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A FUJITA-TYPE RESULT FOR A SEMILINEAR EQUATION IN HYPERBOLIC SPACE

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Abstract

In this paper, we study the positive solutions for a semilinear equation in hyperbolic space. Using the heat semigroup and by constructing subsolutions and supersolutions, a Fujita-type result is established.

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1. Introduction

This paper is devoted to the time-weighted semilinear parabolic problem:

$$\begin{cases} u_t = \Delta_{\mathbb{H}^n} u + e^{\alpha t} u^p + e^{\beta t} u^q & \text{in } \mathbb{H}^n \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{H}^n, \end{cases} \tag{1.1}$$

where $\alpha, \beta \in \mathbb{R}^+$, $p > 1$, $q > 1$, $u_0(x) \geq 0$, $\mathbb{H}^n$ is the n-dimensional hyperbolic space and $\Delta_{\mathbb{H}^n}$ denotes the Laplace–Beltrami operator in $\mathbb{H}^n$. Here we shall derive a Fujita-type result for all positive solutions u.

In [6], Fujita considered the following Cauchy problem:

$$\begin{cases} u_t = \Delta u + u^p & \text{in } \mathbb{R}^n \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{1.2}$$

If $1 < p < 1 + 2/n$, then every nonnegative solution of (1.2) blows up eventually except the trivial solution $u \equiv 0$. If $p > 1 + 2/n$, there are many nonnegative initial values u_0 which give global solutions. The solution of (1.2) in critical exponents $p = 1 + 2/n$ blows up (see [1, 7, 15]). Since then, there have been numerous interesting results on the Fujita-type conclusion to various kinds of equations, such as equations with weight

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functions [10, 11], systems [3, 13] etc. We refer readers to the surveys [5, 8, 12] and the references therein.

An extensive literature has been devoted to the Cauchy problem

$$\begin{cases} u_t = \Delta_{\mathbb{H}^n} u + e^{\alpha t} u^p & \text{in } \mathbb{H}^n \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{H}^n. \end{cases} \tag{1.3}$$

Let $p^*_{\mathbb{H}^n} := 1 + \alpha/\lambda_0$, where $\lambda_0 = (n-1)^2/4$ is the infimum of the L^2-spectrum of $-\Delta_{\mathbb{H}^n}$. Bandle *et al.* [2] proved that if $1 < p < p^*_{\mathbb{H}^n}$, then every nontrivial solution of (1.3) blows up in finite time, if $p > p^*_{\mathbb{H}^n}$, problem (1.3) possesses global solutions for small initial data and, if $p = p^*_{\mathbb{H}^n}$, $\alpha > \frac{2}{3}\lambda_0$, there exist global solutions. If $p = p^*_{\mathbb{H}^n}$, $0 < \alpha \le \frac{2}{3}\lambda_0$, the blow-up does not occur, which was obtained by Wang and Yin [14].

For (1.1), the proof of local solutions relies on the parabolic comparison principle. Furthermore, using the heat semigroup and subsolution and supersolution methods, we get the finite-time blow-up and global existence of positive solutions in different exponents.

Now we address our results.

For $T < +\infty$, the local existence and uniqueness of weak solutions to problem (1.1) is obtained by the comparison principle (see Theorem 2.3).

Then, on global solutions of problem (1.1), we have the following theorem.

THEOREM 1.1. *Set $p_\alpha = 1 + (4\alpha)/(n-1)^2$ and $q_\beta = 1 + (4\beta)/(n-1)^2$.*

(i) *If $1 < p < p_\alpha$ or $1 < q < q_\beta$ and initial data $u_0 \in C^0(\mathbb{H}^n) \cap L^\infty(\mathbb{H}^n)$, every nonnegative solution of (1.1) blows up in finite time.*

(ii) *If $p > p_\alpha$ and $q > q_\beta$, there are global nonnegative solutions to (1.1) for initial data $u_0 \le \phi$, where ϕ is the positive classical solution of $\Delta_{\mathbb{H}^n}\phi = -((n-1)^2/4)\phi$.*

(iii) *If $p = p_\alpha$ and $q = q_\beta$, then there exist global solutions to (1.1) for initial data $u_0 \le g_n(x, 0, t_0)$, where $g_n(x, y, t)$ is the heat kernel of hyperbolic space.*

The organization of this paper is as follows. In Section 2, we present the basic properties of the hyperbolic space and prove the existence of local solutions. The proof of Theorem 1.1 is described in Section 3.

2. Preliminaries and local existence

In this section, we present the basic properties in the hyperbolic space and prove the local existence of weak solutions to problem (1.1).

The hyperbolic space $\mathbb{H}^n$ is equivalent to the unit ball $B_1 \subset \mathbb{R}^n$ endowed with the Poincaré metric

$$ds^2 = \frac{4}{(1-|x|^2)^2}\, dx^2.$$

The geodesic distance between any $x \in \mathbb{H}^n$ and 0 is given by

$$\rho(x) := \int_0^{|x|} \frac{2}{1-s^2}\, ds = \ln\Big(\frac{1+|x|}{1-|x|}\Big).$$

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The volume element of $\mathbb{H}^n$ is

$$d\mu = \frac{2^n}{(1-|x|^2)^n}\, dx_1 \cdots dx_n = (\sinh\rho)^{n-1}\, d\rho\, d\theta,$$

$dx_1 \cdots dx_n = dx$ being the Lebesgue measure in $\mathbb{R}^n$ and (ρ, θ) being polar geodesic coordinates in $\mathbb{H}^n \backslash \{0\}$.

Then the Laplace–Beltrami operator is given by the following equalities:

$$\Delta_{\mathbb{H}^n} = \frac{1}{4}(1-|x|^2)^2 \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{n-2}{2}(1-|x|^2) \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

and

$$\Delta_{\mathbb{H}^n} = \frac{\partial^2}{\partial\rho^2} + (n-1)\coth\rho \frac{\partial}{\partial\rho} + \frac{1}{(\sinh\rho)^2}\Delta_\theta,$$

Δ_θ being the Laplace–Beltrami operator on the $(n-1)$-dimensional sphere of $\mathbb{R}^n$.

From Davies (see [4, Theorem 5.7.2]), the heat kernel $g_n(x, y, t)$ in $\mathbb{H}^n$ satisfies

$$c_n^{-1} h_n(d(x, y), t) \le g_n(x, y, t) \le c_n h_n(d(x, y), t), \tag{2.1}$$

where c_n is a positive constant and

$$h_n(d, t) = (4\pi t)^{-n/2}(1+d)(1+d+t)^{(n-3)/2} e^{-\lambda_0 t - ((n-1)/2)\, d - d^2/4t}, \quad \lambda_0 = \frac{(n-1)^2}{4}.$$

We shall use the following notation common in the theory of semigroups:

$$(e^{-\Delta_{\mathbb{H}^n} t}\psi)(x) := \int_{\mathbb{H}^n} g_n(x, y, t)\psi(y)\, d\mu_y, \quad x \in \mathbb{H}^n, \quad t > 0,$$

with $\psi \in \mathrm{C}(\mathbb{H}^n)$. By the semigroup property, we have for any $x, y, z \in \mathbb{H}^n$, $s, t \in [0, +\infty)$,

$$\int_{\mathbb{H}^n} g_n(x, y, t) g_n(y, z, s)\, d\mu_y = g_n(x, z, s+t) \tag{2.2}$$

and the conservation of probability implies that

$$\int_{\mathbb{H}^n} g_n(x, y, t)\, d\mu_y = 1.$$

Next, we give the proof of the local existence of weak solutions to (1.1). For any $T > 0$, we set $\Omega_T = \mathbb{H}^n \times (0, T)$. The definition of a weak solution of (1.1) is as follows.

Definition 2.1. $u \in \mathrm{C}(\overline{\Omega}_T)$ is called a continuous weak solution of (1.1) in $[0, T]$ if for any $\tau \in (0, T]$,

$$-\int_{\Omega_\tau} u(\psi_t + \Delta_{\mathbb{H}^n}\psi)\, d\mu\, dt = \int_{\mathbb{H}^n} u_0 \psi(\cdot, 0)\, d\mu + \int_{\Omega_\tau} (e^{\alpha t} u^p + e^{\beta t} u^q)\psi\, d\mu\, dt,$$

where any $\psi \in \mathrm{C}_0^{2,1}(\overline{\Omega}_\tau)$.

The following comparison principle is a conclusion of [2, Lemma 2.2].

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Lemma 2.2. *Let w be a continuous function satisfying*

$$\int_{\mathbb{H}^n\times(0,T)} w(\psi_t + \Delta_{\mathbb{H}^n}\psi)\, d\mu\, dt \geq 0 \quad \text{for all positive } \psi \in \mathrm{C}_0^\infty(\Omega_T),$$
$$w(x,t) \leq Ae^{cd(x,0)^2} \quad \text{for some positive } A, c$$

and $w(x,0) = 0$. Then $w \leq 0$ in Ω_T.

Let us prove local existence and uniqueness of weak solutions to problem (1.1).

Theorem 2.3. *Assume that $p > 1$, $q > 1$. Let u_0 be a positive function in $C^0(\mathbb{H}^n) \cap L^\infty(\mathbb{H}^n)$. Then there exists $T > 0$ such that (1.1) has a unique continuous weak solution $u \in \mathrm{L}^\infty(\Omega_T)$.*

Proof. Due to $u_0 \in C^0(\mathbb{H}^n) \cap L^\infty(\mathbb{H}^n)$, there exists $M > 1$ such that $|u_0(x)| < M$ for $x \in \mathbb{H}^n$.

Let T satisfy

$$\int_0^T (e^{\alpha s} + e^{\beta s})\, ds = \frac{1}{(pq-1)M^{pq-1}}.$$

Define

$$\bar{u} = M\Big[1 - (pq-1)M^{pq-1}\int_0^t (e^{\alpha s} + e^{\beta s})\, ds\Big]^{-1/(pq-1)}, \quad t \in [0,T).$$

A simple calculation gives

$$\bar{u}_t = (e^{\alpha t} + e^{\beta t})M^{pq}\Big[1 - (pq-1)M^{pq-1}\int_0^t (e^{\alpha s} + e^{\beta s})\, ds\Big]^{-pq/(pq-1)}.$$

If $p > 1, q > 1$, then $\bar{u}_t \geq \Delta_{\mathbb{H}^n}\bar{u} + e^{\alpha t}\bar{u}^p + e^{\beta t}\bar{u}^q$. In Cartesian coordinates, the problem (1.1) is degenerate on the boundary of B_1. It can be seen that $\bar{u}$ is a classical upper solution and $\underline{u} = 0$ is a lower solution to problem (1.1) with Dirichlet boundary value in the cylinder $B_{1-1/n} \times [0,T)$. For any fixed integer $n_0 \geq 2$, the sequence $\{u_n\}_{n>n_0}$ is uniformly bounded and equicontinuous in $B_{1-1/(n_0-1)} \times [0, T - 1/n_0]$. By choosing diagonal elements, the subsequence has a limit function u which is a continuous weak solution of (1.1).

Suppose that u_1 and u_2 are two bounded weak solutions of (1.1).

If we define $w = u_1 - u_2$, then they satisfy

$$\int_{\Omega_T} (w(\psi_t + \Delta_{\mathbb{H}^n}\psi) + \psi e^{\alpha t}(u_1^p - u_2^p) + \psi e^{\beta t}(u_1^q - u_2^q))\, d\mu\, dt = 0$$

for $\psi \in \mathrm{C}_0^\infty(\Omega_T)$. Set $w_+ = \max\{0, w\}$ and $Q = \{z \in \Omega_T | w_+(z) > 0\}$. Then, for $\psi \in \mathrm{C}_0^\infty(Q)$,

$$\int_Q w_+(\psi_t + \Delta_{\mathbb{H}^n}\psi + \psi e^{\alpha t}p\|u_1\|_\infty^{p-1} + \psi e^{\beta t}q\|u_1\|_\infty^{q-1})\, d\mu\, dt \geq 0,$$

where $p > 1$ and $q > 1$.

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Let $\zeta = e^{At}\psi$. Observe that

$$\begin{aligned} &w_+(\psi_t + \Delta_{\mathbb{H}^n}\psi + \psi e^{\alpha t} p\|u_1\|_\infty^{p-1} + \psi e^{\beta t} q\|u_1\|_\infty^{q-1}) \\ &\quad = e^{-At} w_+(\zeta_t + \Delta_{\mathbb{H}^n}\zeta - A\zeta + \zeta e^{\alpha t} p\|u_1\|_\infty^{p-1} + \zeta e^{\beta t} q\|u_1\|_\infty^{q-1}). \end{aligned}$$

If $A > \max_{[0,T]}\{e^{\alpha t} p\|u_1\|_\infty^{p-1} + e^{\beta t} q\|u_1\|_\infty^{q-1}\}$, then

$$\int_Q e^{-At} w_+[(e^{At}\psi)_t + \Delta_{\mathbb{H}^n}(e^{At}\psi)]\, d\mu\, dt \geq 0.$$

Using the definition of Q and the initial condition $w_+(x,0) = 0$, we have $w_+ = 0$ on the parabolic boundary of Q. By Lemma 2.2, we obtain $w_+ \leq 0$. If we exchange the roles of the solutions, then $w_+ = 0$. □

3. The proof of Theorem 1.1

In this section, our goal is to prove Theorem 1.1. Firstly, the definition of a mild solution of (1.1) is as follows.

Definition 3.1. Let $u_0 \in L^\infty(\mathbb{H}^n)$, $u_0 \geq 0$. By a mild solution to problem (1.1), we mean any nonnegative function $u \in C(\overline{\Omega}_T)$ such that

$$u(x,t) = e^{-\Delta_{\mathbb{H}^n} t} u_0 + \int_0^t e^{-\Delta_{\mathbb{H}^n}(t-s)}(e^{\alpha s} u^p + e^{\beta s} u^q)\, ds$$

for any $t \in [0,T]$.

In terms of the heat kernel, this definition is expressed as

$$u(x,t) = \int_{\mathbb{H}^n} g_n(x,y,t) u_0\, d\mu_y + \int_{\Omega_t} g_n(x,y,t-s)(e^{\alpha s} u^p + e^{\beta s} u^q)\, d\mu_y\, ds. \tag{3.1}$$

A mild solution is a classical solution which is based on classical regularity results and on the estimate (2.1).

In the hyperbolic space, the heat semigroup has the following estimate.

Lemma 3.2 [2]. *Let $u_0 \geq 0$, $u_0 \not\equiv 0$. Then, for any $T_0 > 0$, there exists a function $f(x) > 0$ which depends only on n, T_0 and u_0 such that*

$$(e^{-\Delta_{\mathbb{H}^n} t} u_0)(x) \geq t^{-3/2} e^{-\lambda_0 t} f(x)$$

for any $t \in [T_0, +\infty)$, $x \in \mathbb{H}^n$.

Mancini and Sandeep [9] proved global existence of solutions to a single equation.

Lemma 3.3. *Let $q > 1$ if $n = 2$ and $1 < q < (n+2)/(n-2)$ if $n \geq 3$. Then*

$$\Delta_{\mathbb{H}^n}\xi + \lambda\xi + \xi^q = 0 \tag{3.2}$$

has a positive solution for any $\lambda \leq (n-1)^2/4$.

Next, we give the proof of Theorem 1.1.

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PROOF OF THEOREM 1.1(i). The proof is by contradiction. Let u be a global nonnegative solution of (1.1). Multiplying the equality (3.1) by $g_n(x,z,T-t)$, integrating over $\mathbb{H}^n$ and using (2.2),

$$\begin{aligned}\int_{\mathbb{H}^n} g_n(x,z,T-t)u(z,t)\,d\mu_z\\ = \int_{\mathbb{H}^n} g_n(x,y,T)u_0\,d\mu_y + \int_{\Omega_t} g_n(x,y,T-s)(e^{\alpha s}u^p + e^{\beta s}u^q)\,d\mu_y\,ds.\end{aligned}$$

Set $\Psi_x(t) = \int_{\mathbb{H}^n} g_n(x,z,T-t)u(z,t)\,d\mu_z$. Jensen's inequality implies that

$$(\Psi_x(t))^p \le \int_{\mathbb{H}^n} g_n(x,z,T-t)u^p(z,t)\,d\mu_z.$$

Hence,

$$\int_0^t e^{\alpha s}(\Psi_x(s))^p\,ds \le \Psi_x(t) - \Psi_x(0)$$

and

$$\int_0^t e^{\beta s}(\Psi_x(s))^q\,ds \le \Psi_x(t) - \Psi_x(0).$$

Using a Gronwall-type inequality yields

$$\frac{p-1}{\alpha}(e^{\alpha t}-1) \le \frac{1}{(\Psi_x(0))^{p-1}} - \frac{1}{(\Psi_x(t))^{p-1}},$$

$$\frac{q-1}{\beta}(e^{\beta t}-1) \le \frac{1}{(\Psi_x(0))^{q-1}} - \frac{1}{(\Psi_x(t))^{q-1}}.$$

From Lemma 3.2,

$$f(x)T^{-3/2}e^{-\lambda_0 T} \le \Psi_x(0) \le \left(\frac{1}{p-1}\right)^{1/(p-1)}\left(\frac{\alpha}{e^{\alpha T}-1}\right)^{1/(p-1)}$$

and

$$f(x)T^{-3/2}e^{-\lambda_0 T} \le \Psi_x(0) \le \left(\frac{1}{q-1}\right)^{1/(q-1)}\left(\frac{\beta}{e^{\beta T}-1}\right)^{1/(q-1)}.$$

So,

$$\left(\frac{e^{\alpha T}-1}{\alpha}\right)^{1/(p-1)} T^{-3/2}e^{-\lambda_0 T} \le \left(\frac{1}{p-1}\right)^{1/(p-1)}\frac{1}{f(x)}$$

and

$$\left(\frac{e^{\beta T}-1}{\beta}\right)^{1/(q-1)} T^{-3/2}e^{-\lambda_0 T} \le \left(\frac{1}{q-1}\right)^{1/(q-1)}\frac{1}{f(x)}.$$

If $\alpha/(p-1) > \lambda_0$ or $\beta/(q-1) > \lambda_0$, there is a contradiction. □

Proof of Theorem 1.1(ii). Let $\phi(x)$ be the ground state corresponding to λ_0 ($\lambda_0 = (n-1)^2/4$), that is, ϕ is the positive classical solution of $\Delta_{\mathbb{H}^n}\phi = -\lambda_0\phi$ and $\|\phi\|_\infty < +\infty$ (see [2, Definition A.1 and Lemma A.1]).

Define

$$\bar{u} = e^{-\lambda_0 t}\zeta(t)\phi(x).$$

We have

$$\begin{aligned}&\bar{u}_t - \Delta_{\mathbb{H}^n}\bar{u} - e^{\alpha t}\bar{u}^p - e^{\beta t}\bar{u}^q \\ &\quad = e^{-\lambda_0 t}\phi(x)(\zeta' - e^{(\alpha-(p-1)\lambda_0)t}\phi^{p-1}\zeta^p - e^{(\beta-(q-1)\lambda_0)t}\phi^{q-1}\zeta^q).\end{aligned}$$

When $p \geq q$, let $\zeta(t)$ satisfy the problem

$$\begin{cases}\dot{\zeta} = e^{(\alpha-(p-1)\lambda_0)t}\|\phi\|_\infty^{p-1}\zeta^p + e^{(\beta-(q-1)\lambda_0)t}\|\phi\|_\infty^{q-1}\zeta^p, \\ \zeta(0) = 1.\end{cases}$$

A simple computation yields

$$\zeta(t) = \Big(1 - (p-1)\|\phi\|_\infty^{p-1}\frac{e^{(\alpha-(p-1)\lambda_0)t} - 1}{\alpha-(p-1)\lambda_0} - (p-1)\|\phi\|_\infty^{q-1}\frac{e^{(\beta-(q-1)\lambda_0)t} - 1}{\beta-(q-1)\lambda_0}\Big)^{-1/(p-1)}.$$

If $\alpha-(p-1)\lambda_0 < 0$ and $\beta-(q-1)\lambda_0 < 0$ are such that

$$\frac{\|\phi\|_\infty^{p-1}}{\alpha-(p-1)\lambda_0} + \frac{\|\phi\|_\infty^{q-1}}{\beta-(q-1)\lambda_0} > -\frac{1}{p-1},$$

then $\bar{u}$ is a global supersolution to (1.1) for $u_0 \leq \phi$.

When $p \leq q$, we choose $\zeta(t)$ to satisfy the problem

$$\begin{cases}\dot{\zeta} = e^{(\alpha-(p-1)\lambda_0)t}\|\phi\|_\infty^{p-1}\zeta^q + e^{(\beta-(q-1)\lambda_0)t}\|\phi\|_\infty^{q-1}\zeta^q, \\ \zeta(0) = 1.\end{cases}$$

Similarly, if $\alpha-(p-1)\lambda_0 < 0$ and $\beta-(q-1)\lambda_0 < 0$ are such that

$$\frac{\|\phi\|_\infty^{p-1}}{\alpha-(p-1)\lambda_0} + \frac{\|\phi\|_\infty^{q-1}}{\beta-(q-1)\lambda_0} > -\frac{1}{q-1},$$

then $\bar{u}$ is a global supersolution to (1.1) for $u_0 \leq \phi$.

Hence, if $\alpha-(p-1)\lambda_0 < 0$ and $\beta-(q-1)\lambda_0 < 0$ are such that

$$\frac{\|\phi\|_\infty^{p-1}}{\alpha-(p-1)\lambda_0} + \frac{\|\phi\|_\infty^{q-1}}{\beta-(q-1)\lambda_0} > \max\Big\{-\frac{1}{p-1}, -\frac{1}{q-1}\Big\},$$

then $\bar{u}$ is a global supersolution to (1.1) for $u_0 \leq \phi$. □

Proof of Theorem 1.1(iii). (1) $\alpha \geq \beta > \frac{2}{3}\lambda_0$ or $\beta \geq \alpha > \frac{2}{3}\lambda_0$.

We shall construct a global upper solution for (1.1) for $\alpha \geq \beta > \frac{2}{3}\lambda_0$.

Define

$$\bar{u} = \eta(t)g_n(x, 0, t+t_0), \qquad t_0 > 0.$$

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Since $p = 1 + \alpha/\lambda_0$, $q = 1 + \beta/\lambda_0$,

$$\begin{aligned}\bar{u}_t - \Delta_{\mathbb{H}^n}\bar{u} - e^{\alpha t}\bar{u}^p - e^{\beta t}\bar{u}^q &= g_n(\dot{\eta} - e^{\alpha t}g_n^{p-1}\eta^p - e^{\beta t}g_n^{q-1}\eta^q)\\ &= g_n(\dot{\eta} - e^{\alpha t}g_n^{\alpha/\lambda_0}\eta^{1+\alpha/\lambda_0} - e^{\beta t}g_n^{\beta/\lambda_0}\eta^{1+\beta/\lambda_0}).\end{aligned}$$

From (2.1),

$$g_n(x,0,t+t_0) \le c_n(4\pi)^{-n/2}(t+t_0)^{-3/2}e^{-\lambda_0(t+t_0)}[1+d(x,0)]\widetilde{g}(x)e^{-((n-1)/2)\,d(x,0)}$$

for $x \in \mathbb{H}^n$ and $t \in [0,+\infty)$, where

$$\widetilde{g}(x) = \begin{cases} 1, & n \le 3,\\ \left(1 + \dfrac{1+d(x,0)}{t_0}\right)^{-1/2}, & n > 3.\end{cases}$$

Hence, there exists $K \ge 1$ such that

$$g_n(x,y,t+t_0) \le K(t+t_0)^{-3/2}e^{-\lambda_0(t+t_0)}$$

for $x \in \mathbb{H}^n$ and $t \in [0,+\infty)$.

Let η solve the following problem:

$$\begin{cases} \dot{\eta} = K^{(\alpha+\beta)/\lambda_0}(e^{-\alpha t_0}(t+t_0)^{-3\alpha/2\lambda_0} + e^{-\beta t_0}(t+t_0)^{-3\beta/2\lambda_0})\eta^{1+\alpha/\lambda_0},\\ \eta|_{t=0} = \eta(0).\end{cases}$$

A direct calculation gives

$$\begin{aligned}\eta(t)^{-\alpha/\lambda_0} = \eta(0)^{-\alpha/\lambda_0} &+ \frac{K^{(\alpha+\beta)/\lambda_0}e^{-\alpha t_0}}{\dfrac{3}{2} - \dfrac{\lambda_0}{\alpha}}\Big((t+t_0)^{1-3\alpha/2\lambda_0} - t_0^{1-3\alpha/2\lambda_0}\Big)\\ &+ \frac{K^{(\alpha+\beta)/\lambda_0}e^{-\beta t_0}}{\dfrac{3}{2} - \dfrac{\lambda_0}{\beta}}((t+t_0)^{1-3\beta/2\lambda_0} - t_0^{1-3\beta/2\lambda_0}).\end{aligned}$$

If $\eta(0)$ is small enough and $\alpha \ge \beta > \frac{2}{3}\lambda_0$, then η exists and $0 < \eta < 1$ for all t. Then

$$\begin{aligned}&\dot{\eta} - e^{\alpha t}g_n^{\alpha/\lambda_0}\eta^{1+\alpha/\lambda_0} - e^{\beta t}g_n^{\beta/\lambda_0}\eta^{1+\beta/\lambda_0}\\ &\quad \ge \dot{\eta} - K^{\alpha/\lambda_0}e^{-\alpha t_0}(t+t_0)^{-3\alpha/2\lambda_0}\eta^{1+\alpha/\lambda_0} - K^{\beta/\lambda_0}e^{-\beta t_0}(t+t_0)^{-3\beta/2\lambda_0}\eta^{1+\beta/\lambda_0}\\ &\quad = (K^{(\alpha+\beta)/\lambda_0} - K^{\alpha/\lambda_0})e^{-\alpha t_0}(t+t_0)^{-3\alpha/2\lambda_0}\zeta^{1+\alpha/\lambda_0}\\ &\qquad + e^{-\beta t_0}(t+t_0)^{-3\beta/2\lambda_0}(K^{(\alpha+\beta)/\lambda_0}\eta^{1+\alpha/\lambda_0} - K^{\beta/\lambda_0}\eta^{1+\beta/\lambda_0}) \ge 0.\end{aligned}$$

Furthermore, we find that $\bar{u}$ is an upper solution of (1.1) if $u_0 \le g_n(x,0,t_0)$. When $\beta \ge \alpha > \frac{2}{3}\lambda_0$, we can get the same result as $\alpha \ge \beta > \frac{2}{3}\lambda_0$. So, global solutions of (1.1) exist for $\alpha \ge \beta > \frac{2}{3}\lambda_0$ or $\beta \ge \alpha > \frac{2}{3}\lambda_0$.

(2) $\alpha \ge \beta$, $0 < \beta \le \frac{2}{3}\lambda_0$ or $\beta \ge \alpha$, $0 < \alpha \le \frac{2}{3}\lambda_0$.

Set $\bar{u} = e^{-\lambda_0 t}\bar{\xi}(x)$.

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By a direct calculation, $\bar{u}$ is an upper solution of (1.1) if and only if the following inequality holds:

$$\Delta_{\mathbb{H}^n}\bar{\xi} + \lambda_0\bar{\xi} + \bar{\xi}^p + \bar{\xi}^q \le 0. \tag{3.3}$$

Assume that $\alpha \ge \beta$, $0 < \beta \le \frac{2}{3}\lambda_0$.

Case 1: $2 \le n < 8$. If $0 < \beta \le \frac{2}{3}\lambda_0$, then $q = 1 + \beta/\lambda_0 \le 1 + \frac{2}{3} < (n+2)/(n-2)$. Using Lemma 3.3, there exists a positive solution to (3.2). Let ξ be the positive solution of (3.2). Set $\xi_\sigma = \xi/\sigma$ for $\sigma = \max\{2^{1/(q-1)}, |\xi|_{L^\infty(\mathbb{H}^n)}\}$. We have $|\xi_\sigma|_{L^\infty(\mathbb{H}^n)} < 1$ and

$$\Delta_{\mathbb{H}^n}\xi_\sigma + \lambda_0\xi_\sigma + \xi_\sigma^p + \xi_\sigma^q \le \frac{1}{\sigma}\Delta_{\mathbb{H}^n}\xi + \frac{1}{\sigma}\lambda_0\xi + \frac{2}{\sigma^q}\xi^q \le \frac{1}{\sigma}(\Delta_{\mathbb{H}^n}\xi + \lambda_0\xi + \xi^q) = 0.$$

Thus, $\bar{\xi}(x) = \xi_\sigma(x)$ satisfies (3.3).

Case 2: $n \ge 8$. If $0 < \beta \le (4/(n-2))\lambda_0$, then $q = 1 + \beta/\lambda_0 < (n+2)/(n-2)$. Following Case 1, we can construct positive solutions of (3.3).

If $(4/(n-2))\lambda_0 \le \beta \le \frac{2}{3}\lambda_0$, then $q \ge (n+2)/(n-2)$. Choosing

$$\widetilde{q} < \frac{n+2}{n-2} \le q,$$

let ξ be the positive solution of the following equation:

$$\Delta_{\mathbb{H}^n}\xi + \lambda\xi + \xi^{\widetilde{q}} = 0.$$

We can construct a positive solution $\xi_\sigma = \xi/\sigma$, where $\sigma = \max\{2^{1/(\widetilde{q}-1)}, |\xi|_{L^\infty(\mathbb{H}^n)}\}$ as Case 1. Thus,

$$\begin{aligned}\Delta_{\mathbb{H}^n}\xi_\sigma + \lambda_0\xi_\sigma + \xi_\sigma^p + \xi_\sigma^q &\le \frac{1}{\sigma}\Delta_{\mathbb{H}^n}\xi + \frac{1}{\sigma}\lambda_0\xi + \frac{2}{\sigma^{\widetilde{p}}}\xi^{\widetilde{p}}\\ &\le \frac{1}{\sigma}(\Delta_{\mathbb{H}^n}\xi + \lambda_0\xi + \xi^{\widetilde{p}}) = 0.\end{aligned}$$

Thus, $\bar{\xi}(x) = \xi_\sigma(x)$ satisfies (3.3). Hence, when $\alpha \ge \beta$ and $0 < \beta \le \frac{2}{3}\lambda_0$, (1.1) has an upper solution. Similarly, if $\beta \ge \alpha$ and $0 < \alpha \le \frac{2}{3}\lambda_0$, there exists an upper solution to (1.1). This completes the proof. □

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