A DIRECT PROOF OF LEUTBECHER'S LEMMA

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Using the theory of group extensions, A. Leutbecher [1] proved this

LEMMA. *Let G be a group, and w some 2-cocycle of a trivial G-module M. The cohomology class of w will contain symmetric cocycles if and only if w is semisymmetric.*

Here we have called w symmetric or semisymmetric according as $w(h, g) = w(g, h)$ for all $g, h \in G$ or only for those with $hg = gh$. In one direction, the proof reduces to observing that 2-coboundaries of trivial G-modules are semisymmetric. The nontrivial part of the lemma also admits of a straightforward proof, as follows.

If w is any cocycle of a trivial G-module, following H. Petersson [2, p. 47] we define, inductively for $n \geqq 2$,

$$w(g_1, \ldots, g_n, g_{n+1}) = w(g_1, \ldots, g_n) + w(g_1 g_2 \cdots g_n, g_{n+1})$$

for $g_j \in G$ $(1 \leqq j \leqq n+1)$. Then

$$w(g_1, \ldots, g_j, g_{j+1}, \ldots, g_n) = w(g_1, \ldots, g_j g_{j+1}, \ldots, g_n) + w(g_j, g_{j+1}).$$

If, in particular, $g_{j+1} = h^{-1}$ is inverse to $g_j = h$, and if e denotes the neutral element of G, we have

$$\begin{aligned} w(g_1, \ldots, h, h^{-1}, \ldots, g_n) &= w(g_1, \ldots, e, \ldots, g_n) + w(h, h^{-1}) \\ &= w(g_1, \ldots, g_{j-1}, g_{j+2}, \ldots, g_n) + w(e, e) + w(h, h^{-1}), \end{aligned}$$

where we have used $w(g, e) = w(e, g) = w(e, e)$ $(g \in G)$.

PROPOSITION 1. *If w is semisymmetric and if $f, g, h \in G$ are such that f commutes with $h^{-1}g$, then*

$$w(h, f, h^{-1}) - w(h, h^{-1}) = w(g, f, g^{-1}) - w(g, g^{-1}).$$

Indeed, semisymmetry gives $w(h^{-1}, g, f) = w(f, h^{-1}, g)$, and so

$$\begin{aligned} w(h, f, h^{-1}) - w(h, h^{-1}) &= w(h, h^{-1}gfg^{-1}h, h^{-1}) - w(h, h^{-1}) \\ &= w(h, h^{-1}, g, f, g^{-1}, h, h^{-1}) - w(h^{-1}, g, f, g^{-1}, h) - w(h, h^{-1}) \\ &= w(h, h^{-1}) + 2w(e, e) + w(g, f, g^{-1}) - w(f, h^{-1}, g, g^{-1}, h) \\ &= w(g, f, g^{-1}) - w(g, g^{-1}), \end{aligned}$$

where we have used $w(h^{-1}, h) = w(h, h^{-1})$ $(h \in G)$. Proposition 1 is thus established.

PROPOSITION 2. *If w is semisymmetric, then there is a 1-cochain $c : G \to M$ such that*

$$c(hg) - c(gh) = w(h, g) - w(g, h) \qquad (g, h \in G).$$

To prove this we choose, using Zermelo's axiom, a representative in each class of

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conjugate elements of G and then define, consistently according to Proposition 1, c on the class of $f \in G$ by

$$c(gfg^{-1}) = w(g, f, g^{-1}) - w(g, g^{-1}) \qquad (g \in G).$$

If $g, h \in G$ are such that gh is in the class represented by f, we have

$$gh = ufu^{-1}, \qquad hg = hufu^{-1}h^{-1}$$

for some $u \in G$. Then

$$\begin{aligned} c(hg) - c(gh) &= w(hu, f, u^{-1}h^{-1}) - w(hu, u^{-1}h^{-1}) - w(u, f, u^{-1}) + w(u, u^{-1}) \\ &= w(h, u, f, u^{-1}, h^{-1}) - w(h, u, u^{-1}, h^{-1}) - w(u, f, u^{-1}) + w(u, u^{-1}) \\ &= w(h, ufu^{-1}, h^{-1}) - w(h, h^{-1}) - w(e, e) \\ &= w(h, g, h, h^{-1}) - w(g, h) - w(h, h^{-1}) - w(e, e) \\ &= w(h, g) - w(g, h); \end{aligned}$$

so Proposition 2 is proved.

To establish the lemma, we derive from c the 2-coboundary $b = \partial c$,

$$b(g, h) = c(g) + c(h) - c(gh) \qquad (g, h \in G).$$

Then, by Proposition 2,

$$b(g, h) - b(h, g) = w(h, g) - w(g, h) \qquad (g, h \in G),$$

and $w + b$, which is cohomologous to w, will be symmetric.

REFERENCES

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