon w(t).\end{aligned}$$ □

THEOREM 3.6. *For each $\gamma \in \hat{G}$, we have $S_w(\gamma) \subseteq \bigcap_{m=1}^{\infty} J_w^m(\gamma)$.*

PROOF. Since $J_w^m(\gamma) = \gamma J_w^m(1)$ and $S_w(\gamma) = \gamma S_w(1)$, we may take $\gamma = 1$. For a fixed $g \in L_w^1(G)$ satisfying $\hat{g}(1) = 1$, choose a compact set $K_n \subset G$ such that $\int_{G\setminus K_n} |g(s)|w(s)\, d\mu(s) < 1/n$ and set $H = \bigcup_{n=1}^{\infty} K_n$. Let T_g be the operator on $L_w^1(G)$ defined by

$$T_g f = -\int_H (\Delta_{-s} f) g(s)\, d\mu(s).$$

By Lemma 3.5, the integrand is weakly measurable and separably-valued on H, and therefore the integral is an absolutely convergent Bochner integral. Moreover, T_g is bounded and maps $L_w^1(G)$ into $J_w^1(1)$ and $J_w^m(1)$ into $J_w^{m+1}(1)$ for each m. Note that $f(t) - f * g(t) = -\int_H (\Delta_{-s} f)(t) g(s)\, d\mu(s)$. So for each $\phi \in L_w^{\infty}(G)$, the dual space of $L_w^1(G)$, it follows from Fubini's theorem that $\int_G \phi(t) T_g f(t) d\mu(t) = \int_G \phi(t)(f - f * g)(t)\, d\mu(t)$. Hence, $T_g f = f - f * g$.

Now take any $f \in L_w^1(G)$ with $\hat{f} = 0$ on a neighbourhood U of 1. Choose $g \in L_w^1(G)$ with $\hat{g}(1) = 1$ and $\mathrm{supp}(\hat{g}) \subseteq U$. So $f * g = 0$ and $f = T_g^m f \in J_w^m(1)$. Hence, $S_w(1) \subseteq J_w^m(1)$ for all $m \in \mathbb{N}$, completing the proof. □

COROLLARY 3.7. *Suppose w has polynomial growth of order N and $\gamma \in \hat{G}$.*

(a) *$J_w^{N+1}(\gamma)$ is the minimal closed ideal of $L_w^1(G)$ with co-spectrum $\{\gamma\}$.*
(b) *$S_w(\gamma) = J_w^{N+1}(\gamma)$.*

PROOF. (a) Since $(\gamma \Delta_t f) * g = \gamma \Delta_t (f * \gamma^{-1} g)$, $J_w^{N+1}(\gamma)$ is a closed ideal. Minimality follows from Corollary 3.4.

(b) Since $S_w(\gamma)$ is an ideal with co-spectrum $\{\gamma\}$ the result follows from (a) and Theorem 3.6. □

4. Spectral analysis

In this section we will assume that $\mathscr{F}$ is a BUC_w-invariant closed subspace of $BC_w(J, X)$.

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Let $\phi \in BC_w(G, X)$. The set $I_w(\phi) = \{f \in L^1_w(G) : \phi * f = 0\}$ is a closed ideal of $L^1_w(G)$ and the *Beurling spectrum* of ϕ is defined to be $\mathrm{sp}_w(\phi) = \mathrm{cosp}(I_w(\phi))$. More generally, following [5, Section 4], set $I_{\mathscr{F}}(\phi) = \{f \in L^1_w(G) : (\phi * f)|_J \in \mathscr{F}\}$. By condition (2.1), $I_{\mathscr{F}}(\phi)$ is a closed translation invariant subspace of $L^1_w(G)$ and is therefore an ideal. We define the *spectrum of ϕ relative to $\mathscr{F}$*, or the *reduced Beurling spectrum*, to be $\mathrm{sp}_{\mathscr{F}}(\phi) = \mathrm{cosp}(I_{\mathscr{F}}(\phi))$. The following lemma may also be found in [18, page 303], [19, page 298] for the spaces $M_a(G)$, $L^1(G)$.

LEMMA 4.1. *For each $\phi \in BUC_w(G, X)$ there is a sequence of approximate units, that is a sequence (g_n) in $L^1_w(G)$ such that $\phi * g_n \to \phi$ in $BUC_w(G, X)$.*

PROOF. Since ϕ is w-uniformly continuous, there is a compact neighbourhood V_n of 0 in G such that $\|\phi_{-s} - \phi\|_{w,\infty} < 1/n$ for all $s \in V_n$. Choose $g_n \in C_c(G)$ with $\mathrm{supp}(g_n) \subseteq V_n$, $g_n \geq 0$ and $\int_G g_n(s)\, d\mu(s) = 1$. So $g_n \in L^1_w(G)$ and for each $t \in G$,

$$\|\phi * g_n(t) - \phi(t)\| = \left\| \int_{V_n} [\phi(t-s) - \phi(t)] g_n(s)\, d\mu(s) \right\| < \frac{1}{n} w(t). \qquad \square$$

The following proposition contains some basic properties of these spectra. The proof is the same as for the Beurling spectrum. See for example [16, page 988] or [32].

PROPOSITION 4.2. *Let $\phi, \psi \in BC_w(G, X)$.*

(a) $\mathrm{sp}_{\mathscr{F}}(\phi_t) = \mathrm{sp}_{\mathscr{F}}(\phi)$ *for all $t \in G$.*
(b) $\mathrm{sp}_{\mathscr{F}}(\phi * f) \subseteq \mathrm{sp}_{\mathscr{F}}(\phi) \cap \mathrm{supp}(\hat{f})$ *for all $f \in L^1_w(G)$.*
(c) $\mathrm{sp}_{\mathscr{F}}(\phi + \psi) \subseteq \mathrm{sp}_{\mathscr{F}}(\phi) \cup \mathrm{sp}_{\mathscr{F}}(\psi)$.
(d) $\mathrm{sp}_{\mathscr{F}}(\gamma\phi) = \gamma\, \mathrm{sp}_{\mathscr{F}}(\phi)$, *provided $\mathscr{F}$ is invariant under multiplication by $\gamma \in \hat{G}$.*
(e) *If $f \in L^1_w(G)$ and $\hat{f} = 1$ on a neighbourhood of $\mathrm{sp}_{\mathscr{F}}(\phi)$, then $\mathrm{sp}_{\mathscr{F}}(\phi * f - \phi) = \emptyset$.*

The following theorem gives our motivation for introducing $\mathrm{sp}_{\mathscr{F}}(\phi)$.

THEOREM 4.3. *Let $\phi \in BUC_w(G, X)$.*

(a) $\mathrm{sp}_{\mathscr{F}}(\phi) = \emptyset$ *if and only if $\phi|_J \in \mathscr{F}$.*
(b) *If $\Delta^k_t \phi|_J \in \mathscr{F}$ for all $t \in G$ and some $k \in \mathbb{N}$, then $\mathrm{sp}_{\mathscr{F}}(\phi) \subseteq \{1\}$.*
(c) *If w has polynomial growth of order N, then $\mathrm{sp}_{\mathscr{F}}(\phi) \subseteq \{\gamma_1, \ldots, \gamma_n\}$ if and only if $\phi = \psi + \sum_{j=1}^n \eta_j \gamma_j$ for some $\psi, \eta_j \in BUC_w(G, X)$ with $\psi|_J \in \mathscr{F}$ and $\Delta_t \eta_j|_J \in \mathscr{F}$ for each $t \in G^{N+1}$.*

PROOF. (a) Suppose $\phi|_J \in \mathscr{F}$. By Proposition 2.4, $(\phi * f)|_J \in \mathscr{F}$ for each $f \in L^1_w(G)$. So $I_{\mathscr{F}}(\phi) = L^1_w(G)$ and $\mathrm{sp}_{\mathscr{F}}(\phi) = \emptyset$. Conversely, if $\mathrm{sp}_{\mathscr{F}}(\phi) = \emptyset$ then $(\phi * f)|_J \in \mathscr{F}$ for all $f \in L^1_w(G)$. By Lemma 4.1, ϕ has approximate units and so $\phi|_J \in \mathscr{F}$.

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(b) Assume $\Delta_t^k\phi|_J \in \mathscr{F}$ for all $t \in G$ and some $k \in \mathbb{N}$. If $g \in L_w^1(G)$ then $(\phi * \Delta_t^k g)|_J = \int_G g(s)(\Delta_t^k \phi_{-s})|_J d\mu(s) \in \mathscr{F}$ and so $\Delta_t^k g \in I_{\mathscr{F}}(\phi)$. But $\widehat{\Delta_t^k g}(\gamma) = (\gamma(t) - 1)^k \hat{g}(\gamma)$ is zero for all $t \in G$ and $g \in L_w^1(G)$ only when $\gamma = 1$. So $\mathrm{sp}_{\mathscr{F}}(\phi) \subseteq \{1\}$.

(c) Firstly, if $\mathrm{sp}_{\mathscr{F}}(\phi) = \{1\}$ then, by Corollary 3.7 (a),

$$\left\{\Delta_t g : g \in L_w^1(G), t \in G^{N+1}\right\} \subseteq I_{\mathscr{F}}(\phi)$$

and so $(\Delta_t\phi * g)|_J = (\phi * \Delta_t g)|_J \in \mathscr{F}$. Taking approximate units we conclude $\Delta_t\phi|_J \in \mathscr{F}$ for each $t \in G^{N+1}$. More generally, assume $\mathrm{sp}_{\mathscr{F}}(\phi) = \{\gamma_1, \ldots, \gamma_n\}$. Choosing $f_j \in L_w^1(G)$ such that $\hat{f_j} = 1$ in a neighbourhood of $\{\gamma_j\}$ and $\mathrm{supp}(\hat{f_j}) \cap \mathrm{sp}_{\mathscr{F}}(\phi) = \{\gamma_j\}$, set $\eta_j = \gamma_j^{-1}(\phi * f_j)$ and $f = f_1 + \cdots + f_n$. We find $\eta_j \in BUC_w(G, X)$, $\mathrm{sp}_{\mathscr{F}}(\eta_j) = \{1\}$ and hence $\Delta_t\eta_j|_J \in \mathscr{F}$ for each $t \in G^{N+1}$. Moreover, $\hat{f} = 1$ in a neighbourhood of $\mathrm{sp}_{\mathscr{F}}(\phi)$ and so by (e) above, $\psi = \phi - \phi * f \in \mathscr{F}$. Also $\phi = \psi + \sum_{j=1}^n \eta_j\gamma_j$ as required. Conversely, for ϕ of the form stated we have $\mathrm{sp}_{\mathscr{F}}(\phi) \subseteq \bigcup_{j=1}^n \gamma_j \, \mathrm{sp}_{\mathscr{F}}(\eta_j)$. But for each $t \in G^{N+1}$ and $f \in L_w^1(G)$ we have $(\eta_j * \Delta_t f)|_J = (\Delta_t\eta_j * f)|_J \in \mathscr{F}$, by Proposition 2.4. So, by Corollary 3.7, $S_w(1) \subseteq I_{\mathscr{F}}(\eta_j)$ and therefore $\mathrm{sp}_{\mathscr{F}}(\eta_j) \subseteq \{1\}$. Hence, $\mathrm{sp}_{\mathscr{F}}(\phi) \subseteq \{\gamma_1, \ldots, \gamma_n\}$. □

COROLLARY 4.4. *Assume w has polynomial growth and, if w is unbounded, $\mathscr{F} \supseteq C_{w,0}(J, X)$. If $\phi \in BUC_w(G, X)$, $\mathrm{sp}_{\mathscr{F}}(\phi) \subseteq \{1\}$ and $\phi|_J \in E_w(J, X)$ then $\phi|_J - M(\phi/w)w \in \mathscr{F}$.*

PROOF. By Theorem 4.3 (c), $\Delta_t\phi|_J \in \mathscr{F}$ for all $t \in J^{N+1}$. Therefore the result follows from Theorem 2.2. □

LEMMA 4.5. *Let $\mathscr{F}$ be a Λ_w-class and $\phi \in BUC_w(G, X)$. Assume either* (a) *w has polynomial growth*; *or* (b) *$\Delta_t(\gamma^{-1}\phi)|_J \in \mathscr{F}$ for all $t \in J^{n(\gamma)}$, $\gamma \in \mathrm{sp}_{\mathscr{F}}(\phi)$ and some $n(\gamma) \in \mathbb{N}$. Also assume that $\gamma^{-1}\phi$ is w-ergodic on J and $M((1/w)\gamma^{-1}\phi|_J)w \in \mathscr{F}$ for all $\gamma \in \mathrm{sp}_{\mathscr{F}}(\phi)$. Then $\mathrm{sp}_{\mathscr{F}}(\phi)$ contains no isolated points.*

PROOF. Suppose γ is an isolated point of $\mathrm{sp}_{\mathscr{F}}(\phi)$. Take an open neighbourhood U of γ in $\hat{G}$ such that $U \cap \mathrm{sp}_{\mathscr{F}}(\phi) = \{\gamma\}$. Choose $f \in L_w^1(G)$ such that $\hat{f}(\gamma) \neq 0$ and $\mathrm{supp}(\hat{f}) \subseteq U$. Then $\mathrm{sp}_{\mathscr{F}}(\phi * f) \subseteq \{\gamma\}$ and so $\mathrm{sp}_{\mathscr{F}}(\gamma^{-1}(\phi * f)) \subseteq \{1\}$. By Corollary 2.5, $\gamma^{-1}(\phi * f) = (\gamma^{-1}\phi) * (\gamma^{-1} f)$ is w-ergodic on J and

$$M((1/w)\gamma^{-1}(\phi * f)|_J) = M((1/w)\gamma^{-1}\phi)\hat{f}(\gamma).$$

If (a) holds, then $\gamma^{-1}(\phi * f)|_J - M((1/w)\gamma^{-1}\phi|_J)w \in \mathscr{F}$ by Corollary 4.4. If (b) holds, then $\Delta_t(\gamma^{-1}(\phi * f))|_J \in \mathscr{F}$ for all $t \in J^{n(\gamma)}$, $\gamma \in \mathrm{sp}_{\mathscr{F}}(\phi)$ by Proposition 2.4. By the difference Theorem 2.2 we again conclude $\gamma^{-1}(\phi * f)|_J - M((1/w)\gamma^{-1}\phi|_J)w \in \mathscr{F}$. Hence $(\phi * f)|_J \in \mathscr{F}$ which means $\gamma \notin \mathrm{sp}_{\mathscr{F}}(\phi)$. This is a contradiction and so $\mathrm{sp}_{\mathscr{F}}(\phi)$ contains no isolated points. □

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Recall that a subset of a topological space is called *perfect* if it is closed and has no isolated points. It is *residual* if it is closed and has no non-empty perfect subsets. Thus, a subset of the reals (or any locally compact Hausdorff second countable space [31, page 28]) is residual if and only if it is closed and countable. Moreover, a residual set without isolated points is empty.

PROPOSITION 4.6. *Let $\mathscr{F}$ be a BUC_w-invariant closed subspace of $BUC_w(J, X)$ containing $C_{w,0}(J, X)$ and let $\phi \in BUC_w(G, X)$. Then $\mathrm{sp}_{\mathscr{F}/w}(\phi/w) \subseteq \mathrm{sp}_{\mathscr{F}}(\phi)$, with equality if $\mathrm{sp}_{\mathscr{F}}(\phi)$ is residual.*

PROOF. Let $f \in I_{\mathscr{F}}(\phi)$. So $(\phi * f)|_J \in \mathscr{F}$ and by (2.7), $((\phi/w) * f)|_J \in \mathscr{F}/w$. Hence $f \in I_{\mathscr{F}/w}(\phi/w)$, showing $I_{\mathscr{F}}(\phi) \subseteq I_{\mathscr{F}/w}(\phi/w)$ and $\mathrm{sp}_{\mathscr{F}/w}(\phi/w) \subseteq \mathrm{sp}_{\mathscr{F}}(\phi)$.

Now assume that $\mathrm{sp}_{\mathscr{F}}(\phi) = \{1\}$. Given $\gamma \in \hat{G}$, $\gamma \neq 1$ there exists $f \in L^1_w(G)$ such that $\hat{f}(\gamma) \neq 0$ and $(\phi * f)|_J \in \mathscr{F}$. By (2.7), $((\phi/w) * f)|_J \in (\mathscr{F}/w)$. Hence $\gamma \notin \mathrm{sp}_{\mathscr{F}/w}(\phi/w)$ showing $\mathrm{sp}_{\mathscr{F}/w}(\phi/w) \subseteq \{1\}$. But $\mathrm{sp}_{\mathscr{F}/w}(\phi/w) \neq \emptyset$ as $\phi \notin \mathscr{F}$, so $\mathrm{sp}_{\mathscr{F}/w}(\phi/w) = \{1\}$.

Finally let γ be an isolated point of $\mathrm{sp}_{\mathscr{F}}(\phi)$. Choose $f \in L^1_w(G)$ such that $\hat{f} = 1$ in a neighbourhood U of γ and $\mathrm{supp}\hat{f}) \cap \mathrm{sp}_{\mathscr{F}}(\phi) = \{\gamma\}$. Then $\mathrm{sp}_{\mathscr{F}}(\phi * f) = \{\gamma\}$ and it follows from the previous paragraph that $\mathrm{sp}_{\mathscr{F}/w}((\phi * f)/w) = \{\gamma\}$. Now

$$\mathrm{sp}_{\mathscr{F}/w}\left(\frac{\phi * f}{w}\right) \subseteq \mathrm{sp}_{\mathscr{F}/w}\left(\frac{\phi * f}{w} - \frac{\phi}{w} * f\right) \cup \mathrm{sp}_{\mathscr{F}/w}\left(\frac{\phi}{w} * f - \frac{\phi}{w}\right) \cup \mathrm{sp}_{\mathscr{F}/w}\frac{\phi}{w}.$$

By (2.7), $\mathrm{sp}_{\mathscr{F}/w}((\phi * f)/w) - (\phi/w) * f) = \emptyset$. Moreover, we can choose $g \in L^1(G)$ such that $\hat{g}(\gamma) = 1$ and $\mathrm{supp}(\hat{g}) \subseteq U$. Hence $((\phi/w) * f - (\phi/w)) * g = 0$ and so $\gamma \notin \mathrm{sp}((\phi/w) * f - (\phi/w))$. Thus $\gamma \in \mathrm{sp}_{\mathscr{F}/w}(\phi/w)$. If $\mathrm{sp}_{\mathscr{F}}(\phi)$ is residual, then each of its points is either isolated or a limit of isolated points. Since these spectra are closed, $\mathrm{sp}_{\mathscr{F}}(\phi) \subseteq \mathrm{sp}_{\mathscr{F}/w}(\phi/w)$. □

The following is a generalization of a theorem of Loomis [23], who considered the case $w = 1$ and $\mathscr{F} = AP(G, \mathbb{C})$, the space of almost periodic functions (see also [22, page 92]). For the general case of bounded functions, see [5, Section 4] and [2, 3, 9, 33]. A similar result is proved in [11, Theorem 6.1] under different assumptions on w and ϕ.

THEOREM 4.7. *Let $\mathscr{F}$ be a Λ_w-class and $\phi \in BUC_w(G, X)$. Assume that $\gamma^{-1}\phi$ is w-ergodic on J and $M((1/w)\gamma^{-1}\phi|_J)w \in \mathscr{F}$ for all $\gamma \in \mathrm{sp}_{\mathscr{F}}(\phi)$. If $\mathrm{sp}_{\mathscr{F}}(\phi)$ is residual, then $\phi|_J \in \mathscr{F}$.*

PROOF. By Proposition 4.6, $\mathrm{sp}_{\mathscr{F}/w}(\phi/w)$ is residual. By Lemma 4.5 applied to the function ϕ/w, $\mathrm{sp}_{\mathscr{F}/w}(\phi/w)$ contains no isolated points. Hence $\mathrm{sp}_{\mathscr{F}/w}(\phi/w) = \emptyset$ and by Theorem 4.3 (a), $(\phi/w)|_J \in \mathscr{F}/w$ giving the result. □

Before completing this section we compare $\mathrm{sp}_{\mathscr{F}}(\phi)$ for the case $\mathscr{F} = C_0(\mathbb{R}_+, X)$ and the Beurling specrtra of orbits of representations with spectra used by other authors. For a strongly measurable bounded function $\phi : \mathbb{R}_+ \to X$, its *Laplace transform* $\tilde{\phi}$, defined by $\tilde{\phi}(z) = \int_0^\infty e^{-tz}\phi(t)\,dt$, is holomorphic for $\mathrm{Re}(z) > 0$. A point $\lambda \in i\mathbb{R}$ is a *regular point* if $\tilde{\phi}$ has a holomorphic extension to a neighbourhood of λ. The *singular set*, or set of points in $i\mathbb{R}$ which are not regular points is denoted $\sigma^+(\phi)$. It is known (see [1, 4] and the references therein) that if $\phi \in BUC(\mathbb{R}_+, X)$ and $\sigma^+(\phi) = \emptyset$, then $\phi \in C_0(\mathbb{R}_+, X)$. Moreover, $\sigma^+(\phi) \subseteq \alpha(\mathrm{sp}(\phi))$ where $\alpha : \hat{\mathbb{R}} \to \mathbb{R}$ is the natural isomorphism given by $\alpha(\gamma_s) = s$, where $\gamma_s(t) = e^{ist}$ for $s, t \in \mathbb{R}$.

COROLLARY 4.8. *Let $\phi \in BUC_w(\mathbb{R}, X)$ and $\mathscr{F} = C_{w,0}(\mathbb{R}_+, X)$. If $\mathrm{sp}_{\mathscr{F}}(\phi)$ is residual, then $\alpha(\mathrm{sp}_{\mathscr{F}}(\phi)) \subseteq \sigma^+((\phi/w)|_J)$.*

PROOF. By Proposition 4.6 it suffices to take $w = 1$. Again we begin with the case $\mathrm{sp}_{\mathscr{F}}(\phi) = \{1\}$. If $0 \notin \sigma^+(\phi|_J)$, then by the Ingham inequality [1, Lemma 3.1, (3.1)], $P\phi(t) = \int_0^t \phi(s)\,ds$ is bounded. By [6, Proposition 2.2], ϕ is ergodic. Now for each $t \in \mathbb{R}$ we have $\mathrm{sp}_{\mathscr{F}}(\Delta_t\phi) \subseteq \mathrm{sp}_{\mathscr{F}}(\phi) = \{1\}$ and so, by Theorem 4.3 (c), $\Delta_t\phi|_J \in \mathscr{F}$. By Theorem 2.2, $\phi \in \mathscr{F}$, contradicting $\mathrm{sp}_{\mathscr{F}}(\phi) = \{1\}$. Thus $0 = \alpha(1) \in \sigma^+(\phi|_J)$.

Now let γ be an isolated point of $\mathrm{sp}_{\mathscr{F}}(\phi)$. Choose $f \in L^1(\mathbb{R})$ such that $\hat{f} = 1$ in a neighbourhood U of γ and $\mathrm{supp}(\hat{f}) \cap \mathrm{sp}_{\mathscr{F}}(\phi) = \{\gamma\}$. Then, $\mathrm{sp}_{\mathscr{F}}(\phi * f) = \{\gamma\}$ and by the previous paragraph, $\alpha(\gamma) \in \sigma^+((\phi * f)|_J)$. But

$$\sigma^+((\phi * f)|_J) \subseteq \alpha(\mathrm{sp}(\phi * f - \phi)) \cup \sigma^+(\phi|_J).$$

We can choose $g \in L^1(\mathbb{R})$ such that $\hat{g}(\gamma) = 1$ and $\mathrm{supp}(\hat{g}) \subseteq U$. Hence

$$(\phi * f - \phi) * g = 0$$

and so $\gamma \notin \mathrm{sp}(\phi * f - \phi)$. Thus $\alpha(\gamma) \in \sigma^+(\phi|_J)$.

Each $\gamma \in \mathrm{sp}_{\mathscr{F}}(\phi)$ is either isolated or a limit of isolated points and $\sigma^+(\phi|_J)$ is closed, so the proposition is proved. □

We have been unable to determine whether in general

$$\alpha(\mathrm{sp}_{\mathscr{F}}(\phi)) \subseteq \sigma^+((\phi/w)|_J).$$

However, in using these spectra it is frequently assumed that $\mathrm{sp}_{\mathscr{F}}(\phi)$ or $\sigma^+(\phi|_J)$ is residual. See for example [2, 3, 9] and Section 5 below. In any case, $\mathrm{sp}_{\mathscr{F}}(\phi)$ is the optimal spectrum for determining membership of $\mathscr{F}$. It also has the advantage of being defined for functions on groups more general than $\mathbb{R}$.

Finally let $T : G \to L(X)$ be a strongly continuous representation dominated by a non-quasianalytic weight w. Following [24] we define the *Fourier transform* with respect to T of a function $f \in L^1_w(G)$ at $x \in X$ by $\hat{f}(T)x = \int_G f(s)T(-s)x\,d\mu(s)$. Let $A_1(T)$ be the smallest closed unital subalgebra of $L(X)$ containing each such operator $\hat{f}(T)$. The maximal ideal space $M_1(T)$ of $A_1(T)$ is homeomorphically embedded in $\hat{G} \cup \{\infty\}$ the one point compactification of $\hat{G}$. The image of this embedding is denoted by $\sigma_1(T)$ and is called the *ring spectrum* of T (see [24, page 132]).

THEOREM 4.9. *Let $T : G \to L(X)$ be a strongly continuous representation dominated by a non-quasianalytic weight w. Then $\mathrm{sp}_w(T(\cdot)x) \subset \sigma_1(T)$ for each $x \in X$.*

PROOF. Let $\gamma \in \hat{G} \setminus \sigma_1(T)$. Choose an open neighbourhood U of γ in $\hat{G}$ such that $\overline{U}$ is compact and $\overline{U} \cap \sigma_1(T) = \emptyset$. Also choose $f \in L^1_w(G)$ such that $\hat{f}(\gamma) = 1$ and $\mathrm{supp}(\hat{f}) \subset U$. Since $\hat{f}$ is 0 on a neighbourhood of $\sigma_1(T)$, it follows from [24, Lemma 2.2] that $\hat{f}(T) = 0$. Hence $0 = \hat{f}(T)T(\cdot)x = f * T(\cdot)x$ showing $f \in I_w(T(\cdot)x)$. Hence $\gamma \notin \mathrm{sp}_w(T(\cdot)x)$. □

5. Applications

In this section we apply the results of the previous sections, firstly to the convolution equations, secondly to the orbits of C_0-semigroups of operators and finally to the orbits of representations. Consider the equation

$$k * \phi + \sum_{j=1}^{m} a_j \phi_{t_j} = \lambda\phi + \psi \quad \text{on } G, \tag{5.1}$$

where $\phi, \psi \in BUC_w(G, X)$, $k \in L^1_w(G)$, $t_j \in G$ and $a_j, \lambda \in \mathbb{C}$.

The convolution operator $B : BUC_w(G, X) \to BUC_w(G, X)$ defined by

$$B\phi = k * \phi + \sum_{j=1}^{m} a_j \phi_{t_j}$$

has characteristic function $\theta_B : \hat{G} \to \mathbb{C}$ defined by

$$\theta_B(\gamma) = \hat{k}(\gamma) + \sum_{j=1}^{m} a_j \gamma(t_j).$$

The inverse image $\theta_B^{-1}(\lambda)$ is sometimes called the spectrum of (5.1). See [15], [30, page 289]. Our aim is to determine the point spectrum $\sigma_p(B)$ of B.

THEOREM 5.1. *Suppose* (5.1) *holds and $\psi|_J \in \mathcal{F}$ for some BUC_w-invariant closed subspace $\mathcal{F}$ of $BUC_w(J, X)$. Then $\mathrm{sp}_{\mathcal{F}}(\phi) \subseteq \theta_B^{-1}(\lambda)$.*

PROOF. Take any $\gamma \in \hat{G}\backslash\theta_B^{-1}(\lambda)$ and choose $f \in L^1_w(G)$ such that $\hat{f}(\gamma) \neq 0$. If $g = k * f + \sum_{j=1}^m a_j f_{t_j} - \lambda f$ then by (5.1), $\phi * g = \psi * f$. By Proposition 2.4, $(\psi * g)|_J \in \mathscr{F}$ and so $g \in I_{\mathscr{F}}(\phi)$. Since $\hat{g}(\gamma) \neq 0$ we conclude that $\gamma \notin \mathrm{sp}_{\mathscr{F}}(\phi)$. □

The following is an immediate consequence of Theorem 4.3. For the case $w = 1$, $\mathscr{F} = C_0(G, \mathbb{C})$, $\lambda = 0$ and ϕ slowly oscillating, part (a) is a classical tauberian theorem of Pitt [29]. See also [32, 7.2.7].

COROLLARY 5.2. *Suppose the conditions of Theorem* 5.1 *are satisfied.*

(a) *If* $\theta_B^{-1}(\lambda) = \emptyset$, *then* $\phi|_J \in \mathscr{F}$.

(b) *If* $\theta_B^{-1}(\lambda) = \{\gamma_1, \dots, \gamma_n\}$ *and* w *has polynomial growth of order* N, *then* $\phi = \psi + \sum_{j=1}^n \eta_j \gamma_j$ *for some* $\psi, \eta_j \in BUC_w(G, X)$ *with* $\psi|_J \in \mathscr{F}$ *and* $\Delta_t \eta_j|_J \in \mathscr{F}$ *for each* $t \in G^{N+1}$.

In our next application we use $\theta_B(\hat{G})$, the range of θ_B. It is well-known that the closure of $\hat{k}(\hat{G})$ is the spectrum of k as an element of the Banach algebra $L^1_w(G)$.

COROLLARY 5.3. *The operator* B *has point spectrum* $\sigma_p(B) = \theta_B(\hat{G})$. *If also* w *has polynomial growth and* $\theta_B^{-1}(\lambda) = \{\gamma_1, \dots, \gamma_n\}$, *then every eigenfunction corresponding to* λ *is of the form* $\phi = \sum_{j=1}^n p_j \gamma_j$ *for some polynomials* $p_j \in P_w(G, X)$.

PROOF. Suppose $B\phi = \lambda\phi$ for some $\phi \in BUC_w(G, X)$. Applying Theorem 5.1 with $\mathscr{F} = \{0\}$ we find $\mathrm{sp}_w(\phi) \subseteq \theta_B^{-1}(\lambda)$. So if $\lambda \notin \theta_B(\hat{G})$ then the only solution of $B\phi = \lambda\phi$ is $\phi = 0$, showing $\sigma_p(B) \subseteq \theta_B(\hat{G})$. But for each $\gamma \in \hat{G}$, $B\gamma = \theta_B(\gamma)\gamma$. So $\sigma_p(B) = \theta_B(\hat{G})$. The second assertion follows from Lemma 4.1. □

COROLLARY 5.4. *Suppose* w *has polynomial growth,* $\phi \in BUC_w(G, X)$, $\hat{k}^{-1}(0)$ *is residual and* $\mathscr{F}$ *is a* Λ_w*-class. Then* $(k * \phi)|_J \in \mathscr{F}$ *if and only if* $\mathrm{sp}_{\mathscr{F}}(\phi) \subseteq \hat{k}^{-1}(0)$. *If also* (5.1) *holds with* $a_j = 0$, $\lambda \neq 0$ *and* $\psi|_J \in \mathscr{F}$, *then* $\phi|_J \in \mathscr{F}$ *if and only if* $\mathrm{sp}_{\mathscr{F}}(\phi) \subseteq \hat{k}^{-1}(0)$.

PROOF. If $(k * \phi)|_J \in \mathscr{F}$ then $\mathrm{sp}_{\mathscr{F}}(\phi) \subseteq \hat{k}^{-1}(0)$ by Theorem 5.1. Conversely, if $\mathrm{sp}_{\mathscr{F}}(\phi) \subseteq \hat{k}^{-1}(0)$ and $\gamma \in \mathrm{sp}_{\mathscr{F}}(\phi)$ then $k \in I_w(\gamma)$. By Corollary 3.2, $\gamma^{-1}(k * \phi) \in E_{w,0}(G, X)$. By Lemma 4.5, $\mathrm{sp}_{\mathscr{F}}(k * \phi)$ has no isolated points. But $\mathrm{sp}_{\mathscr{F}}(k * \phi) \subseteq \mathrm{sp}_{\mathscr{F}}(\phi) \subseteq \hat{k}^{-1}(0)$ which is residual. So $\mathrm{sp}_{\mathscr{F}}(k * \phi) = \emptyset$ showing $(k * \phi)|_J \in \mathscr{F}$. The statement concerning (5.1) is now obvious. □

EXAMPLES 5.5. (a) Take $G = \mathbb{R}$, $B\phi = k * \phi$ and $w(t) = (1 + |t|)^N$. If $\lambda \in \hat{k}(\hat{\mathbb{R}})$ and $\hat{k}^{-1}(\lambda) = \{\gamma_1, \dots, \gamma_n\}$ then every eigenfunction of B corresponding to λ is of the form $\phi = \sum_{j=1}^n p_j \gamma_j$ for some polynomials $p_j(t) = \sum_{i=0}^N a_{ji} t^i$, $a_{ji} \in X$. But then

$$k * \phi(t) = \lambda\phi(t) + \sum_{j=1}^n \sum_{l=0}^N b_{jl} t^l \gamma_j(t),$$

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where

$$b_{jl} = \sum_{m=1}^{N-l} a_{j,m+l} \binom{m+l}{l} (-1)^m \widehat{(t^m k)}(\gamma_j) \quad \text{and} \quad (t^m k)(t) = t^m k(t).$$

So ϕ is an eigenfunction if and only if each $b_{jl} = 0$. It follows that the eigenspace corresponding to λ is a direct sum $E(\lambda) = \sum_{j=1}^{n} E(\gamma_j)$ where $E(\gamma_j) = \operatorname{span}\{t^l \gamma_j : 0 \leq l \leq m(\gamma_j)\} \otimes X$. Here $m(\gamma_j) = \min(m-1, N)$, where m is the smallest positive integer for which $\widehat{(t^m k)}(\gamma_j) \neq 0$. In particular, if $\dim X = n < \infty$, then $\dim E(\lambda) = \sum_{j=1}^{n} (m(\gamma_j) + 1)n$.

(b) As a particular example, take $X = \mathbb{C}$ and $k(t) = \max(\min(1+t, 1-t), 0)$ for $t \in \mathbb{R}$. Hence

$$\hat{k}(\gamma_s) = \left(\frac{\sin(s/2)}{s/2}\right)^2, \quad \text{where} \quad \gamma_s(t) = \exp(i\,st).$$

So $\sigma_p(B) = \{\lambda : 0 \leq \lambda \leq 1\}$. It is easy to see that $m(\gamma)$ is always 0 or 1. Also, if $0 < \lambda \leq 1$ then $\hat{k}^{-1}(\lambda)$ is finite and $E(\lambda)$ has a basis consisting of those γ for which $\hat{k}(\gamma) = \lambda$ and, if $N \geq 1$, those $t\gamma$ for which $\hat{k}(\gamma) = \lambda$ and $\widehat{tk}(\gamma) = 0$. On the other hand, $E(0)$ is infinite dimensional. In particular, there are no characters γ for which $k * t^2\gamma = \lambda t^2 \gamma$. However, $k * k * t^m \gamma_{2\pi} = 0$ for $m = 0, 1, 2, 3$.

(c) Suppose (5.1) holds and $\lambda \notin \theta_B(\hat{G})$. Then $\phi \in C_{w,0}(G, X)$ if and only if $\psi \in C_{w,0}(G, X)$. Similarly, $\phi|_J \in \mathscr{F}_{w,0}(J, X)$ if and only if $\psi|_J \in \mathscr{F}_{w,0}(J, X)$.

(d) Take $G = \mathbb{R}$, $J = \mathbb{R}_+$, $w(t) = (1 + |t|)^N$ and $\mathscr{F} = C_{w,0}(\mathbb{R}_+, X)$ a Λ_w-class. Assume $\phi \in BUC_w(\mathbb{R}, X)$ and $\hat{k}^{-1}(0)$ is residual. By Corollary 5.4, $(k * \phi)|_J \in \mathscr{F}$ if and only if $\operatorname{sp}_{\mathscr{F}}(\phi) \subseteq \hat{k}^{-1}(0)$.

Finally, we present the following application of our results. For the case of bounded semigroups, that is $w = 1$, part (a) appears in [6, Theorem 4.1 (ii)]. The Beurling-Domar condition (1.3) is not used in the proof.

THEOREM 5.6. *Let w be a weight on $\mathbb{R}$, $x \in X$ and A be the generator of a C_0-semigroup of operators $T(t)$, $t \geq 0$ which is dominated by w.*

(a) *If $\sigma(A) \cap i\mathbb{R}$ is residual and $\sigma_p(A^*) \cap i\mathbb{R}$ is empty, then $(1/w)T(\cdot)x \in C_0(\mathbb{R}_+, X)$.*

(b) *If $\sigma(A) \cap i\mathbb{R}$ is finite, then $(1/w)T(\cdot)x = \sum_{j=1}^{n} \eta_j \gamma_j$, where $\eta_j \in BUC(\mathbb{R}_+, X)$, $\Delta_t \eta_j \in C_0(\mathbb{R}_+, X)$ for all $t \in \mathbb{R}_+$ and $\gamma_j(t) = e^{\lambda_j t}$ for $\lambda_j \in \sigma(A) \cap i\mathbb{R}$.*

PROOF. Note first that $\|T(t+h)x - T(t)x\| \leq cw(t)\|T(h)x - x\|$ and so $T(\cdot)x$ is w-uniformly continuous. Now let $J = \mathbb{R}_+$, $\mathscr{F} = C_0(J, X)$ and $v = \tilde{w}$, where $\tilde{w} : J \to X$ corresponds to w as in Remark 2.8. Then $(1/w)T(\cdot)x - (1/v)T(\cdot)x \in \mathscr{F}$ and so $(1/v)T(\cdot)x$ is uniformly continuous. Next, we may assume $x \in D(A^2)$, the domain of A^2, since this space is dense in X. We define

$$\phi(t) = \begin{cases} \dfrac{v(0)}{v(t)} T(t)x, & \text{for } t \geq 0; \\ x \cos t + \left(A - \dfrac{v'(0)}{v(0)}\right) x \sin t, & \text{for } t < 0, \end{cases}$$

$$\psi(t) = \begin{cases} -\dfrac{v(0)v'(t)}{v^2(t)} T(t)x, & \text{for } t \geq 0; \\ -(1 + A^2)x \sin t + \dfrac{v'(0)}{v(0)}(A \sin t - \cos t)x, & \text{for } t < 0. \end{cases}$$

Then $\phi, \psi \in BUC(\mathbb{R}, X)$, $\phi' = A\phi + \psi$ on $\mathbb{R}$ and $\psi|_J \in \mathscr{F}$. By [6, Theorem 3.3] $\mathrm{sp}_{\mathscr{F}}(\phi) \subseteq \sigma(A) \cap i\mathbb{R}$. For part (a), $\mathrm{sp}_{\mathscr{F}}(\phi)$ is residual and by Examples 5.5 (c), $\gamma\phi$ is ergodic on J with mean 0 for all $\gamma \in \hat{\mathbb{R}}$. Hence, applying Theorem 4.7 with $w = 1$ we conclude $\phi|_J \in \mathscr{F}$. For part (b), $\mathrm{sp}_{\mathscr{F}}(\phi) \subseteq \{\gamma_j : 1 \leq j \leq n\}$ for some $\gamma_j(t) = e^{\lambda_j t}$, $\lambda_j \in \sigma(A) \cap i\mathbb{R}$. By Theorem 4.3 (c), $\phi = \psi + \sum_{j=1}^{n} \psi_j \gamma_j$ for some $\psi, \psi_j \in BUC(\mathbb{R}, X)$ with $\psi|_J \in \mathscr{F}$ and $\Delta_t \psi_j|_J \in \mathscr{F}$ for each $t \in \mathbb{R}$. Now replace ψ_1 by $\gamma_1^{-1}\psi_1$ and set $\eta_j = (v/w)(\psi_j/v(0))|_J$. The theorem follows readily. □

THEOREM 5.7. *Let $T : G \to L(X)$ be a non-trivial strongly continuous representation dominated by a weight w satisfying* (1.1)-(1.5). *Then either the ring spectrum $\sigma_1(T)$ is non-residual or the unitary point spectrum $\sigma_{up}(T^*)$ is non-empty.*

PROOF. Assume $\sigma_1(T)$ is residual and $\sigma_{up}(T^*) = \emptyset$. Let $\mathscr{F} = \{0\}$, a Λ_w-class. By Theorem 2.6, each orbit $T(\cdot)x$ is totally w-ergodic and $M((\gamma/w)T(\cdot)x) = 0$ for all $\gamma \in \hat{G}$. By Theorem 4.9, $\mathrm{sp}_w(T(\cdot)x) \subset \sigma_1(T)$ and so $\mathrm{sp}_{\mathscr{F}}(T(\cdot)x)$ is residual for each $x \in X$. By Theorem 4.7, $T(\cdot)x \in \mathscr{F}$, showing that T is trivial. □

REMARK 5.8. (a) The assumptions of Theorem 5.6 (b) are readily satisfied in practice. For example, if $X = \mathbb{C}^2$ and $A = \left[\begin{smallmatrix} i & 1 \\ 0 & i \end{smallmatrix}\right]$ then $T(t) = e^{it}\left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right]$ is dominated by $w(t) = 1 + |t|$ and $\sigma(A) = \sigma_p(A^*) = \{i\}$. Moreover,

$$\frac{1}{w}T(\cdot)x = \eta\gamma, \quad \text{where} \quad \eta(t) = \left(\frac{x_1 + tx_2}{1+t}, \frac{x_2}{1+t}\right) \text{ and } \gamma(t) = e^{it}.$$

Clearly, $\Delta_t \eta \in C_0(\mathbb{R}_+, X)$ for all $t \in \mathbb{R}_+$. In general, if w has polynomial growth of order N, then the conclusion of Theorem 5.6 (b) can be strengthened to $T(\cdot)x = \sum_{j=1}^{n} \eta_j \gamma_j$, where $\eta_j \in BUC_w(\mathbb{R}_+, X)$, $\Delta_t \eta_j \in C_0(\mathbb{R}_+, X)$ for all $t \in \mathbb{R}_+^{N+1}$ and $\gamma_j(t) = e^{\lambda_j t}$ for $\lambda_j \in \sigma(A) \cap i\mathbb{R}$.

(b) A proof of Theorem 5.7, under different assumptions on the weight, is contained in the proof of [11, Theorem 3.2]. In particular, the authors there require that $\log w(nt) = o\left(\sqrt{n}\right)$ as $n \to \infty$ for each $t \in G$. Their proof is different from ours—instead of exploiting ergodicity they use Šilov's idempotent theorem and a theorem of Zarrabi [35]. The result for representations of groups is then used to prove global stability for semigroup representations $T : J \to L(X)$ dominated by a weight w satisfying

$$\liminf_{t\in J} \frac{w(s+t)}{w(t)} \geq 1 \quad \text{for each } s \in J \tag{5.2}$$

The key ingredient is [11, Proposition 3.1] which exploits a method developed by several authors (see [10, 11, 28]) of associating with T a limit representation which extends to a group representation $U : G \to L(Y)$ on a different space Y. This representation is dominated by an associated reduced weight given by

$$w_1(s) = \begin{cases} \limsup_{t\in J}(w(s+t)/w(t)) & \text{for } s \in J; \\ \inf\{w_1(t) : t \in J, s \leq t\} & \text{for } s \in G. \end{cases}$$

Applying the same argument with the symmetric reduced weight $w_2(s) = w_1(s) + w_1(-s)$, we obtain the following as a consequence of Theorem 5.7.

COROLLARY 5.9. *Let $T : J \to L(X)$ be a strongly continuous representation dominated by a weight w satisfying* (5.2). *Suppose w_2 satisfies* (1.1)–(1.5), *the ring spectrum $\sigma_1(T)$ is residual and the unitary point spectrum $\sigma_{up}(T^*)$ is empty. Then* $\lim_{t\in J}(1/w(t))\|T(t)x\| = 0$ *for each $x \in X$.*

(c) Let $w(t) = e^{t^p}$ on $\mathbb{R}_+$. If $0 < p < 1$ then $w_1(t) = 1$ and if $p = 1$ then $w_1(t) = \max(1, e^t)$. To our knowledge, no examples have been given of weights satisfying (5.2) for which the reduced weight is non-quasianalytic and different from 1. In general, if w satisfies (5.2) and $w_1 = 1$, then $\lim_{t\in J}(w(s+t)/w(t)) = 1$ for each $s \in J$. So for each $\varepsilon > 0$, there exists $u \in J$ such that

$$|w(s+u+t) - w(u+t)| < \varepsilon w(u+t) \quad \text{for all } t \in J.$$

Hence $|(\Delta_s w/w)_u| < \varepsilon$. This proves that $|\Delta_s w| \in E_{w,0}(J, \mathbb{C})$, which is condition (2.4). If also $J = \mathbb{R}_+$, then

$$\lim_{t\to\infty} \frac{w(s+t)}{w(t)} = \lim_{t\in J} \frac{w(s+t)}{w(t)} = 1 \quad \text{for each } s \in J.$$

Hence $|\Delta_s w| \in C_{w,0}(J, \mathbb{C})$.

(d) As a final example, consider the weight $w(t) = (1 + |\sin t|)(1 + |t|)$ on $\mathbb{R}_+$. This satisfies neither (5.2) nor (2.4). However, $w_1(t) = 1 + |\sin t|$ on $\mathbb{R}_+$. Moreover, $\Delta_s w \in E_w(\mathbb{R}_+, \mathbb{C})$ for each $s \in \mathbb{R}_+$, but $M((1/w)\Delta_s w)$ is not always 0.

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