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SOME EXAMPLES OF COMPRESSIBLE GROUP ALGEBRAS AND OF NONCOMPRESSIBLE GROUP ALGEBRAS

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Dedicated to Professor Hisao Tominaga on his 60th birthday.

A ring R with centre $Z(R)$ is called *compressible* if $Z(eRe) = eZ(R)e$ for any idempotent e of R. In this paper we shall give some examples of compressible group algebras and of noncompressible group algebras. These examples show that it is very difficult to judge the compressibility of a group algebra.

1. Introduction

Let R be a ring and let $Z(R)$ be the centre of R. An idempotent e of R is called *compressible* in R if $Z(eRe) = eZ(R)e$. A ring R is also called *compressible* if every idempotent of R is compressible. Such a ring has been studied by several authors (see [*1*-*6* and *10*]). In this paper we shall give examples of group algebras which are and which are not compressible. It seems to be very difficult to give characterizations of compressible modular group albegras.

Throughout this paper let p be a fixed prime number, let K be a field of characteristic p, let G be a finite p-solvable group and let KG be the group algebra of G over K. For the subset S of G,

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$\hat{S}$ denotes the sum of all elements of S in KG . Let F be a finite field of order q^r where q is a power of p and let A be a cyclic subgroup in the multiplicative group F^* of F such that $GF(q)^*A = F^*$. We shall define permutations and permutation groups on F.

$$w : x \to x^q , \quad v_a : x \to ax \text{ for } a \in A,$$

$$u_b : x \to x + b \text{ for } b \in F, \quad W = \langle w \rangle ,$$

$$V = \{v_a \mid a \in A\} , \quad U = \{u_b \mid b \in F\} ,$$

$$G = \langle W, V, U \rangle = WVU \text{ and } H = \langle V, U \rangle = VU .$$

2. Noncompressible group algebras

The purpose of this section is to show that the group algebra KG is not compressible.

LEMMA 1. *The element* $e = |V|^{-1}\hat{V}$ *is an idempotent of* KG *and* $we = ew$ *is contained in the centre* $Z(eKGe)$ *of* $eKGe$.

Proof. It is easy to see e is an idempotent of KG , H is a semidirect product of U by V and G is a semidirect product of H by W . It follows from $F^* = GF(q)^*A$ that for every element c of F , there exist elements $a \in A$ and $b \in GF(q)$ such that $c = ab$. Thus we have the equation $eu_ce = ev_au_bv_a^{-1}e = eu_be$. It follows from this equation that w and eu_ce commute for every element u_c of U . This implies that ew is contained in the centre $Z(eKGe)$ of $eKGe = eKWUe$.

The next result is the aim of this section.

PROPOSITION 1. *If* $r > 1$, *then the group algebra* KG *is not compressible.*

Proof. Assume false. Then we have from Lemma 1 that $w\hat{H}$ is contained in $Z(KG)\hat{H}$. Hence it suffices to prove $Z(KG)\hat{H} = K\hat{H}$. Let C be a conjugate class of G such that $\hat{C}\hat{H}$ is not zero. Since C is contained in a residue class w^sH for some $r > s \geq 0$, it follows from

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the equation $0 \neq \hat{C}\hat{H} = |C|w^s\hat{H}$ that a centralizer $C_G(w^s h)$ contains a p-subgroup U of G for some $h \in H$. This implies $s = 0$, completing the proof.

In the case $r = p = q = 2$, G is isomorphic to the symmetric group S_4 of order 24. There exists also a group $T = SL(2, 3)$ of order 24 such that KT is not compressible if $p = 2$.

EXAMPLE. Let $p = 2$, let $Q = \langle x, y \mid x^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1}\rangle$ be the quaternion group of order 8 and let g be the automorphism of Q defined by $x^g = xy$ and $y^g = x$. The $\langle g\rangle$-orbits of Q are $D_1 = \{1\}$, $D_2 = \{x^2\}$, $D_3 = \{x, xy, y\}$ and $D_4 = x^2D_3$. Let T be the semidirect product of Q by a cyclic group $\langle g\rangle$ of order 3 with respect to this action and let $e = 1 + g + g^2$. Then e is an idempotent and $eKTe = \sum_{i=1}^4 K\hat{D}_i e$ is commutative since x^2 is central in Q and $D_4 = x^2D_3$. But $\hat{D}_3 e$ is not contained in $Z(KT)e = (K\hat{D}_1 + K\hat{D}_2 + K(\hat{D}_3 + \hat{D}_4))e$. This group T is isomorphic to $SL(2, 3)$ and $T/\langle x^2\rangle$ is isomorphic to A_4. This shows T is an example such that $K(T/\langle x^2\rangle)$ is compressible (see Proposition 2) and KT is not compressible.

3. Compressible group algebras

In this section we shall give some examples of compressible modular group algebras.

Two idempotents e and f in a ring are said to be equivalent if ther exist elements x and y with $xy = e$ and $yx = f$. In this case, e is compressible if and only if f is.

Let us now again consider the permutation groups defined in the introduction, but this time let us assume that $A = F^*$ and K contains a primitive nth root of 1 where $n = |V|$. Then we shall show KH is compressible where $H = VU$. Let $\{\theta_1, \theta_2, \ldots, \theta_n\}$ be the set of all distinct linear characters of V and we set $e_s = -\sum_{v\in V} \theta_s(v^{-1})v$. Then

$\{e_1, e_2, \ldots e_n\}$ is the set of all primitive idempotents of KV since $|V| \equiv -1 \ (mod\ p)$ and $ve_s = \theta_s(v)e_s$ (see also [7, p.236, (33.8) Theorem]). We start with the following lemma.

LEMMA 2. *$e_s u e_t \neq 0$ and $e_s u e_s = (1 - \hat{U})e_s$ for every $u \in U - 1$.*

Proof. We should note that if $vuv' = u$ for $u \in U - 1$ and v, $v' \in V$, then $v = v' = 1$. Thus we can see that the coefficient of u appearing in an element $e_s u e_t$ is equal to 1. This completes the proof of the first assertion. Noting that $e_s v^{-1} u v e_s = e_s u e_s$ for all $v \in V$ by the equation $ve_s = e_s v = \theta_s(v)e_s$ and V is transitive and fixed-point-free on $U - 1$, we can see that, for $u \in U - 1$,

$$e_s u e_s = -\sum_{v \in V} e_s v^{-1} u v e_s = e_s(1 - \hat{U})e_s = (1 - \hat{U})e_s .$$

PROPOSITION 2. *Assume that $A = F^*$ and K contains a primitive $|V|$th root of 1. Then KH is compressible where $H = VU$.*

Proof. Let e be an idempotent of KH. Then e is equivalent to an idempotent f in KV since the kernel of the natural homomorphism $KH \to KH/U$ is contained in the radical of KH. We may assume $f = \sum_{s=1}^{m} e_s$ for some $m \leqq n$. It is evident that $fZ(KH)f$ is contained in $Z(fKHf)$. Thus it remains only to prove $Z(fKHf)$ is contained in $fZ(KH)$. Let $y = \sum_{v \in V, u \in U} a_{v,u} vu$ be an element of $Z(fKHf)$. We should remark that $e_s y e_t = y e_s e_t = 0$ for $s \neq t$ since $e_s = f e_s f$ is contained in $fKHf$. This together with Lemma 2 yields the following.

$$\begin{aligned} y = fyf &= \sum_s e_s y e_s \\ &= \sum_{s,v,u} a_{v,u}\, v e_s u e_s \\ &= \sum_{s,v} a_{v,1}\theta_s(v)e_s + \sum_{s,v,u\neq 1} a_{v,u} v(1 - \hat{U})e_s \\ &= \sum_s c_s e_s - \sum_{s,v} d_v v\hat{U} e_s \end{aligned}$$

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$$= z - (\sum_v d_v v\hat{U})f$$

where $1 \leqq s \leqq m$, v and u run over V and U , respectively, $c_s = \sum_{v,u} a_{v,u}\theta_s(v)$, $d_v = \sum_{u\neq 1} a_{v,u}$ and $z = \sum_s c_s e_s$. Since $\sum_{v\in V} d_v v\hat{U}$ is contained in $Z(KH)$, we have that z is an element of $Z(fKHf)$ and so z commutes with $e_s u e_t$ for all $s, t \leqq m$. It is easy to see $z = cf$ for some $c \in K$ from the first assertion of Lemma 2 .

Next we shall prove that the group algebra of a finite p-nilpotent group over K is compressible.

LEMMA 3. *Let R be a primary ring (see [8, p. 56]). Then R is compressible.*

Proof. It follows from [8, Theorem 3.9.1] that R is a full matrix ring $(S)_n$ of some degree n over a completely primary ring S . Let $J(S)$ be the Jacobson radical of S . We consider the natural homomorphism $- : R = (S)_n \to \overline{R} = (\overline{S})_n$ where $\overline{S} = S/J(S)$. The kernel of $-$ is the radical of R (see [8, Theorem 1.7.3]). Let $E_t \in R = (S)_n$ be the matrix with 1 in the (t, t) position and 0's elsewhere. Then it follows easily that every left $\overline{R}$-module $\overline{RE}_t$ is irreducible and $\{E_t \mid 1 \leqq t \leqq n\}$ is a set of orthonormal idempotents. Let E be an arbitary idempotent of R . Then $\overline{RE}$ is isomorphic to a direct sum of some irreducible modules $\overline{RE}_t$ since $\overline{RE}$ is a submodule of a completely reducible $\overline{R}$-module $\overline{R}$. And hence E is equivalent to a sum of some orthogonal idempotents E_t ([8, Proposition 3.7.4 and 3.8.1]). Thus we may assume E is a sum of some idempotents E_t . There is a ring isomorphism between ERE and $(S)_k$ for some $k \leqq n$ such that $Z(R)E$ is isomorphic to $Z((S)_k)$ under this isomorphism. This completes the proof.

Osima's theorem [9, Theorem 1] together with Lemma 3 yields the following.

PROPOSITION 3. *The group algebra of a finite p-nilpotent group over K is compressible.*

References

[1] E.P. Armendariz, H.K. Koo and J.K. Park, "Compressible group algebras", *Comm. Algebra* 13 (1985), 1763-1777.

[2] E.P. Armendariz, H.K. Koo and J.K. Park, "Compressible endomorphism rings of projective modules", (Preprint).

[3] E.P. Armendariz and J.K. Park, "Compressible matrix rings", *Bull. Austral. Math. Soc.* 30 (1984) 295-298.

[4] E.P. Armendariz and J.K. Park, "Compressible P.I. group algebras", (Preprint).

[5] S.K. Berberian, "The center of a corner of a ring", *J. Algebra* 71 (1981), 515-523.

[6] G.M. Bergman, "Some examples of non-compressible rings", *Comm. Algebra* 12 (1984), 1-8.

[7] C.W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras* (Interscience, New York, 1962).

[8] N. Jacobson, *The structure of rings* (Amer. Math. Soc., Providence, 1956).

[9] M. Osima, "On primary decomposable group rings", *Proc. Phy.-Math. Soc. Japan (3)* 24 (1943), 1-9.

[10] A. Page, "Handwritten note to S.K. Berberian", (Dated 25 June 1982).

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