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BALLS INTERSECTION PROPERTIES OF BANACH SPACES

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Necessary and sufficient conditions for a Banach space with the Mazur intersection property to be an Asplund space are given. It is proved that Mazur intersection property is determined by the separable subspaces of the space. Corresponding problems for a space to have the ball-generated property are considered. Some comments on possible renorming so that a space having the Mazur intersection property are given.

A normed space X is said to have the *Mazur's intersection property* (MIP) if for every bounded closed convex set K in X, there exists a family of closed balls, $\{B_i\}_{i\in I}$, such that $K = \bigcap_{i\in I} B_i$. For a summary of the (MIP), we refer to [3]. A set K in X is said to be *ball-generated* [5] if $K = \bigcap_{i\in I} K_i$ where each K_i is a finite union of closed balls in X. We say that X has the *ball-generated property* (BGP) if every bounded closed convex set in X is ball-generated.

The problem raised in [4] whether every Banach space with the (MIP) is an Asplund space is still open. In this note, we give some necessary and sufficient conditions for a Banach space with (MIP) to be an Asplund space. We show that the (MIP) is determined by separable subspaces of the space. Similar problems for space with the (BGP) are considered. We make some comments on possible renorming for a Banach space to have the (MIP).

For a Banach space X, let $B_X = \{x : x \in X, \|x\| \leqslant 1\}$ and $S_X = \{x : x \in X, \|x\| = 1\}$. For a set K in X, let $\overline{co}K$ be the closed convex hull of K and let $[K]$ be the closed linear subspace in X spanned by K. If K is a subset in the dual space X^*, let $\overline{co}^*K$ be the weak* closed convex hull of K. A weak* slice of K is a set $S(x, K, \alpha) = \{x^* : x^* \in K, x^*(x) > \sup_{y^*\in K} y^*(x) - \alpha\}$ where $x \in X$ and $\alpha > 0$. K is said to be *weak* dentable* if for every $\varepsilon > 0$, there exists a weak* slice of K such that diam $S < \varepsilon$. For a point $x^* \in K$, x^* is said to be a *weak* strongly exposed point* of K if there exists x in X such that $x^*(x) > y^*(x)$ for all $y^* \neq x^*$ in K and for any sequence $\{x_n^*\}$ in K, $\lim_n x_n^*(x) = x^*(x)$ implies that $\lim_n \|x_n^* - x^*\| = 0$. x^* is called

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a *weak* denting point* of K if for every $\varepsilon > 0$, there is a weak* slice S of K containing x^* and diam $S < \varepsilon$. x^* is called a *weak*-weak point of continuity* (respectively *point of continuity*) of K if the identity mapping $Id : (K, \text{weak}^*) \to (K, \text{weak})$ (respectively, $Id : (K, w) \to (K, \|\cdot\|)$) is continuous at x^*.

1.

We first use the technique in [2] to obtain a basic result on Asplund spaces with the Mazur intersection property.

Theorem 1. *Let X be an Asplund space with the Mazur intersection property. Then for any subspace Y in X, there exists a subspace Z in X containing Y with the same density as Y and Z has the Mazur intersection property.*

Proof: Let D be the duality mapping of X, that is, $D(x) = \{x^* : \|x^*\| = 1, \langle x^*, x\rangle = \|x\|\}$, $x \in S(X)$. D is norm–weak* upper semi-continuous.

Since X is Asplund, every non-empty bounded subset of X^* is weak* dentable. By a theorem of Jayne and Rogers [9, Theorem 8] there exist continuous functions $f_n : (X, \|\cdot\|) \to (X^*, \|\cdot\|)$, $n \in \mathbb{N}$ such that $f_0(x) \equiv \lim_n f_n(x)$ exists in $(X^*, \|\cdot\|)$ and $f_0(x) \in D(x)$, $x \in X$. Define $f(x) = \{f_1(x), f_2(x), \dots\}$, $x \in X$. Then, [2, p.146], f is norm–norm lower semi-continuous and for every subspace Z in X

$$(\star) \qquad Z^* = \left[x^* |_Z : x^* \in f(x), x \in Z\right]$$

where $x^* |_Z$ is the restriction of x^* on Z.

Let $Y \neq \{0\}$ be a subspace of X and let $\alpha =$ density Y. Let $Z_0 = Y$. Then density $f(Z_0) \leqslant \alpha$. Since X has the (MIP), $B_{X^*} = \overline{A}$ where A is the set of all weak* denting points of B_{X^*}, [4]. Choose $A_0 \subset A$, $\overline{A}_0 \supset S_{[f(Z_0)]}$ and card $A_0 \leqslant \alpha$. Next, choose a subspace $Z_1 \supset Z_0$, density $Z_1 = \alpha$, $\|x^*\| = \|x^* |_{Z_1}\|$ for all $x^* \in [A_0]$ and every x^* in A_0 is a Z_1-denting point of B_{X^*}, that is, for all $\varepsilon > 0$, there is a slice of B_{X^*} containing x^* which is determined by some element in Z_1 and has diameter less that ε. Continue by induction, there exist subspaces Z_n in X and subsets A_n in A such that for all $n = 0, 1, 2, \dots$,

(i) density $Z_n \leqslant \alpha$ and card $A_n \leqslant \alpha$;

(ii) $Z_{n+1} \supset Z_n$, $A_{n+1} > A_n$ and $\overline{A_{n+1}} \supset S_{[A_n]}$;

(iii) $\overline{A_n} \supset S_{[f(Z_n)]}$;

(iv) $\|x^*\| = \|x^* |_{Z_{n+1}}\|$, $x^* \in [A_n]$;

(v) Every element in A_n is a Z_{n+1}-denting point of B_{X^*}.

Set $Z = \overline{\bigcup_n Z_n}$, $E = \left[\bigcup_{n\geqslant 0} A_n\right]$ and $T : E \to Z^*$ by $T(x^*) = x^* |_Z$, $x^* \in E$. Then Z and Y have the same density, $E \supset \left[\bigcup_{n\geqslant 0} f(Z_n)\right]$, $S_E = \overline{\bigcup_{n\geqslant 0} A_n}$, T is an isometry and

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every element in $T(\bigcup_{n\geqslant 0} A_n)$ is a weak* denting point of B_{Z^*}. Since f is norm–norm lower semi-continuous, $\overline{f(\bigcup_{n\geqslant 0} Z_n)} \supset f(\overline{\bigcup_{n\geqslant 0} Z_n}) = f(Z)$. Hence $E \supset f(Z)$. By ($\star$), $Z^* = [T(f(Z))] = T(E)$. It follows that $T(\bigcup_{n\geqslant 0} A_n)$ is a dense subset of S_{Z^*} and so the set of weak* denting points of B_{Z^*} is dense in S_Z. Therefore, by [4], Z has the (MIP). ∎

A Banach space X is called nicely smooth [6] if for all x^{**} in X^{**}, $\bigcap_{x\in X} B_{X^{**}}(x, \|x^{**}-x\|) = \{x^{**}\}$ where $B_{X^{**}}(x,r)$ is the closed ball in X^{**} with center x and radius r. Equivalently, [6, Lemma 6] X is nicely smooth if and only if X^* contains no proper closed norming subspace of X. It follows that we have the following lemma.

Lemma 2. *Let X be a Banach space. If for any separable subspace Y in X, there exists a separable nicely smooth subspace Z in X containing Y, then X is Asplund.*

Proof: Let Y be a separable subspace in X and let Z be a separable nicely smooth subspace of X containing Y. Since Z is separable and nicely smooth, so Z^* is separable. Hence Y^* is separable and so X is Asplund. ∎

Lemma 3. [5, Theorem 8.3]. *Every ball–generated Banach space is nicely smooth.*

Combine Theorem 1 and Lemma 1 and 2, we obtain:

Theorem 4. *Let X be a Banach space with the Mazur intersection property. Then the following are equivalent.*

(1) *X is Asplund;*
(2) *For every subspace Y of X, there exists a subspace Z in X with the same density of Y, $Z \supset Y$ and Z has the Mazur intersection property.*
(3) *For every separable subspace Y of X, there exists a separable subspace Z of X containing Y and Z has the Mazur intersection property.*

Theorem 5. *Let X be an infinite dimensional normed space. If every infinite dimensional separable subspace of X has the Mazur intersection property, then X has the Mazur intersection property.*

Proof: Suppose X fails to have the (MIP). By [4, Theorem 2.1], there is a support mapping $\lambda : S_X \to S_{X^*}$, a dense subset A in S_X and $x^* \in S_{X^*}$ such that $d(x^*, \lambda(A)) > \varepsilon$ for some $\varepsilon > 0$. Choose a countable subset $A_1 \subset A$, $\sup x^*(A_1) = 1$ and $\dim[A_1] = \infty$. Since A_1 is countable and $d(x^*, \lambda(A)) > \varepsilon$, there exists a countable subset A_2 in A, $A_2 \supset A_1$, $\overline{A_2} \supset S_{[A_1]}$ and $\sup(x^* - \lambda(x))(A_2) > \varepsilon$ for all x in A_1. Continue by

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induction, there is a sequence of countable subsets $\{A_n\}$ in A such that for all $n \in \mathbb{N}$,

(i) $A_{n+1} \supset A_n$ and $\overline{A_{n+1}} \supset S_{[A_n]}$;

(ii) $\sup(x^* - \lambda(x))(A_{n+1}) > \varepsilon$ for all x in A_n.

Let $Y = \left[\bigcup_n A_n\right]$. Then Y is an infinite dimensional separable subspace of X and $x^* |_Y \in S_{Y^*}$. Let λ_0 be any support mapping on Y such that $\lambda_0(x) = \lambda(x) |_Y$, $x \in \bigcup_n A_n$. By (i), $\bigcup_n A_n$ is dense in S_Y and, by (ii), $d\left(x^* |_Y, \lambda_0\left(\bigcup_n A_n\right)\right) \geqslant \varepsilon$. Hence, by [4, Theorem 2.1], Y fails to have the (MIP). ∎

2.

In this section, we consider the spaces with the ball-generated property.

Lemma 6. [4, Lemma 2.2]. *Let A be a bounded set in a normed space X and let x be an element in X. If A and x can be strictly separated by a weak* denting point of B_{X^*}, then there is a closed ball in X containing A but not containing x.*

Theorem 7. *Let X be a Banach space. Consider the following statements.*

(1) *X^* is the closed linear span of the weak* strongly exposed points of B_{X^*}.*

(2) *X^* is the closed linear span of the weak* denting points of B_{X^*}.*

(3) *X has the ball-generated property.*

(4) *X is nicely smooth.*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$. If X is Asplund, then all are equivalent.

Proof: It is clear that $(1) \Rightarrow (2)$.

$(2) \Rightarrow (3)$. Let $A \neq \phi$ be a bounded closed convex set in X and let $x_0 \in X \backslash A$. Since the set of weak* denting points of B_{X^*} is symmetric and since X^* is the closed linear span of weak* denting points of B_{X^*}, there exists $\varepsilon > 0$, weak* denting points x_i^* of B_{X^*}, $\lambda_i > 0$, $i = 1, 2, \dots, n$, $\sum_{i=1}^{n} \lambda_i = 1$ and $\inf_{x \in A} \sum_{i=1}^{n} \lambda_i x_i^*(x) > \sum_{i=1}^{n} \lambda_i x_i^*(x_0) + \varepsilon$. Let $A_i = \{x : x \in A, x_i^*(x) > x_i^*(x_0) + \varepsilon\}$, $i = 1, \dots, n$. By Lemma 4, for each $i = 1, \dots, n$, there is a closed ball B_i, $B_i \supset A_i$ and $x_0 \notin B_i$. Hence $x_0 \notin \bigcup_{i=1}^{n} B_i$ and $\bigcup_{i=1}^{n} B_i \supset A$. Therefore A is ball-generated.

By Lemma 3, we have $(3) \Rightarrow (4)$.

Finally, if X is Asplund, then the set $F = \{x : \|x\| = 1, \|\cdot\|$ is Frechét differentiable at $x\}$ is dense in S_X. Let D be the duality mapping of X. Then $D(F)$ norms X and every element in $D(F)$ is a weak* strongly exposed point of B_X. If X is nicely smooth, then $X^* = [D(F)] = [\text{weak}^* \text{ strongly exposed points of } B_{X^*}]$. ∎

Corollary 8. *Let X be a Banach space. If the duality mapping D :*

$(S_X, \|\cdot\|) \to (S_{X^*}, w)$ is upper semi-continuous, then X has the ball-generated property.

PROOF: If $D : (S_X, \|\cdot\|) \to (S_{X^*}, w)$ is upper semi-continuous, by [**8**, Lemma 6 and Theorem 10], then B_{X^*} is the closed convex hull of its weak* strongly exposed points. Hence X^* is the closed linear span of the weak* strongly exposed points of B_{X^*}. It follows that X has the ball-generated property. ∎

COROLLARY 9. *For a Banach space X, X is reflexive if and only if X^* has the ball-generated property.*

PROOF: If X^* has the ball-generated property, then X^{**} contains no proper closed norming subspace of X^*. Hence $X^{**} = X^*$ and X is reflexive. The other direction is well-known (for example [**3**]). ∎

COROLLARY 10. *For a Banach space X, the following are equivalent.*

(1) *For any subspace Y in X, Y^* is the closed linear span of weak* strongly exposed points of B_{Y^*} (respectively,*
   - *(1a) Y^* is the closed linear span of weak* denting points of B_{Y^*};*
   - *(1b) Y^* is the closed linear span of weak* points of continuity of B_{Y^*};*
   - *(1c) Y^* is the closed linear span of weak*–weak points of continuity of B_{Y^*});*

(1′) *For any separable subspace Y of X, Y^* is the closed linear span of weak* strongly exposed points of B_{Y^*} (respectively,*
   - *(1′a) Y^* is the closed linear span of weak* denting points of B_{Y^*};*
   - *(1′b) Y^* is the closed linear span of weak* points of continuity of B_{Y^*};*
   - *(1′c) Y^* is the closed linear span of weak*–weak points of continuity of B_{Y^*});*

(2) *Every subspace of X has the ball-generated property;*

(2′) *Every separable subspace of X has the ball-generated property;*

(3) *Every subspace of X is nicely smooth;*

(3′) *Every separable subspace of X is nicely smooth.*

REMARK. In the case that X is an Asplund space, then X has the ball-generated property if and only if X^* is the closed linear span of weak* strongly exposed points of B_{X^*} (we don't know whether this is true in general). Using the argument similar to Theorem 1 and the fact that in case that all separable subspaces of X have the ball-generated property, then X is Asplund and X has the ball-generated property if and only if X^* is the closed linear span of the weak* strongly exposed points of B_{X^*}, the following results can be obtained. We omit the detail.

Theorem 11. *If a Banach space X has the ball-generated property, then the following are equivalent.*

(1) *X is Asplund.*
(2) *For any subspace Y in X there exists a subspace Z in X with the same density of Y such that Z contains Y and Z has the ball-generating property.*
(3) *For any separable subspace Y in X, there is a separable subspace Z in X containing Y and Z has the ball-generating property.*

Theorem 12. *Let X be an infinite dimensional Banach space. If all separable infinite dimensional subspaces of X have the ball-generated property, then X has the ball-generated property.*

3.

In [**1**], Deville proves an interesting renorming theorem for a Banach space to have the Mazur intersection property. In this section, we give some observations and a slightly simpler proof of his result.

Let $(X, |\cdot|)$ be a Banach space and let K be a bounded closed convex set in X with $0 \in K$. Put

$$\psi(x) = |x|^2 + \int_{-\infty}^{\infty} d^2(x, tK) e^{-|t|} dt, \quad x \in X$$

where $d(x, tK)$ is the distance from x to tK in $(X, |\cdot|)$. Let $\|\cdot\|$ be the gauge function of $\{x : x \in X, \psi(x) \leqslant 1\}$. Then $\|\cdot\|$ is an equivalent norm on X and as in the proof of [**1**, Lemma 5], has the following property.

Lemma 13. *For all t in $\mathbb{R}$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in S_{(X,\|\cdot\|)} \cap tK$ and $y \in B_{(X,\|\cdot\|)}$ if $\|x + y/2\| > 1 - \delta$ then $d(y, tK) < \varepsilon$.*

For a set A in X, let $\mathbb{R}A = \{\alpha x : \alpha \in \mathbb{R}, x \in A\}$, ext $A = \{x : x$ is an extreme point of $A\}$, pc $A = \{x : x$ is a point of continuity of $A\}$, dent $A = \{x : x$ is a denting point of $A\}$ and LUR $A = \{x : x \in A$ and for any sequence $\{x_n\}$ in A, $\lim_n \|x + x_n/2\| = \|x\|$ implies that $\lim_n \|x - x_n\| = 0\}$.

Proposition 14. *Let K be a bounded closed convex set of a Banach space $(X, |\cdot|)$ with $0 \in K$. Let $\|\cdot\|$ be the norm constructed above. Then*

(1) $S_{(X,\|\cdot\|)} \cap \mathbb{R}$ *ext* $K \subset$ *ext* $B_{(X,\|\cdot\|)}$.
(2) $S_{(X,\|\cdot\|)} \cap \mathbb{R}$ *pc* $K \subset$ *pc* $B_{(X,\|\cdot\|)}$.
(3) $S_{(X,\|\cdot\|)} \cap \mathbb{R}$ *dent* $K \subset$ *dent* $B_{(X,\|\cdot\|)}$.
(4) $S_{(X,\|\cdot\|)} \cap \mathbb{R}$ *LUR* $K \subset$ *LUR* $B_{(X,\|\cdot\|)}$.

PROOF: (1) Let $\|x\| = 1$ and $x \in t$ ext K for some $t \in \mathbb{R}$. Then $t \neq 0$. Assume that $x = 1/2(y+z)$ for some $\|y\| = \|z\| = 1$ and $y \neq z$. Then $\|x + y/2\| = \|x + z/2\| = 1$. By Lemma 12, $y, z \in tK$. But this implies that x/t is not an extreme point of K.

(2) Suppose $x = ty$, $\|x\| = 1$ and $y \in \text{pc } K$. If $\{x_\lambda\}$ is a net in $B_{(X,\|\cdot\|)}$ such that weak–lim $x_\lambda = x$ then $\lim \|x + x_\lambda/2\| = 1$. By Lemma 12, there exists $\{y_\lambda\}$ in K such that $\lim |x_\lambda - ty_\lambda| = 0$. Thus weak–lim $y_\lambda = y$. But $y \in \text{pc } K$, so $\lim_\lambda \|y_\lambda - y\| = 0$. Hence $\lim \|x_\lambda - x\| = 0$. This shows that $x \in \text{pc } B_{(X,\|\cdot\|)}$.

(3) Follows from (1) and (2) and the fact [10] that x is a denting point of a bounded closed convex set C if and only if x is an extreme point and also a point of continuity of C.

(4) Let $x = ty$, $\|x\| = 1$ and $y \in \text{LUR } K$. Suppose $\|x_n\| = 1$ and $\lim_n \|x + x_n/2\| = 1$. By Lemma 12, there exists $\{y_n\}$ in K such that $\lim_n |x_n - ty_n| = 0$. Since K is convex, we have

$$\begin{aligned}\frac{1}{2}\left(|x| - |x_n|\right)^2 &\leqslant |x|^2 + |x_n|^2 - 2\left|\frac{x + x_n}{2}\right|^2 \\ &\leqslant \psi(x) + \psi(x_n) - 2\psi\left(\frac{x + x_n}{2}\right) \\ &= 2 - 2\psi\left(\frac{x + x_n}{2}\right) \longrightarrow 0.\end{aligned}$$

Hence $\lim_n |x_n| = |x| = \lim_n |x + x_n/2|$. Thus $\lim_n |y_n| = |y| = \lim_n |y + y_n/2|$. Since $y \in \text{LUR } K$, $\lim_n \|y_n - y\| = 0$. Thus $\lim_n \|x_n - x\| = 0$. This completes the proof that $x \in \text{LUR } B_{(X,\|\cdot\|)}$. □

COROLLARY 15. *A Banach space X admits an equivalent norm such that the denting (respectively, extreme; points of continuity; or locally uniformly rotund) points of the unit ball is dense on the unit sphere of X if and only if there exists a bounded closed convex set K in X with $0 \in K$ and the set $\mathbb{R}$ dent K (respectively, $\mathbb{R}$ ext K, $\mathbb{R}$ pc K, or $\mathbb{R}$ LUR K) is dense on X.*

In the dual spaces, using similar argument, we have

PROPOSITION 16. *For a Banach space $(X, |\cdot|)$, let K be a weak* compact convex set in X^* with $0 \in K$. Define*

$$\psi(x^*) = |x^*| + \int_{-\infty}^{\infty} d^2(x^*, tK)e^{-|t|}dt, x^* \in X^*,$$

and let $\|\cdot\|$ be the gauge function of $\{x^* : x^* \in X^*, \psi(x^*) \leqslant 1\}$. Then

(1) *$\|\cdot\|$ is an equivalent dual norm on X^*.*

(2) $S_{(X^*,\|\cdot\|)} \cap \mathbb{R}$ *weak* pc* $K \subset$ *weak* pc* $B_{(X^*,\|\cdot\|)}$.

(3) $S_{(X^*,\|\cdot\|)} \cap \mathbb{R}$ *weak* dent* $K \subset$ *weak* dent* $B_{(X^*,\|\cdot\|)}$.

(4) $S_{(X^*,\|\cdot\|)} \cap \mathbb{R}$ *weak*–weak pc* $K \subset$ *weak*–weak pc* $B_{(X^*,\|\cdot\|)}$.

(5) $S_{(X^*,\|\cdot\|)} \cap \mathbb{R}$ *weak*–weak dent* $K \subset$ *weak*–weak dent* $B_{(X^*,\|\cdot\|)}$.

Corollary 17. *A Banach space X admits an equivalent norm $\|\cdot\|$ such that $S_{(X,\|\cdot\|)} = \overline{\text{weak}^* \text{ dent } B_{(X,\|\cdot\|)}}$ (respectively, (i) $X^* = [\text{ weak}^* \text{ dent } B_{(X,\|\cdot\|)}]$; or (ii) $X^* = \overline{\mathbb{R} \text{ ext } B_{(X,\|\cdot\|)}}^{\sigma_c}$ where σ_c is the topology on X^* which is uniformly convergence on all compact subsets of X) if and only if there exists a weak* compact convex set K in X^* such that $X^* = \overline{\mathbb{R} \text{ weak}^* \text{ dent } K}$ (respectively, (i) $X^* = [\text{weak}^*. \text{ dent } K]$; or (ii) $X^* = \overline{\mathbb{R} \text{ ext } K}^{\sigma_c}$).*

A Banach space X is said to have the property (CI) [14] if every compact convex set in X is an intersection of closed balls. Using the result in [14], we have the following.

Corollary 18. *A Banach space X admits an equivalent norm $\|\cdot\|$ such that $(X, \|\cdot\|)$ has the Mazur intersection property (respectively, (i) the ball–generating property; or (ii) the property (CI)) if and only if (respectively, if) there exists a weak* compact convex set K in X^* such that $X^* = \overline{\mathbb{R} \text{ weak}^* \text{ dent } K}$ (respectively, (i) $X^* = [\text{weak}^* \text{ dent } K]$; or (ii) $X^* = \overline{\mathbb{R} \text{ ext } K}^{\sigma_c}$).*

Corollary 19. [1, Theorem 4]. *Let X and Y be Banach spaces. If Y has the Mazur intersection property and there exists an operator $T : X \to Y$ such that both T^* and T^{**} are injective, then X admits an equivalent norm $\|\cdot\|$ such that $(X, \|\cdot\|)$ has the Mazur intersection property.*

Proof: Let $K = T^*(B_{Y^*})$. Then $0 \in K$, K is weak* compact convex in X^* and weak* dent $K \supset T^*$ (weak* dent B_{Y^*}). Since $Y^* = \overline{\mathbb{R} \text{ weak}^* \text{ dent } B_{Y^*}}$ and T^{**} is injective, we have $X^* = \overline{T^* Y^*}$. Hence $X^* = \overline{\mathbb{R} \text{ weak}^* \text{ dent } K}$. Thus X admits an equivalent norm with (MIP). ∎

Problems. Let X and Y be Banach spaces and let $T : X \to Y$ be bounded operator with T^* and T^{**} injective. If Y has the ball–generated property (respectively, property (CI)), does X admit an equivalent norm with the ball–generated property (respectively, property (CI))?

4.

The density of the weak* denting points of B_{X^*} in S_{X^*} plays an important role on the geometry of X. Let us consider four kinds of density in this respect.

(I) Every point of S_{X^*} is a weak* denting point of B_{X^*}.

(II) The set of weak* denting points of B_{X^*} is dense in S_{X^*}.

(III) B_{X^*} is the closed convex hull of weak* denting points of B_{X^*}.

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(IV) X^* is the closed linear span of the weak* denting points of B_{X^*}.

In [7, Theorem 3.1], it is proved that (I) is a necessary and sufficient condition for X to be Φ–ANP–I for some norming set Φ of X (since the definition of ANP is rather lengthy, we refer to [7, or 8] for definition) which in turn implies that X is Frechét differentiable and Hahn–Banach smooth [8].

It is well–known [4] that (II) is a necessary and sufficient condition for X to have the Mazur intersection property.

In [8], it is proved that (III) is a necessary condition for the duality mapping on X to be weakly upper semi–continuous.

In this paper, we have proved that (IV) is a sufficient condition for X to have the ball–generated property and so X is nicely smooth.

It may be interesting to find geometric property on X that has (I) (respectively, (III) or (IV)) as necessary and sufficient condition.

Added in Proof. After the paper was accepted, G. Godefroy and S. Sersouri kindly informed the authors that the answer concerning the problem in the case of property (CI) is affirmative. Namely, let X and Y be Banach spaces and let $T : X \to Y$ be a bounded operator with T and T^{**} injective. If Y has the property (CI) then X admits an equivalent nrom with property (CI) (see [13]).

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