ON THE BEHAVIOUR OF A SERIES ASSOCIATED WITH THE CONJUGATE SERIES OF A FOURIER SERIES

R. MOHANTY AND B. K. RAY

1. Introduction.

1.1. *Definition.* Let $\lambda \equiv \lambda(\omega)$ be continuous, differentiable, and monotonic increasing in $(0, \infty)$ and let it tend to infinity as $\omega \to \infty$. Suppose that $\sum a_n$ (we write $\sum$ for $\sum_1^\infty$ throughout the present paper) is a given infinite series and let

$$C_r(\omega) = \sum_{n \leqq \omega} \{\lambda(\omega) - \lambda(n)\}^r a_n \qquad (r \geqq 0).$$

The series $\sum a_n$ is said to be summable $|R, \lambda, r|$, where $r > 0$, if[1]

$$\int_A^\infty \left| d\left[\frac{C_r(\omega)}{\{\lambda(\omega)\}^r}\right]\right| < \infty,$$

where A is a fixed positive number (**6**, Definition B). Now, for $r > 0$, $m < \omega < m + 1$,

$$\frac{d}{d\omega}\left[\frac{C_r(\omega)}{\{\lambda(\omega)\}^r}\right] = \frac{r\lambda'(\omega)}{\{\lambda(\omega)\}^{r+1}} \sum_{n \leqq \omega} \{\lambda(\omega) - \lambda(n)\}^{r-1} \lambda(n) a_n.$$

Hence, $\sum a_n$ is summable $|R, \lambda, r|$, where $r > 0$, if

$$\int_A^\infty \frac{r\lambda'(\omega)}{\{\lambda(\omega)\}^{r+1}} \left| \sum_{n \leqq \omega} \{\lambda(\omega) - \lambda(n)\}^{r-1} \lambda(n) a_n \right| d\omega < \infty.$$

1.2. Let $f(t)$ be Lebesgue integrable over $(-\pi, \pi)$ and periodic with period 2π and let

(1.2.1) $\quad f(t) \sim \frac{1}{2}a_0 + \sum (a_n \cos nt + b_n \sin nt) \equiv \frac{1}{2}a_0 + \sum A_n(t).$

Then the series conjugate to (1.2.1) at $t = x$ is

(1.2.2) $\quad \sum (b_n \cos nx - a_n \sin nx) \equiv \sum B_n(x).$

We write

$$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}, \quad h(t) = \frac{\psi(t)}{\log k/t}, \quad \text{where}^2\ k > \pi,$$

$$\Psi(t) = \int_t^\pi \frac{\psi(u)}{u}\, du = \psi_1(t) \log k/t,$$

$$\psi_1^*(t) = \frac{1}{t}\int_0^t \psi(u)\, du, \qquad h_1^*(t) = \frac{1}{t}\int_0^t h(u)\, du.$$

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[1]We write $\int_a^b |dg(x)| = \int_a^b dG(x) = G(b) - G(a)$, where $G(x)$ denotes the total variation of $g(x)$ in some closed interval $[c, x]$, c being independent of x.

[2]k is introduced merely since $\log t^{-1} = 0$ when $t = 1$.

Stieltjes integrals. The Stieltjes integrals employed are to be regarded as Lebesgue-Stieltjes integrals. Stieltjes integrals at the origin are to be interpreted in the sense $\lim_{\epsilon\to+0}\int_\epsilon$.

1.3. For the conjugate series, the following results are known.

THEOREM A (**6**, Theorem 5). *If* (i) $\psi(t)\log k/t$ *is of bounded variation in*[3] $(0, \pi)$ *and* (ii) $|\psi(t)|/t$ *is integrable in* $(0, \pi)$, *then the conjugate series of* $f(t)$, *at* $t = x$, *is summable* $|R, e^{\omega^\alpha}, 1|$, *where* $0 < \alpha < 1$.

THEOREM B (**1**, Theorem 4). *If* (i) $\psi(t)$ *is of bounded variation in* $(0, \pi)$ *and* (ii) $|\psi(t)|/t$ *is integrable in* $(0, \pi)$, *then the conjugate series of* $f(t)$, *at* $t = x$ *is summable* $|C, \delta|$, $\delta > 0$.

THEOREM C (**1**, Theorem 1). *If* $|\psi(t)|/t$ *is integrable in* $(0, \pi)$, *then the conjugate series of* $f(t)$, *at* $t = x$, *is summable* $|C, 1 + \delta|$, $\delta > 0$.

THEOREM D (**6**, Theorem 6). *If* (i) $\psi_1^*(t)\log k/t$ *is of bounded variation in* $(0, \pi)$ *and* (ii) $|\psi_1^*(t)|/t$ *is integrable in* $(0, \pi)$, *then the conjugate series of* $f(t)$, *at* $t = x$, *is summable* $|R, e^{(\log \omega)^2}, 2|$.

THEOREM E (**12**, Dini's test). *If* $|\psi(t)|/t$ *is integrable in* $(0, \pi)$, *then the conjugate series of* $f(t)$, *at* $t = x$, *is convergent.*

1.4. In the present paper we show that the series $\sum B_n(x)/\log(n + 1)$ behaves more or less like the conjugate series (1.2.2) as far as Theorems A, B, C, D, and E are concerned, the function $h(t)$ in the present problem playing the role of $\psi(t)$ in the corresponding one for the conjugate series. To be more precise, we prove the following theorems.

THEOREM I. *If* (i) $h(t)\log k/t$ *is of bounded variation in* $(0, \pi)$ *and* (ii) $|h(t)|/t$ *is integrable in* $(0, \pi)$, *then the series* $\sum B_n(x)/\log(n + 1)$ *is summable* $|R, e^{\omega^\alpha}, 1|$, *where* $0 < \alpha < 1$.

THEOREM II. *If* (i) $h(t)$ *is of bounded variation in* $(0, \pi)$ *and* (ii) $|h(t)|/t$ *is integrable in* $(0, \pi)$, *then the series* $\sum B_n(x)/\log(n + 1)$ *is summable* $|C, \delta|$, $\delta > 0$.

THEOREM III. *If* $|h(t)|/t$ *is integrable in* $(0, \pi)$, *then the series* $\sum B_n(x)/\log(n + 1)$ *is summable* $|C, 1 + \delta|$, $\delta > 0$.

THEOREM IV. *If* (i) $h_1^*(t)\log k/t$ *is of bounded variation in* $(0, \pi)$ *and* (ii) $|h_1^*(t)|/t$ *is integrable in* $(0, \pi)$, *then the series* $\sum B_n(x)/\log(n + 1)$ *is summable* $|R, e^{(\log \omega)^2}, 2|$.

For the analogue of Theorem I for the series $\sum A_n(x)/\log(n + 1)$, see (**6**, Theorem 2). In § 6, we obtain the following result which has no analogue for the conjugate series. One may note that summability $|R, \log \omega, 1|$ of the series $\sum B_n(x)/\log(n + 1)$ is a local property of the generating function, whereas the same for $\sum B_n(x)$ is a non-local one.

[3]Here and elsewhere, the interval $(0, \pi)$ is open at the origin.

THEOREM V. *If $|h(t)|/t$ is integrable in $(0, \pi)$, then the series $\sum B_n(x)/\log(n+1)$ is summable $|R, \log \omega, 1|$.*

In § 7, we prove the following convergence criterion for the series $\sum B_n(x)/\log(n+1)$ which is an analogue of Dini's test (Theorem E of the present paper) for the conjugate series.

THEOREM VI. *If $|h(t)|/t$ is integrable in $(0, \pi)$, then the series $\sum B_n(x)/\log(n+1)$ is convergent.*

The following general convergence criterion has been conjectured (see § 7 of the present paper, for the final conjecture made by the referee) by the referee for the series $\sum B_n(x)/\log(n+1)$; we have not been able to establish it, and thus it remains an open question. It may be noted that Theorem VI would be a corollary of the conjecture given below as the hypothesis of Theorem VI implies both conditions (i) (trivial) and (ii) (by Lemma 5 of the present paper) of the conjecture.

Conjecture. If

(i) $$\int_{\to 0}^{\pi} \frac{\psi(u)}{u \log k/u}\, du$$

exists as a Cauchy integral at the origin and is finite and

(ii) $$\int_t^{\pi} \frac{|\psi(u)|}{u}\, du = o\left(\log \frac{1}{t}\right) \quad \text{as } t \to +0,$$

then the series $\sum B_n(x)/\log(n+1)$ is convergent.

Note. Let us consider the odd function $f(t)$ defined by

$$f(t) = \psi(t) = -\tfrac{1}{2}t \qquad (0 \leqq t \leqq \pi),$$

and elsewhere by periodicity. It is easy to see that

$$\psi(t) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin nt.$$

Thus, the series $\sum B_n(x)/\log(n+1)$ at the origin is $\sum (-1)^{n-1}/n \log(n+1)$. By the above example, it is evident that Theorems I, II, and IV are best possible in the sense that their hypotheses do not imply absolute convergence of the series $\sum B_n(x)/\log(n+1)$. Furthermore, in the conclusion of Theorem III, δ cannot be replaced by zero, as $|C, 1|$ summability of $\sum B_n(x)/\log(n+1)$ is not a local property of the generating function.

2. Theorem I.

2.1. LEMMA 1 (**6**, Lemma 10). *If $\eta > 0$, then necessary and sufficient conditions that* (i) *$h(t) \log k/t$ should be of bounded variation in $(0, \eta)$ and* (ii) *$|h(t)|/t$ should be integrable in $(0, \eta)$ are that*

$$\int_0^{\eta} \log \frac{k}{t}\, |dh(t)| < \infty \quad \textit{and} \quad h(+0) = 0. \tag{2.1.1}$$

By Lemma 1, Theorem I is equivalent to the following result.

THEOREM Ia. *If* (i) $h(+0) = 0$, *and* (ii) $\int_0^\pi \log(k/t)|dh(t)| < \infty$, *then the series* $\sum B_n(x)/\log(n+1)$ *is summable* $|R, e^{\omega^\alpha}, 1|$, *where* $0 < \alpha < 1$.

2.2. For $0 < \alpha < 1$, we set

$$\xi(\omega, t) = \sum_{n \leqq \omega} e^{n^\alpha} \frac{\sin nt}{\log(n+1)}, \tag{2.2.1}$$

$$\eta(\omega, t) = \sum_{n \leqq \omega} e^{n^\alpha} \frac{\cos nt}{n \log(n+1)}, \tag{2.2.2}$$

$$g(\omega, t) = \int_0^t \log \frac{k}{u} \xi(\omega, u)\, du, \tag{2.2.3}$$

$$E(\omega, t) = \int_t^\pi \log \frac{k}{u} \xi(\omega, u)\, du. \tag{2.2.4}$$

For our proof we shall need the following estimates:

$$\sum_{n \leqq \omega} \frac{e^{n^\alpha}}{\log(n+1)} = O\left\{\frac{e^{\omega^\alpha} \omega^{1-\alpha}}{\log \omega}\right\}, \tag{2.2.5}$$

$$\sum_{n \leqq \omega} \frac{e^{n^\alpha}}{n} = O\left\{\frac{e^{\omega^\alpha}}{\omega^\alpha}\right\}, \tag{2.2.6}$$

$$\sum_{n \leqq \omega} \frac{e^{n^\alpha}}{n \log(n+1)} = O\left\{\frac{e^{\omega^\alpha}}{\omega^\alpha \log \omega}\right\}, \tag{2.2.7}$$

$$\eta(\omega, t) = O\left\{t^{-1} \frac{e^{\omega^\alpha}}{\omega \log \omega}\right\}, \tag{2.2.8}$$

$$g(\omega, t) = O\left\{t \log \frac{k}{t} e^{\omega^\alpha} \frac{\omega^{1-\alpha}}{\log \omega}\right\}, \tag{2.2.9}$$

$$g(\omega, \pi) = O\left\{\frac{e^{\omega^\alpha}}{\omega^\alpha}\right\}, \tag{2.2.10}$$

$$E(\omega, t) = O\left\{t^{-1} \log \frac{k}{t} \frac{e^{\omega^\alpha}}{\omega \log \omega}\right\}, \tag{2.2.11}$$

$$E(\omega, t) = O\left\{\log \frac{k}{t} \cdot \frac{e^{\omega^\alpha}}{\omega^\alpha \log \omega}\right\}. \tag{2.2.12}$$

The estimates (2.2.5), (2.2.6), and (2.2.7) can be proved by adopting techniques similar to those used in (**6**, proof of (2.1.5)).

Proof of (2.2.8). By Abel's lemma, for $m \leqq \omega < m + 1$,

$$\eta(\omega, t) = \sum_1^m \frac{e^{n^\alpha}}{n \log(n+1)} \cos nt = O\left[\frac{e^{\omega^\alpha}}{\omega \log \omega} t^{-1}\right].$$

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Proof of (2.2.9). Using (2.2.5), we have that

$$g(\omega, t) = O\left[\frac{e^{\omega^\alpha}\omega^{1-\alpha}}{\log \omega}\int_0^t \log\frac{k}{u}\,du\right] = O\left[t\log\frac{k}{t}\frac{e^{\omega^\alpha}\omega^{1-\alpha}}{\log \omega}\right].$$

Proof of (2.2.10). Since

$$\begin{aligned}\int_0^\pi \log\frac{k}{u}\sin nu\,du &= \int_0^{\pi/n}\log\frac{k}{u}\sin nu\,du + \int_{\pi/n}^\pi \log\frac{k}{u}\sin nu\,du\\ &= O\left(\int_0^{\pi/n}\log\frac{k}{u}\,du\right) + \log\frac{kn}{\pi}\int_{\pi/n}^{\pi'}\sin nu\,du \\ &\qquad\qquad\left(\frac{\pi}{n} \leqq \pi' \leqq \pi\right)\\ &= O\left(\frac{\log n}{n}\right) + \log\frac{kn}{\pi}\frac{(\cos\pi - \cos n\pi')}{n} = O\left(\frac{\log n}{n}\right),\end{aligned}$$

we have that

$$\begin{aligned}g(\omega, \pi) &= \int_0^\pi \log\frac{k}{u}\sum_{n\leqq\omega}\frac{e^{n^\alpha}}{\log(n+1)}\sin nu\,du\\ &= \sum_{n\leqq\omega}\frac{e^{n^\alpha}}{\log(n+1)}\left(\int_0^\pi\log\frac{k}{u}\sin nu\,du\right)\\ &= O\left(\sum_{n\leqq\omega}\frac{e^{n^\alpha}}{n}\right)\\ &= O\left(\frac{e^{\omega^\alpha}}{\omega^\alpha}\right), \quad \text{by using (2.2.6).}\end{aligned}$$

Proof of (2.2.11). By the application of the second mean value theorem, we have, for $t \leqq \mu \leqq \pi$,

$$\begin{aligned}E(\omega, t) &= \log\frac{k}{t}\int_t^\mu\left(\sum_{n\leqq\omega}\frac{e^{n\alpha}}{\log(n+1)}\sin nu\right)du\\ &= \log\frac{k}{t}\,[\eta(\omega, t) - \eta(\omega, \mu)]\\ &= O\left[t^{-1}\log\frac{k}{t}\frac{e^{\omega^\alpha}}{\omega\log\omega}\right], \quad\text{by using (2.2.8).}\end{aligned}$$

Proof of (2.2.12). As in (2.2.11), we have that

$$\begin{aligned}E(\omega, t) &= \log\frac{k}{t}\,[\eta(\omega, t) - \eta(\omega, \mu)]\\ &= O\left[\log\frac{k}{t}\sum_{n\leqq\omega}\frac{e^{n^\alpha}}{n\log(n+1)}\right]\\ &= O\left[\log\frac{k}{t}\cdot\frac{e^{\omega^\alpha}}{\omega^\alpha\log\omega}\right], \quad\text{by using (2.2.7).}\end{aligned}$$

2.3. *Proof of Theorem* Ia. We have that

$$B_n(x) = \frac{2}{\pi}\int_0^\pi \psi(t)\sin nt\,dt = \frac{2}{\pi}\int_0^\pi h(t)\log\frac{k}{t}\sin nt\,dt.$$

Integrating by parts, we have that

$$B_n(x) = \frac{2}{\pi}\left[-h(t)\int_t^\pi \log\frac{k}{u}\sin nu\,du\right]_0^\pi + \frac{2}{\pi}\int_0^\pi dh(t)\int_t^\pi \log\frac{k}{u}\sin nu\,du$$

$$= \frac{2}{\pi}\int_0^\pi dh(t)\int_t^\pi \log\frac{k}{u}\sin nu\,du;$$

the integrated part vanishes since $h(+0) = 0$ and the integral

$$\int_t^\pi \log(k/u)\sin nu\,du$$

is finite. Thus, we have that

$$\frac{B_n(x)}{\log(n+1)} = \frac{2}{\pi}\int_0^\pi dh(t)\int_t^\pi \log\frac{k}{u}\cdot\frac{\sin nu}{\log(n+1)}\,du. \tag{2.3.1}$$

By the definition, the series $\sum B_n(x)/\log(n+1)$ is summable $|R, e^{\omega^\alpha}, 1|$, where $0 < \alpha < 1$, if

$$I = \int_e^\infty \alpha\omega^{\alpha-1}e^{-\omega^\alpha}\,d\omega\left|\sum_{n\leqq\omega} e^{n^\alpha}\frac{B_n(x)}{\log(n+1)}\right| < \infty. \tag{2.3.2}$$

Substituting for $B_n(x)/\log(n+1)$ from (2.3.1), we have, by (2.2.1) and (2.2.4), that

$$I \leqq \int_0^\pi |dh(t)|\int_e^\infty \alpha\omega^{\alpha-1}e^{-\omega^\alpha}\,d\omega|E(\omega, t)|. \tag{2.3.3}$$

Now by virtue of condition (ii) of Theorem Ia, it is sufficient to show that, uniformly for $0 < t \leqq \pi$,

$$P(t) = \int_e^\infty \alpha\omega^{\alpha-1}e^{-\omega^\alpha}|E(\omega, t)|\,d\omega = O(\log k/t).$$

Let $\tau_1 = t^{-1}$, $\tau_2 = t^{-(1-\alpha)^{-1}}$ and write

$$P(t) = \int_e^{\tau_1} + \int_{\tau_1}^{\tau_2} + \int_{\tau_2}^\infty = P_1 + P_2 + P_3, \quad \text{say}.$$

It is easy to see that P_1 vanishes if $1 \geqq t \geqq e^{-1}$, and P_1 and P_2 both vanish if $t > 1$. For $0 < t < e^{-1}$, we have that

$$\begin{aligned}
P_1 &= \int_e^{\tau_1} \alpha\omega^{\alpha-1}e^{-\omega^\alpha}|E(\omega, t)|\,d\omega \\
&= \int_e^{\tau_1} \alpha\omega^{\alpha-1}e^{-\omega^\alpha}|g(\omega, \pi) - g(\omega, t)|\,d\omega \\
&< \int_e^{\tau_1} \alpha\omega^{\alpha-1}e^{-\omega^\alpha}|g(\omega, \pi)|\,d\omega + \int_e^{\tau_1} \alpha\omega^{\alpha-1}e^{-\omega^\alpha}|g(\omega, t)|\,d\omega \\
&= P_{1,1} + P_{1,2}, \quad \text{say}.
\end{aligned}$$

Employing the inequalities (2.2.10) and (2.2.9) for $P_{1,1}$ and $P_{1,2}$, respectively, it can be shown that

$$P_1 = O(\log k/t).$$

Using the estimates (2.2.12) and (2.2.11) for P_2 and P_3, respectively, we obtain $P_2 = O(\log k/t)$ and $P_3 = O(\log k/t)$ and this completes the proof of Theorem Ia.

3. Theorem II.

3.1. LEMMA 2. *If* $\eta > 0$ *and* $\rho > 0$, $t^\rho h(t) = H(t)$, *then necessary and sufficient conditions that* (i) $h(t)$ *should be of bounded variation in* $(0, \eta)$ *and* (ii) $|h(t)|/t$ *should be integrable in* $(0, \eta)$ *are that*

$$(3.1.1) \qquad \int_0^\eta t^{-\rho}|dH(t)| < \infty \quad \textit{and} \quad H(+0) = 0.$$

The proof is similar to that of Lemma 1. By Lemma 2, Theorem II is equivalent to the following theorem.

THEOREM IIa. *If* (i) $H(+0) = 0$ *and* (ii) $\int_0^\pi t^{-\rho}|dH(t)| < \infty$, *then the series* $\sum B_n(x)/\log(n+1)$ *is summable* $|C, \delta|$, $\delta > 0$.

As Lemma 2 is of the necessary and sufficient type and holds for any $\rho > 0$, without loss of generality we can assume[4] that $\rho > 1$ while proving Theorem IIa. It is well known that $|C, \delta| \sim |R, n, \delta|$, and by the first theorem of consistency (see Lemma 3 of the present paper), $|R, n, \delta|$ implies $|R, n, \delta'|$ for $\delta' > \delta$. Thus, without loss of generality we can assume that $0 < \delta < 1$ in the proof of Theorem IIa.

3.2. *Notation.* For $0 < \delta < 1$, we write

$$(3.2.1) \qquad A_n^\delta = \binom{n+\delta}{n} = \frac{(\delta+1)(\delta+2)\ldots(\delta+n)}{n!} \simeq \frac{n^\delta}{\Gamma(\delta+1)},$$

$$(3.2.2) \qquad S_p(n, u) = \sum_{\nu=0}^{p} A_{n-\nu}^{\delta-1}\cos \nu u, \quad p \leqq n,$$

$$(3.2.3) \qquad S_p'(n, u) = \frac{d}{du} S_p(n, u),$$

$$(3.2.4) \qquad H^\delta(n, u) = \frac{1}{A_n^\delta}\sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1}\frac{\cos \nu u}{\log(\nu+2)},$$

$$(3.2.5) \qquad G(n, t) = \int_t^\pi u^{-\rho}\log\frac{k}{u}\frac{d}{du}H^\delta(n, u)\,du.$$

[4]It is convenient to take $\rho > 1$, as will be apparent from the proof of (3.2.11).

We shall need the following inequalities:

$$S_p(n,u) = \begin{cases} O\{p(n-p)^{\delta-1}\} \\ O\{u^{-1}(n-p)^{\delta-1}\} \quad (0 < u \leqq \pi), \end{cases} \tag{3.2.6}$$

$$S_n(n,u) = \begin{cases} O(n^{\delta}) \\ O(u^{-\delta}) \quad (0 < u \leqq \pi), \end{cases} \tag{3.2.7}$$

$$S_p'(n,u) = \begin{cases} O\{p^2(n-p)^{\delta-1}\} \\ O\{u^{-1}p(n-p)^{\delta-1}\} \quad (0 < u \leqq \pi), \end{cases} \tag{3.2.8}$$

$$S_n'(n,u) = \begin{cases} O(n^{1+\delta}) \\ O(nu^{-\delta}) \quad (0 < u \leqq \pi), \end{cases} \tag{3.2.9}$$

$$\frac{d}{du} H^{\delta}(n,u) = O(n/\log n), \tag{3.2.10}$$

$$G(n,t) = O\left\{t^{1-\rho} \log \frac{k}{t} \frac{n}{\log n}\right\}, \tag{3.2.11}$$

$$G(n,t) = O\left[\frac{t^{-\rho} \log k/t}{(\log n)^2}\right] + O\left[\frac{t^{-\rho-\delta} \log k/t}{n^{\delta} \log n}\right]. \tag{3.2.12}$$

Proof of (3.2.6). Since $A_{n-\nu}^{\delta-1}$ increases as ν increases for $0 < \delta < 1$ we have, for $0 \leqq L \leqq L' \leqq p$, that

$$S_p(n,u) = A_{n-p}^{\delta-1} \operatorname*{Max}_{L,L'} \left| \sum_{L}^{L'} \cos \nu u \right| = \begin{cases} O\{p(n-p)^{\delta-1}\} \\ O\{u^{-1}(n-p)^{\delta-1}\}. \end{cases}$$

Proof of (3.2.8) runs parallel to the proof of (3.2.6). See Obrechkoff (**11**) for the proof of (3.2.7) and (3.2.9).

Proof of (3.2.10).[5] We have that

$$\begin{aligned} -A_n^{\delta} \frac{d}{du} H^{\delta}(n,u) &= \sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1} \frac{\nu \sin \nu u}{\log(\nu+2)} \\ &= O\left[\sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1} \frac{\nu}{\log(\nu+2)}\right] \\ &= O\left[\sum_{\nu=0}^{[n/2]} + \sum_{[n/2]+1}^{n}\right] \\ &= O(n^{\delta-1}) \sum_{\nu=0}^{[n/2]} \frac{\nu}{\log(\nu+2)} + O\left(\frac{n}{\log n}\right) \sum_{[n/2]+1}^{n} A_{n-\nu}^{\delta-1} \\ &= O(n^{\delta-1}) O\left(\frac{n^2}{\log n}\right) + O\left(\frac{n}{\log n}\right) O(n^{\delta}). \end{aligned}$$

Now the inequality (3.2.10) follows at once from the above relation.

[5]We are indebted to the referee for supplying us with the present version of the proof for (3.2.10).

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Proof of (3.2.11). By (3.2.10), we have that

$$G(n, t) = O\left[\int_t^\pi u^{-\rho} \log \frac{k}{u} \frac{n}{\log n}\, du\right] = O\left[t^{1-\rho} \log \frac{k}{t} \frac{n}{\log n}\right].$$

Proof of (3.2.12). Here we shall adopt the method employed in (**4**, p. 207). Adopting our notation, we have, by Abel's method of partial summation, that

$$\frac{d}{du} H^\delta(n, u) = \frac{1}{A_n^\delta} \sum_{\nu=0}^{n-1} S_\nu'(n, u) \Delta\left(\frac{1}{\log(\nu+2)}\right) + \frac{1}{A_n^\delta} \frac{S_n'(n, u)}{\log(n+2)},$$

where $\Delta\mu_n = \mu_n - \mu_{n+1}$. Thus, we have that

$$\begin{aligned}(3.2.13)\qquad G(n, t) &= \int_t^\pi u^{-\rho} \log \frac{k}{u} \frac{d}{du} H^\delta(n, u)\, du \\ &= \frac{1}{A_n^\delta} \int_t^\pi u^{-\rho} \log \frac{k}{u}\, du \sum_{\nu=0}^{n-1} S_\nu'(n, u) \Delta\left(\frac{1}{\log(\nu+2)}\right) \\ &\qquad + \frac{1}{A_n^\delta} \int_t^\pi u^{-\rho} \log \frac{k}{u} \frac{S_n'(n, u)}{\log(n+2)}\, du \\ &= G_1(n, t) + G_2(n, t), \quad \text{say.}\end{aligned}$$

Using (3.2.8), we have that

$$\begin{aligned}G_1(n, t) &= \frac{1}{A_n^\delta} \int_t^\pi u^{-\rho} \log \frac{k}{u}\, du\, O\left\{u^{-1} \int_0^n \nu(n-\nu)^{\delta-1} \frac{d\nu}{\nu(\log \nu)^2}\right\} \\ &= O\left[n^{-\delta} \int_t^\pi u^{-\rho-1} \log \frac{k}{u}\, du \int_0^n \frac{(n-\nu)^{\delta-1}}{(\log \nu)^2}\, d\nu\right] \\ &= O\left[n^{-\delta} \int_t^\pi u^{-\rho-1} \log \frac{k}{u}\, du \frac{n^\delta}{(\log n)^2}\right].\end{aligned}$$

Thus, we obtain

$$(3.2.14)\qquad G_1(n, t) = O\left[t^{-\rho} \log \frac{k}{t} (\log n)^{-2}\right].$$

By the mean value theorem, we have, for $t < t' < \pi$, that

$$G_2(n, t) = \frac{t^{-\rho} \log k/t}{A_n^\delta \log(n+2)} \{S_n(n, t') - S_n(n, t)\}.$$

By using (3.2.7), we have that

$$\begin{aligned}(3.2.15)\qquad G_2(n, t) &= \frac{t^{-\rho} \log k/t}{A_n^\delta \log(n+2)} O(t^{-\delta}), \quad \text{since } t < t', \\ &= O\left[t^{-\rho-\delta} \log \frac{k}{t} \frac{1}{n^{-\delta} \log n}\right].\end{aligned}$$

Now combining the results of (3.2.13), (3.2.14), and (3.2.15), the inequality (3.2.12) follows immediately.

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3.3. *Proof of Theorem* IIa. For $n \geqq 1$, we have that

$$B_n(x) = \frac{2}{\pi}\int_0^\pi H(t)t^{-\rho}\log\frac{k}{t}\sin nt\,dt = -\frac{2}{\pi}\int_0^\pi H(t)\beta'(n, t)\,dt,$$

where $\beta(n, t) = \int_t^\pi u^{-\rho}\log(k/u)\sin nu\;du$. Recalling that $\beta(n, \pi) = 0$, $h(+0)$ is finite, and $\beta(n, t) = O(t^{1-\rho}\log k/t)$ (since $\rho > 1$), we have, by integrating by parts, that

$$B_n(x) = \frac{2}{\pi}\int_0^\pi dH(t)\beta(n, t).$$

Thus, we have that

$$\frac{B_n(x)}{\log(n+1)} = \frac{2}{\pi}\int_0^\pi dH(t)\int_t^\pi u^{-\rho}\log\frac{k}{u}\frac{\sin nu}{\log(n+1)}\,du. \tag{3.3.1}$$

It is sufficient to prove that

$$\sum = \sum_{n=1}^{\infty}\frac{|\tau_n{}^\delta|}{n} < \infty, \quad \text{where} \quad \tau_n{}^\delta = \frac{1}{A_n{}^\delta}\sum_{\nu=0}^{n}A_{n-\nu}{}^{\delta-1}\frac{\nu B_\nu(x)}{\log(\nu+2)}.$$

Using (3.3.1), we have that

$$\begin{aligned}
\tau_n{}^\delta &= \frac{1}{A_n{}^\delta}\sum_{\nu=0}^{n}A_{n-\nu}{}^{\delta-1}\frac{\nu}{\log(\nu+2)}\cdot\frac{2}{\pi}\int_0^\pi dH(t)\int_t^\pi u^{-\rho}\log\frac{k}{u}\sin\nu u\,du\\
&= \frac{2}{\pi}\int_0^\pi dH(t)\int_t^\pi u^{-\rho}\log\frac{k}{u}\,du\left[\frac{1}{A_n{}^\delta}\sum_{\nu=0}^{n}A_{n-\nu}{}^{\delta-1}\frac{\nu}{\log(\nu+2)}\sin\nu u\right]\\
&= -\frac{2}{\pi}\int_0^\pi dH(t)\int_t^\pi u^{-\rho}\log\frac{k}{u}\frac{d}{du}H^\delta(n, u)\,du\\
&= -\frac{2}{\pi}\int_0^\pi dH(t)G(n, t).
\end{aligned}$$

$$\begin{aligned}
\sum &< \int_0^\pi |dH(t)|\sum_{n=1}^{\infty}\frac{|G(n, t)|}{n}\\
&= \int_0^\pi |dH(t)|\left(\sum_{n\leqq t^{-1}} + \sum_{n>t^{-1}}\right).
\end{aligned}$$

Using (3.2.11) and (3.2.12) for $\sum_{n\leqq t^{-1}}$ and $\sum_{n>t^{-1}}$, respectively, it can be shown that, uniformly in $0 < t \leqq \pi$,

$$\sum_{n=1}^{\infty}\frac{|G(n, t)|}{n} = O(t^{-\rho}).$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{|\tau_n{}^\delta|}{n} = O\left\{\int_0^\pi t^{-\rho}|dH(t)|\right\} < \infty,$$

by the hypothesis, and this completes the proof of the theorem.

4. Theorem III.

4.1. *Notation.* For $0 < \delta < 1$,

$$R_p(n,t) = \sum_{\nu=0}^{p} A_{n-\nu}^{\ \delta} \cos \nu t, \qquad p \leqq n, \tag{4.1.1}$$

$$R_p'(n,t) = \frac{d}{dt} R_p(n,t), \tag{4.1.2}$$

$$J^{1+\delta}(n,t) = \frac{1}{A_n^{\ 1+\delta}} \sum_{\nu=0}^{n} A_{n-\nu}^{\ \delta} \frac{\cos \nu t}{\log(\nu+2)}. \tag{4.1.3}$$

We shall use the following estimates

$$R_p'(n,t) = \begin{cases} O(n^{\delta} p^2) \\ O(n^{\delta} p t^{-1}), \end{cases} \qquad p < n, \tag{4.1.4}$$

$$R_n'(n,t) = \begin{cases} O(n^{2+\delta}) \\ O(n t^{-1-\delta}), \end{cases} \tag{4.1.5}$$

$$\frac{d}{dt} J^{1+\delta}(n,t) = O\left(\frac{n}{\log n}\right), \tag{4.1.6}$$

$$\frac{d}{dt} J^{1+\delta}(n,t) = O[t^{-1}(\log n)^{-2}] + O[t^{-1-\delta} n^{-\delta} (\log n)^{-1}]. \tag{4.1.7}$$

Proof of (4.1.4). We have, for $0 \leqq N \leqq N' \leqq p$, that

$$|R_p'(n,t)| = \left| -\sum_{\nu=0}^{p} A_{n-\nu}^{\ \delta}\, \nu \sin \nu t \right| = O\left[A_n^{\ \delta} \operatorname*{Max}_{N,N'} \left| \sum_{N}^{N'} \nu \sin \nu t \right| \right]$$

since $A_{n-\nu}^{\ \delta}$ decreases as ν increases for every positive δ. Thus,

$$R_p'(n,t) = \begin{cases} O(n^{\delta} p^2) \\ O(n^{\delta} p t^{-1}). \end{cases}$$

Proof of (4.1.5). We have that

$$\begin{aligned} R_n'(n,t) &= A_n^{\ 1+\delta} \frac{d}{dt} \left[\frac{1}{A_n^{\ 1+\delta}} \sum_{\nu=0}^{n} A_{n-\nu}^{\ \delta} \cos \nu t \right] \\ &= \begin{cases} A_n^{\ 1+\delta} O(n) \\ A_n^{\ 1+\delta} O(n^{-\delta} t^{-1-\delta}), \quad \text{see Obrechkoff } (\mathbf{11}) \end{cases} \\ &= \begin{cases} O(n^{2+\delta}) \\ O(n t^{-1-\delta}). \end{cases} \end{aligned}$$

Proof of (4.1.6). By definition,

$$\begin{aligned} \frac{d}{dt} J^{1+\delta}(n,t) &= \frac{1}{A_n^{\ 1+\delta}} \left[\sum_{\nu=0}^{n-1} R_\nu'(n,t) \Delta\left(\frac{1}{\log(\nu+2)} \right) + \frac{R_n'(n,t)}{\log(n+2)} \right] \\ &= O\left[\frac{1}{A_n^{\ 1+\delta}} \int_0^n n^{\delta} \nu^2 \Delta\left(\frac{1}{\log(\nu+2)} \right) d\nu \right] + O\left[\frac{1}{A_n^{\ 1+\delta}} \frac{n^{2+\delta}}{\log(n+2)} \right], \end{aligned}$$

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using (4.1.4) and (4.1.5). Thus, we have that

$$\frac{d}{dt} J^{1+\delta}(n, t) = O\left[n^{-1} \int_0^n \frac{\nu \, d\nu}{(\log \nu)^2}\right] + O\left[\frac{n}{\log n}\right] = O\left(\frac{n}{\log n}\right).$$

Proof of (4.1.7). Proceeding as in (4.1.6) and also using (4.1.4) and (4.1.5), we obtain

$$\begin{aligned}\frac{d}{dt} J^{1+\delta}(n, t) &= O\left[\frac{1}{A_n^{1+\delta}} \int_0^n n^\delta \nu t^{-1} \Delta\left(\frac{1}{\log(\nu+2)}\right) d\nu\right] \\ &\qquad + O\left[\frac{1}{A_n^{1+\delta} \log(n+2)} n t^{-1-\delta}\right] \\ &= O\left[n^{-1} t^{-1} \int_0^n \frac{d\nu}{(\log \nu)^2}\right] + O\left[\frac{t^{-1-\delta}}{n^\delta \log n}\right] \\ &= O[t^{-1}(\log n)^{-2}] + O[t^{-1-\delta} n^{-\delta} (\log n)^{-1}].\end{aligned}$$

4.2. *Proof of Theorem* III. It is sufficient to prove that

$$\sum_{n=1}^\infty \frac{|\xi_n^{1+\delta}|}{n} < \infty,$$

where $\xi_n^{1+\delta}$ is the nth Cesàro mean of order $1 + \delta$ of the sequence $\{nB_n(x)/\log(n+2)\}$, that is,

$$\begin{aligned}\xi_n^{1+\delta} &= \frac{1}{A_n^{1+\delta}} \sum_{\nu=0}^n A_{n-\nu}^{\delta} \frac{\nu B_\nu(x)}{\log(\nu+2)} \\ &= \frac{1}{A_n^{1+\delta}} \sum_{\nu=0}^n A_{n-\nu}^{\delta} \frac{\nu}{\log(\nu+2)} \cdot \frac{2}{\pi} \int_0^\pi \psi(t) \sin \nu t \, dt \\ &= -\frac{2}{\pi} \int_0^\pi h(t) \log \frac{k}{t} \frac{d}{dt}\left[\frac{1}{A_n^{1+\delta}} \sum_{\nu=0}^n A_{n-\nu}^{\delta} \frac{\cos \nu t}{\log(\nu+2)}\right] \\ &= -\frac{2}{\pi} \int_0^\pi h(t) \log \frac{k}{t} \frac{d}{dt} J^{1+\delta}(n, t) \, dt.\end{aligned}$$

We have that

$$\sum_{n=1}^\infty \frac{|\xi_n^{1+\delta}|}{n} < \int_0^\pi |h(t)| \log \frac{k}{t} \, dt \sum_{n=1}^\infty \frac{|(d/dt) J^{1+\delta}(n, t)|}{n}.$$

Writing

$$\sum_{n=1}^\infty \frac{|(d/dt) J^{1+\delta}(n, t)|}{n} = \sum_{n \leqq t^{-1}} + \sum_{n > t^{-1}},$$

and using the estimates (4.1.6) and (4.1.7) for $\sum_{n \leqq t^{-1}}$ and $\sum_{n > t^{-1}}$, respectively, it can be shown that

$$\sum_{n=1}^\infty \frac{|(d/dt) J^{1+\delta}(n, t)|}{n} = O\{t^{-1}(\log k/t)^{-1}\}, \quad \text{uniformly in } 0 < t \leqq \pi.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{|\xi_n{}^{1+\delta}|}{n} < \int_0^{\pi} |h(t)| \log \frac{k}{t}\, dt\, O\left\{t^{-1}\left(\log \frac{k}{t}\right)^{-1}\right\}$$
$$= O\left(\int_0^{\pi} \frac{|h(t)|}{t}\, dt\right) < \infty,$$

by the hypothesis, and this completes the proof.

5. Theorem IV.

5.1. LEMMA 3(**10**). *If a series* $\sum a_n$ *is summable* $|R, \lambda, r|$, *where* $r > 0$, *it is summable* $|R, \lambda, r'|$, *for* $r' > r$.

LEMMA 4(**2**). *If* (i) *the series* $\sum a_n$ *is summable* $|R, \lambda, r|$, *where* $r > 0$, (ii) μ *is a logarithmico-exponential function of* λ *such that* $\mu = O(\lambda^{\Delta})$, *where* Δ *is a constant, then the series* $\sum a_n$ *is summable* $|R, \mu, r|$.

In view of Lemma 1, Theorem IV can be put in the following equivalent form.

THEOREM IVa. *If* (i) $h_1{}^*(+0) = 0$ *and* (ii) $\int_0^{\pi} \log k/t\, |dh_1{}^*(t)| < \infty$, *then the series* $\sum B_n(x)/\log(n+1)$ *is summable* $|R, e^{(\log \omega)^2}, 2|$.

5.2. To simplify the expressions, we write $e(n)$ and $e(\omega)$ for $\exp\{(\log n)^2\}$ and $\exp\{(\log \omega)^2\}$, respectively. We also write

$$F(\omega, t) = \sum_{n \leqq \omega} \{e(\omega) - e(n)\} e(n) \frac{\sin nt}{\log(n+1)}. \tag{5.2.1}$$

Application of the same technique as that used in (**6**, proof of (5.1.4) and (5.1.7)) will yield the following estimates:

$$F(\omega, t) = O[e^2(\omega)\omega^{-1}t^{-2}], \tag{5.2.2}$$

$$F(\omega, t) = O[e^2(\omega)\omega(\log \omega)^{-2}]. \tag{5.2.3}$$

5.3. *Proof of Theorem* IVa. We have, for $n \geqq 1$, that

$$\begin{aligned} B_n(x) &= \frac{2}{\pi} \int_0^{\pi} h(t) \log \frac{k}{t} \sin nt\, dt \\ &= \frac{2}{\pi} \int_0^{\pi} \left[\frac{d}{dt}\{th_1{}^*(t)\}\right] \log \frac{k}{t} \sin nt\, dt \\ &= \frac{2}{\pi} \int_0^{\pi} h_1{}^*(t) \log \frac{k}{t} \sin nt\, dt + \frac{2}{\pi} \int_0^{\pi} dh_1{}^*(t) t \log \frac{k}{t} \sin nt. \end{aligned}$$

Thus,

$$\frac{B_n(x)}{\log(n+1)} = U_n + V_n,$$

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where

$$(5.3.1) \qquad U_n = \frac{2}{\pi} \int_0^\pi h_1^*(t) \log \frac{k}{t} \frac{\sin nt}{\log(n+1)} dt$$

and

$$(5.3.2) \qquad V_n = \frac{2}{\pi} \int_0^\pi dh_1^*(t) t \log \frac{k}{t} \frac{\sin nt}{\log(n+1)} .$$

By Theorem I, the series $\sum U_n$ is summable $|R, e^{\omega^\alpha}, 1|$, where $0 < \alpha < 1$. Hence, by Lemmas 3 and 4, $\sum U_n$ is summable $|R, e(\omega), 2|$. Therefore, it suffices to consider the $|R, e(\omega), 2|$ summability of $\sum V_n$. By familiar arguments, it is evident that we need only establish that, uniformly in $0 < t \leqq \pi$,

$$Q(t) = \int_e^\infty \frac{\log \omega}{\omega^2} e^{-2}(\omega) |F(\omega, t)| \, d\omega = O(t^{-1}).$$

As in the proof of (**6**, Theorem 6A) writing $Q(t) = \int_e^\tau + \int_\tau^\infty$, for $\tau = (k/t) \log k/t$ and using the inequalities (5.2.3) and (5.2.2) over the intervals (e, τ) and (τ, ∞), respectively, it can be proved that $Q(t) = O(t^{-1})$. This will complete the proof of our theorem.

6. Theorem V.

6.1. LEMMA 5. *Define*

(A) $$\int_0^\pi \frac{|\psi(t)|}{t \log k/t} dt < \infty;$$

(a) $\psi_1(t)$ *is of bounded variation in* $(0, \pi)$;

(b) $$\int_0^\pi \frac{|\psi_1(t)|}{t \log k/t} dt < \infty;$$

(c) $$\int_t^\pi \frac{|\psi(u)|}{u} du = o\left(\log \frac{1}{t}\right) \quad as\ t \to 0.$$

(i)[6] (A) *is equivalent to* (a) *and* (b),
(ii) (A) *implies* (c).

Proof of (i). As in (**9**, Lemma 2), we can prove that (A) implies (a). We now proceed to show that (A) implies (b). We shall use the following identity:

$$\frac{\psi(t)}{t \log k/t} = \frac{\psi_1(t)}{t \log k/t} - \frac{d}{dt} \{\psi_1(t)\}.$$

(a) is true whenever (A) holds. Hence, by the above identity, (b) holds whenever (A) holds. Also from the identity, it follows that (a) and (b) together imply (A). Thus, we have proved (i).

[6]We are grateful to the referee for kindly obtaining this equivalence relation.

Proof of (ii). Denoting $\int_0^u (|\psi(v)|/v \log k/v)dv$ by $\theta(u)$, we have by integration by parts, that

$$\begin{aligned}\int_t^\pi \frac{|\psi(u)|}{u}\,du &= \int_t^\pi \frac{|\psi(u)|}{u \log k/u} \log k/u\, du \\ &= [\theta(u) \log k/u]_t^\pi + \int_t^\pi \frac{\theta(u)}{u}\,du \\ &= \theta(\pi) \log k/\pi - \theta(t) \log k/t + \int_t^\pi \frac{\theta(u)}{u}\,du \\ &= o\left(\log \frac{1}{t}\right) + o\left(\int_t^\pi \frac{du}{u}\right) = o\left(\log \frac{1}{t}\right) \quad \text{as } t \to +0,\end{aligned}$$

since $\theta(\pi)$ is finite and $\theta(t) = o(1)$ as $t \to 0$. This completes the proof of (ii).

By Lemma 5(i), it is easy to see that Theorem V is equivalent to the following.

THEOREM Va. *If* (i) $\psi_1(t)$ *is of bounded variation in* $(0, \pi)$ *and* (ii) $|\psi_1(t)|/t \log k/t$ *is integrable in* $(0, \pi)$, *then* $\sum B_n(x)/\log(n+1)$ *is summable* $|R, \log \omega, 1|$.

In (**7**, Theorem B), the analogous result for the series $\sum A_n(x)/\log(n+1)$ was proved.

6.2. *Proof of Theorem* V. We have, for $n \geqq 1$, that

$$\frac{B_n(x)}{\log(n+1)} = \frac{2}{\pi} \int_0^\pi h(t) \log \frac{k}{t} \frac{\sin nt}{\log(n+1)}\,dt. \tag{6.2.1}$$

We have to show that $\sum B_n(x)/\log(n+1)$ is summable $|R, \log \omega, 1|$, or what is the same thing,

$$K = \int_e^\infty \frac{d\omega}{\omega(\log \omega)^2} \left| \sum_{n \leqq \omega} \log(n+1) \frac{B_n(x)}{\log(n+1)} \right| < \infty. \tag{6.2.2}$$

Using (6.2.1), we have that

$$K \leqq \frac{2}{\pi} \int_0^\pi |h(t)| \log \frac{k}{t} |S(t)|\,dt,$$

where

$$S(t) = \int_e^\infty \frac{d\omega}{\omega(\log \omega)^2} \left| \sum_{n \leqq \omega} \sin nt \right|.$$

We have that

$$\begin{aligned}S(t) &= \int_e^{k/t} + \int_{k/t}^\infty \\ &= \int_e^{k/t} \frac{d\omega}{\omega(\log \omega)^2} O(\omega) + \int_{k/t}^\infty \frac{d\omega}{\omega(\log \omega)^2} O(t^{-1}) \\ &= O\{t^{-1}(\log k/t)^{-1}\} \quad \text{for } 0 < t \leqq \pi.\end{aligned}$$

Therefore,

$$K = \int_0^\pi |h(t)| \log \frac{k}{t}\, dt\, O\left\{t^{-1}\left(\log \frac{k}{t}\right)^{-1}\right\}$$

$$= O\left(\int_0^\pi \frac{|h(t)|}{t}\, dt\right) < \infty,$$

by the hypotheses, and thus the proof of Theorem V is complete.

7. Theorem VI.

7.1. It is known (**3**, Theorem 37) that summability by logarithmic means is equivalent to the summability $(R, \log n, 1)$. An infinite series $\sum_{n=0}^\infty u_n$ with $U_n = u_0 + u_1 + u_2 + \ldots + u_n$ (or the sequence $\{U_n\}$) is said to be summable by logarithmic means to the sum U, if

$$\operatorname*{Lim}_{n\to\infty} \frac{U_0 + U_1/2 + U_2/3 + \ldots + U_n/(n+1)}{\log(n+1)} = U.$$

LEMMA 6. *If*

$$\int_t^\pi \frac{|\psi(u)|}{u}\, du = o\left(\log \frac{1}{t}\right) \quad \text{as } t \to 0,$$

then $\bar{S}_n = o(\log n)$, *where* $\bar{S}_n$ *is the nth partial sum of the series* (1.2.2).

This result is due to Misra (**5**, Theorem C(b)).

LEMMA 7. *If a series* $\sum u_n$ *is summable by logarithmic means, then a necessary and sufficient condition that it should be convergent is that the sequence* $\{n \log n \cdot u_n\}$ *is summable by logarithmic means to the value zero* (**8**, Lemma 5).

7.2. *Proof of Theorem* VI. Using Theorem V, we find that with our hypothesis, the series $\sum_2^\infty B_n(x)/\log n$ is summable $|R, \log n, 1|$, and *a fortiori*, summable $(R, \log n, 1)$. By Lemma 5 (ii), our hypothesis satisfies the condition of Lemma 6, and hence the summability by logarithmic means of the sequence $\{nB_n(x)\}$, i.e., the sequence $\{n \log n \cdot [B_n(x)/\log n]\}$, to the value zero, or in other words, as the conclusion of Lemma 6 is ensured by the hypothesis. Thus, it follows that our hypothesis ensures at the same time the summability by logarithmic means of the series $\sum_2^\infty B_n(x)/\log n$ as also the summability by logarithmic means to the value zero of the sequence $\{nB_n(x)\}$, i.e. the sequence $\{n \log n \cdot B_n(x)/\log n\}$, and hence by Lemma 7 the series $\sum_2^\infty B_n(x)/\log n$ converges.

7.3. *Concluding remarks.* From the proof of Theorem VI, it is apparent that if $\sum B_n(x)/\log(n+1)$ is summable $(R, \log n, 1)$ and

$$(*) \qquad \int_t^\pi \frac{|\psi(u)|}{u}\, du = o\left(\log \frac{1}{t}\right) \quad \text{as } t \to 0,$$

then the series $\sum B_n(x)/\log(n+1)$ converges.

As an improvement of the statement made above, the referee finally suggests to prove the following conjecture regarding the convergence of the series $\sum B_n(x)/\log(n+1)$. As we have not been able to confirm the same by a proof, it still remains as an open problem.

Final conjecture. If $\sum B_n(x)/\log(n+1)$ is summable (R) (that is, summable $(R, \log \omega, r)$ for any unspecified r) and $(*)$ holds, then the series $\sum B_n(x)/\log(n+1)$ is convergent.

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Ravenshaw College,
Cuttack (Orissa), India