IDEALS IN TOPOLOGICAL RINGS

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1. Introduction. We present here an investigation of the theory of one-sided ideals in a topological ring R. One of our aims is to discuss the question of "left" properties versus "right" properties. A problem of this sort is to decide if (a) all the modular maximal right ideals of R are closed if and only if all the modular maximal left ideals of R are closed. It is shown that this is the case if R is a quasi-Q-ring, that is, if R is bicontinuously isomorphic to a dense subring of a Q-ring (for the notion of a Q-ring see **(6)** or § 2). All normed algebras are quasi-Q-rings. Also (a) holds if R is a semi-simple ring with dense socle.

Another such problem is a problem of Kaplansky **(6)** to determine if R is a Q_r-ring if and only if R is a Q_l-ring. This is true for all quasi-Q-rings. These facts suggest the desirability of a systematic investigation of quasi-Q-rings. These rings have some interesting properties not shared by all topological rings. These involve the notion of a maximal-closed modular right (left) ideal (i.e. maximal in the set of all closed modular right (left) ideals). Examples show that this notion differs from that of a closed modular maximal right (left) ideal. If R is a quasi-Q-ring, then every modular right (left) ideal which is not dense is contained in a maximal-closed modular right (left) ideal (but not necessarily in a closed maximal right (left) ideal. That this is false in general is shown (see 2.5) by the ring L^ω of Arens **(1)**. These considerations lead to the problem, only partially resolved here, of whether the intersection of all the closed maximal (or of the maximal-closed) modular right ideals is equal to the like intersection for left ideals.

In § 3 a thorough study is made of rings with no nilpotent one-sided ideals. The key result here connecting "left" properties with "right" properties is that, for such a ring, every modular maximal right ideal has a non-zero left annihilator if and only if every modular maximal left ideal has a non-zero right annihilator. Some applications to the theory of normed algebras are made in § 4.

2. Maximal-closed ideals. Let R be a topological ring. A right ideal I in R is called a *maximal-closed modular right ideal* if it is maximal in the set of proper closed modular right ideals of R. Examples of such right ideals which are not maximal right ideals are given below. As these examples are in (real) topological algebras, we start off with the following observation.

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2.1. LEMMA. *Let B be a topological algebra and M a maximal-closed modular right ideal of B as a topological ring. Then M is closed under scalar multiplication.*

Proof. Suppose otherwise that for some $x \in M$ and some scalar c, $cx \notin M$. Then the right ideal generated by cx and M is dense in B. Let j be a left identity for B modulo M, U a symmetric neighbourhood of zero, and V any neighbourhood of zero such that $Vj \subset U$. There exists $y \in B$, an integer k, and $z \in M$ such that $j - k(cx) - (cx)y - z \in V$. Then

$$j^2 - x[k(cj) + cyj] - zj \in Vj \subset U.$$

Therefore $j^2 + U$ contains an element of M. It follows that $j^2 \in M$ and, since $j^2 - j \in M$, that $j \in M$, which is impossible. This argument is patterned after one in **(10)** which shows that modular maximal right ring ideals are algebra ideals.

2.2. DEFINITION. *A topological ring R is a quasi-Q_r-ring (quasi-Q-ring) if it is bicontinuously isomorphic to a dense subring of a Q_r-ring (Q-ring) R_0.* We consider R as embedded in R_0.

2.3. LEMMA. *Every real normed algebra B is a quasi-Q-ring.*

Proof. The completion of B is a Banach algebra and hence a Q-algebra **(11**, p. 18**)** so that B is a quasi-Q-ring. In the same way any metric ring in the sense of **(6**, p. 153**)** is a quasi-Q-ring.

We adopt the algebraic conventions of **(5** and **11)**. In particular we employ the "circle operation" $x \circ y = x + y - xy$, call the element x right quasi-regular or r.q.r. (left quasi-regular or l.q.r.) if there exist $y \in R$ such that $x \circ y = 0$ ($y \circ x = 0$), and say that x is quasi-regular (q.r.) if it is both r.q.r. and l.q.r. As in **(6)** we say that R is a Q_r-ring (Q-ring) if its r.q.r. elements (q.r. elements) form an open set.

2.4. THEOREM. *Let R be a quasi-Q_r-ring. Then any modular right ideal I of R which is not dense is contained in a maximal-closed modular right ideal of R.*

Proof. Let j be a left identity for R modulo I and let K be the closure of I in R_0. Clearly $jx - x \in K$ for all $x \in R_0$ so that K is a modular right ideal of R_0. If $j \in K$, then j lies in the closure $\bar{I}$ of I in R and $\bar{I} = R$, which is impossible. Therefore, K is contained in a modular maximal right ideal M of R_0 which must be closed **(14**, Theorem 1.6**)**. Let $\mathfrak{A}$ be the collection of all modular maximal right ideals of R_0 containing K and let $\mathfrak{B} = \{M \cap R | M \in \mathfrak{A}\}$. Clearly $j \notin M$ for each $M \in \mathfrak{A}$ and each $M \cap R \in \mathfrak{B}$ is a proper modular right ideal of R. Partially order $\mathfrak{B}$ by set-inclusion and let U be a symmetric neighbourhood of zero in R_0 consisting entirely of r.q.r. elements of R_0. Note that j is a left identity for R_0 modulo M for each $M \in \mathfrak{A}$. The arguments of **(14**, Theorem 1.6**)** show that if M possesses an element in $j + U$, then $j \in M$, which is impossible.

Let $\mathfrak{C}$ be a chain in $\mathfrak{B}$ and let N be the union of the elements of $\mathfrak{C}$. Then N is a right ideal of R and j is a left identity for R modulo N. Each $M \cap R$ in $\mathfrak{C}$ fails to contain any element of $j + U$ as, therefore, does N. Thus, the closure N_0 of N in R_0 is a proper modular right ideal of R_0 which is contained in a modular maximal right ideal N_1 of R_0. Clearly $N_1 \in \mathfrak{A}$ and $N_1 \cap R$ is an upper bound for $\mathfrak{C}$ in $\mathfrak{B}$. By Zorn's maximal principle there then exists a maximal member $N_2 \cap R$ of $\mathfrak{B}$ where $N_2 \in \mathfrak{A}$.

We claim that $N_2 \cap R$ is a maximal-closed modular right ideal of R (containing I). For suppose that I_1 is a proper closed modular right ideal of R, $I_1 \supset N_2 \cap R$ and $I_1 \neq N_2 \cap R$. Clearly $I_1 \supset I$ and j is a left identity for R modulo I_1. Arguing as above we find a modular maximal right ideal K_1 of R_0, $I_1 \subset K_2$. Since $K_1 \supset K$ we have $K_1 \in \mathfrak{A}$ and $K_1 \cap R \supset I_1$ contrary to the maximality of $N_1 \cap R$. This completes the proof.

2.5. *Example.* We provide an example of a topological ring where the conclusion of Theorem 2.4 fails. Consider the topological ring L^ω of **(1)**. This is the intersection of all the L^p-spaces based on the interval [0, 1]. If we set

$$||f||_p = \left(\int_0^1 |f(x)|^p dx \right)^{1/p}$$

the ring L^ω can be metrized **(1**, p. 33**)** by the distance formula

$$(f, g) = \sum_{p=1}^{\infty} \frac{2^{-p}||f - g||_p}{1 + ||f - g||_p} \tag{2.1}$$

so that $f_n \to f$ in L^ω if and only if $f_n \to f$ in each L^p, $p \geqslant 1$. The multiplication in L^ω is pointwise (a.e.) so that we have a commutative real topological algebra with an identity. It has been observed **(7**, p. 455, footnote**)** that L^ω has no closed maximal ideals. We need the stronger statement, which we prove next, that any ideal $\mathfrak{J}$ of L^ω which is not dense is properly contained in a closed ideal $\neq L^\omega$ so that $\{0\}$ is a modular ideal contained in no maximal-closed ideal.

We may suppose that $\mathfrak{J}$ is closed. For each $f \in L^\omega$ let $\alpha(f)$ be the measure of $\{t \in [0, 1] \mid f(t) = 0\}$. Let $a = \inf \alpha(f)$, where f ranges over $\mathfrak{J}$. We claim that $a > 0$. For suppose otherwise. Take $\epsilon > 0$. There exists $f \in \mathfrak{J}$ such that $\alpha(f) < \epsilon/2$. Now

$$\alpha(f) = \lim_{n \to \infty} m\{t \in [0, 1] \mid |f(t)| \leqslant 1/n\}, \tag{2.2}$$

where $m(S)$ is the Lebesgue measure of S. Thus we may select an integer n so that

$$m\{t \in [0, 1] \mid |f(t)| \leqslant 1/n\} < \epsilon. \tag{2.3}$$

Consider the function g defined to be zero on the set W of (2.3) and $1/f$ on the complement of W. Clearly $g \in L^\omega$, $gf \in \mathfrak{J}$, and $gf = 1$ outside W while $gf = 0$ on W. Let 1 denote the function identically one. We see from (2.1) that

$$(2.4) \qquad (1, gf) = \sum_{p=1}^{\infty} \frac{2^{-p}[m(W)]^{1/p}}{1+[m(W)]^{1/p}} \leqslant \sum_{p=1}^{\infty} \frac{2^{-p}\epsilon^{1/p}}{1+\epsilon^{1/p}}.$$

Then, since the latter expression approaches zero as $\epsilon \to 0$, we see that $1 \in \mathfrak{J}$ or $\mathfrak{J} = L^{\omega}$, which is impossible.

We next show that there exists $g \in \mathfrak{J}$ with $\alpha(g) = a$. This is trivial if $a = 1$; suppose $a < 1$. Let $\{b_n\}$ be any sequence, $b_n \downarrow 0$, $a + b_n < 1$. For each integer n select $f_n \in \mathfrak{J}$ with $\alpha(f_n) < a + b_n$. We fix n and note that, by (2.2), there corresponds an integer q such that $m(W_n) < a + b_n$, where $W_n = \{t \mid |f_n(t)| \leqslant 1/q\}$. Then, by multiplication by a suitable function, we see that the characteristic function g_n of the complement of W_n lies in $\mathfrak{J}$. Observe that

$$(2.5) \qquad \left\| \sum_{n=r+1}^{s} 2^{-n} g_n \right\|_p \leqslant 2^{-r}.$$

Then from (2.1) and (2.5) we see that

$$(2.6) \qquad \left(\sum_{n=r+1}^{s} 2^{-n} g_n, 0 \right) \leqslant 2^{-r}.$$

Since L^{ω} is a complete metric space, the function

$$g = \sum_{n=1}^{\infty} 2^{-n} g_n$$

lies in $\mathfrak{J}$. Now $g(t) = 0$ if and only if every $g_n(t) = 0$ which makes $\alpha(g) \leqslant a$. Since $g \in \mathfrak{J}$ we see $\alpha(g) = a$.

Let $Z = \{t \mid g(t) = 0\}$. For any $f \in \mathfrak{J}$, $h = f^2 + g^2 \in \mathfrak{J}$ and $h(t) \neq 0$ for $t \notin Z$. It follows that f must vanish almost everywhere on Z. Consider a subset T of Z where $m(T) = a/2$. Clearly $\mathfrak{J}$ is properly contained in the set $\mathfrak{K}$ consisting of all functions in L^{ω} vanishing on T. We show that $\mathfrak{K}$ is a closed ideal.

That $\mathfrak{K}$ is a proper ideal is trivial. Let $f_n \in \mathfrak{K}$ and $f_n \to f$. Note that

$$(2.7) \qquad \|f_n - f\|_p \geqslant \left(\int_T |f(t)|^p dt \right)^{1/p}.$$

Let b be the value of the right hand side of (2.7) for $p = 1$. From (2.7) and (2.1) we see that

$$(2.8) \qquad (f_n, f) \geqslant b(2 + 2b)^{-1}.$$

Since $f_n \to f$, we see that $b = 0$ or $f \in \mathfrak{K}$.

We also wish to record that L^{ω} is semi-simple.

2.6. *Example.* We give an example of a maximal-closed modular left ideal which is not a modular maximal left ideal where the ring is a quasi-Q-ring.

Let $\mathfrak{X}$ be a real normed linear space which is *not* complete and let $\mathfrak{X}_c$ denote its completion. We let $\mathfrak{E}(\mathfrak{X})$ be the algebra of all bounded linear operators on $\mathfrak{X}$ and $\mathfrak{F}_0(\mathfrak{X})$ be the subalgebra consisting of all $T \in \mathfrak{E}(\mathfrak{X})$ with finite-dimensional range. Likewise we consider $\mathfrak{C}(\mathfrak{X}_c)$ and $\mathfrak{F}_0(\mathfrak{X}_c)$. Each $T \in \mathfrak{C}(\mathfrak{X})$ defines uniquely an extension to a bounded linear operator on $\mathfrak{X}_c$ (an element of $\mathfrak{C}(\mathfrak{X}_c)$). This extension we also denote by T. Note that if $T \in \mathfrak{F}_0(\mathfrak{X})$ its extension has the same range.

Consider $\mathfrak{X}$ as embedded in $\mathfrak{X}_c$ and let $w \in \mathfrak{X}_c$, $w \notin \mathfrak{X}$. We show, by example, that it is possible to have $U \in \mathfrak{C}(\mathfrak{X})$, where U is the limit in norm of a sequence in $\mathfrak{F}_0(\mathfrak{X})$ and where U as an element of $\mathfrak{C}(\mathfrak{X}_c)$ has the property that $U(w) = w$.

To see that such an arrangement is possible, let $\mathfrak{X}_c = l_1$ and let $\mathfrak{X}$ be the set of all sequences in l_1 with only a finite number of non-zero co-ordinates. Let

$$w = \left(1, \frac{1}{2!}, \ldots, \frac{1}{n!}, \ldots\right)$$

and define the operator U on $\mathfrak{X}_c$ by the rule that, if $x = \{c_k\}$, then

$$U(x) = (c_1, c_1/2, c_2/3, \ldots, c_{n-1}/n, \ldots).$$

Then $U(\mathfrak{X}) \subset \mathfrak{X}$ so that $U \in \mathfrak{C}(\mathfrak{X})$ and $U(w) = w$. Moreover, if we define T_k by the rule

$$T_k(x) = (c_1, c_1/2, c_2/3, \ldots, c_{k-1}/k, 0, 0, 0 \ldots),$$

then we see that $||U - T_k|| \to 0$ with each $T_k \in \mathfrak{F}_0(\mathfrak{X})$.

Now let $\mathfrak{B}$ be the subalgebra of $\mathfrak{C}(\mathfrak{X})$ generated by $\mathfrak{F}_0(\mathfrak{X})$ and U. Clearly each element of $\mathfrak{B}$ has the form

$$\sum_{k=1}^{n} a_k U^k + T, \tag{2.9}$$

where each a_k is a scalar and $T \in \mathfrak{F}_0(\mathfrak{X})$.

Now let $M = \{T \in \mathfrak{B} \mid T(w) = 0\}$. We shall show that M is a maximal-closed modular left ideal and not a modular maximal left ideal of $\mathfrak{B}$.

Clearly $M \neq \mathfrak{B}$ and M is a closed left ideal in $\mathfrak{B}$. Let $V \in \mathfrak{B}$. Since $(VU - V)(w) = 0$, we see that U is a right identity for $\mathfrak{B}$ modulo M so that M is a modular left ideal. We show that M is not a maximal left ideal of $\mathfrak{B}$. First there exists a bounded linear functional x^* on $\mathfrak{X}_c$ such that $x^*(w) = 1$. Note that as x^* cannot vanish identically on $\mathfrak{X}$ there exists $y \in \mathfrak{X}$ with $x^*(y) = 1$. If we set $T_0(x) = x^*(x)y$, we then obtain an element $T_0 \in \mathfrak{F}_0(\mathfrak{X})$ with $T_0(w) = y \neq 0$. This shows that $M \not\supset \mathfrak{F}_0(\mathfrak{X})$. Consider now the left ideal $\mathfrak{L}$ of $\mathfrak{B}$ generated by M and T_0. We claim that $U \notin \mathfrak{L}$ so that M is not maximal. For if $U \in \mathfrak{L}$, we can write $U = VT_0 + T$, where $V \in \mathfrak{B}$ and $T \in M$. Then $w = V(y) + T(w)$. But $T(w) = 0$ and $V(y) \in \mathfrak{X}$ and $w \notin \mathfrak{X}$, which is impossible.

We must show that if $V \in \mathfrak{B}$, $V \notin M$, then the left ideal generated by V and M is dense in $\mathfrak{B}$. Consider $M \cap \mathfrak{F}_0(\mathfrak{X})$. We establish that this is a maximal left ideal of $\mathfrak{F}_0(\mathfrak{X})$. For let $T_1 \in \mathfrak{F}_0(\mathfrak{X})$, $T_1 \notin M \cap \mathfrak{F}_0(\mathfrak{X})$, and let T_2 be arbitrary in $\mathfrak{F}_0(\mathfrak{X})$. We have $T_1(w) = v \neq 0$, $v \in \mathfrak{X}$. Then there exists $T_3 \in \mathfrak{F}_0(\mathfrak{X})$ such that $T_3(v) = T_2(w)$ as we know that $T_2(w) \in \mathfrak{X}$. Then

$$T_3 T_1 - T_2 \in M \cap \mathfrak{F}_0(\mathfrak{X}).$$

Next as $V \notin M$, $V(w) \neq 0$. There exists $x^* \in \mathfrak{X}_c^*$ such that $x^*(V(w)) \neq 0$. Set $R(x) = x^*(x)y$, where $y \neq 0$ in $\mathfrak{X}$. We see that $RV \in \mathfrak{F}_0(\mathfrak{X})$, $RV(w) \neq 0$, and $RV \notin M \cap \mathfrak{F}_0(\mathfrak{X})$. Thus the left ideal generated by RV and $M \cap \mathfrak{F}_0(\mathfrak{X})$ contains all of $\mathfrak{F}_0(\mathfrak{X})$. Therefore, the left ideal generated by M and V is dense in $\mathfrak{B}$ since U is the limit of elements in $\mathfrak{F}_0(\mathfrak{X})$.

If all the modular maximal right ideals of R are closed, then the notions of maximal-closed and closed maximal modular right ideals are the same. The notions can coincide for R even if this is not so as the following theorem shows.

2.7. THEOREM. *Let R be a real commutative normed algebra. Then every proper closed modular ideal I is contained in a closed modular maximal ideal.*

Proof. Let j be an identity for R modulo I and let R_0 be the completion of R. As in the proof of Theorem 2.4, I is contained in a modular maximal ideal M of R_0 and M must be closed in R_0. Then there exists (**11**, p. 109), a non-trivial homomorphism γ of R_0 into the complex field with kernel M and $\gamma(j) = 1$. Then γ restricted to R is a non-trivial homomorphism of R into the complex field with kernel $M \cap R$. Then $M \cap R$ is a modular maximal ideal of R, closed in R and containing I.

In connection with Theorem 2.7 it should be pointed out that there R can be semi-simple with all its modular ideals dense. Let R be the set of all polynomials of the form

$$a = \sum_{k=1}^{n} a_k t^k,$$

where each a_k is real, made into a normed algebra by setting

$$||a|| = \sum_{k=1}^{n} |a_k|/k!\,.$$

The completion R_c of R is the Banach algebra of all power series

$$a = \sum_{k=1}^{\infty} a_k t^k$$

for which

$$||a|| = \sum_{k=1}^{\infty} |a_k|/k!$$

converges. As shown (**6**, p. 158), R_c is a radical algebra. Thus if R has a closed modular ideal $\neq R$, then by the proof of Theorem 2.7, R_c has a modular maximal ideal, which is impossible.

2.8. LEMMA. *The following statements concerning a topological ring R are equivalent:*

(a) *R is a Q_r-ring;*

(b) *R is a quasi-Q_r-ring and all the modular maximal right ideals of R are closed.*

Proof. In view of (**14**, Theorem 1.6), (b) follows from (a). Assume (b). Suppose $x \in R$ and x is r.q.r. in R_0. Then $\{xy - y \mid y \in R_0\} = R_0$ and $I = \{xy - y \mid y \in R\}$ is dense in R_0 and therefore dense in R. By (b), $I = R$ so that x is r.q.r. in R. There exists a neighbourhood U of zero in R_0 containing only r.q.r. elements of R_0. Then $U \cap R$ is a neighbourhood of zero in R all of whose elements must be r.q.r. in R. Therefore (**6**, Lemma 2) R is a Q_r-ring.

2.9. Theorem. *Let R be a quasi-Q-ring. Then the following statements concerning R are equivalent*:

(a) *R is a Q_r-ring*;

(b) *R is a Q_l-ring*;

(c) *the modular maximal right ideals of R are closed*;

(d) *the modular maximal left ideals of R are closed.*

Proof. By Lemma 2.8, (a) $\leftrightarrow$ (c) and (b) $\leftrightarrow$ (d). Assume (c). The proof of Lemma 2.8 shows that there exists a neighbourhood V of zero in R containing only elements q.r. in R_0 and r.q.r. in R. Let $x \in V$. Then we have $y \in R$, $z \in R_0$, such that $x \circ y = 0 = z \circ x$. But this implies that $z = y$ so that x is q.r. in R and (b) follows from (c).

An example of a quasi-Q-ring with none of these properties is the set of all polynomials with real coefficients defined on $[0, 1]$ with the sup norm.

2.10. *Example.* We give an example of a commutative semi-simple topological ring where all the modular maximal ideals are closed but which is not a Q-ring. Consider the ring R of all real-valued continuous functions on $[0, 1]$, where the neighbourhoods of zero are the sets of the form $U_1 \cap \ldots \cap U_n$, where $U_k = R$ or U_k is a maximal ideal of R. That we have a topological ring is shown in (**4**, pp. 11—12). Note that $\{0\}$ is not a neighbourhood of zero. This ring is not a Q-ring since otherwise an ideal $\neq \{0\}$ would contain only quasi-regular elements and so be in the radical of R. On the other hand, the maximal ideals are all closed (**4**, p. 12) in this topology.

We adopt the following notation. For a topological ring R let $\mathfrak{P}_r$ ($\mathfrak{P}_l$) be the intersection of the maximal-closed modular right (left) ideals of R. Let $\mathfrak{D}_r$ ($\mathfrak{D}_l$) be the intersection of the closed modular maximal right (left) ideals of R.

2.11. *Example.* We show that $\mathfrak{P}_r \neq \mathfrak{D}_r$ is possible. We exhibit a commutative semi-simple topological algebra E with identity, where $\{0\}$ is the sole closed ideal (and so there are no closed maximal ideals). Let E be the set of all real polynomials in x made into a metric space by the metric (2.1) of L^ω. Let $f \in E$, $f \not\equiv 0$. It suffices to show that there exists a sequence $\{p_n\}$ in E where $fp_n \to 1$ in the metric of L^ω.

Let $x_1, \ldots, x_r$ be the distinct zeros of $f(x)$ in $[0, 1]$ in increasing order. (The case of no zeros follows by the reasoning below; we assume $r \geqslant 1$.) Choose an integer N so large that for all $n \geqslant N$ we have

(a) $f(x)$ has no zeros in $[-n^{-1}, 0)$ and $(1, 1+n^{-1}]$;

(b) $2n^{-1} < \max|x_{i+1} - x_i|$, $i = 1, \ldots, r-1$;

(c) $f(x)$ is monotonic in each of the intervals $[x_j - n^{-1}, x_j]$, $[x_j, x_j + n^{-1}]$, $j = 1, \ldots, r$.

Fix $n \geqslant N$. Let U be the union of the $2r$ intervals of (c) and let V be the complement of U in $[0, 1]$. The union of U and $[0, 1]$ is either $[0, 1]$ or a slightly larger closed interval. On it define the *continuous* function g_n by the rules: (1) $g_n = 1/f$ on V, (2) on an interval of the form $[x_j - n^{-1}, x_j]$ set

$$g_n(x) = f(x)/[f(x_j - n^{-1})]^2,$$

and (3) on an interval of the form $[x_j, x_j + n^{-1}]$ set

$$g_n(x) = f(x)/[f(x_j + n^{-1})]^2.$$

(Note that $f(x_j) = 0$, which makes these requirements consistent and g_n continuous.)

Then $fg_n - 1 = 0$ on V. On an interval of the form $[x_j - n^{-1}, x_j]$, we have

$$|1 - fg_n(x)| = |1 - \{f(x)/f(x_j - n^{-1})\}^2| \leqslant 1$$

in view of (c). Likewise $|1 - fg_n(x)| \leqslant 1$ on intervals of the form $[x_j, x_j + n^{-1}]$. Then, by a simple computation,

$$||1 - fg_n||_p \leqslant (2r/n)^{1/p}.$$

Therefore $fg_n \to 1$ in the metric of L^ω.

We can extract a subsequence $\{h_n\}$ from $\{g_n\}$ such that $(1, fh_n) < (2n)^{-1}$. For each n there exists, by the Weierstrass approximation theorem, a sequence $\{p_k^{(n)}\}$ of polynomials converging to h_n uniformly on $[0, 1]$. Then $p_k^{(n)} \to h_n$ and $fp_k^{(n)} \to fh_n$ in the metric of L^ω. Then we can find a polynomial q_n where $(fq_n, fh_n) < (2n)^{-1}$ or $(1, fq_n) < n^{-1}$.

2.12. LEMMA. *In a quasi-Q_r-ring R every element of $\mathfrak{P}_r$ is* r.q.r. *in R_0.*

Proof. Let $x \in \mathfrak{P}_r$. If x is not r.q.r. in R_0, then $I = \{xy - y \mid y \in R\}$ is not dense in R_0 and so not dense in R. Theorem 2.4 shows that there exists a maximal-closed modular right ideal M of R such that $I \subset M$. Since x is a left identity for R modulo M, $x \notin M$. This is a contradiction.

2.13. LEMMA. *Let R be a quasi-Q_r-ring, $x \in R$, and suppose that each element of the right ideal generated by x is* r.q.r. *in R_0. Then $x \in \mathfrak{D}_r$.*

Proof. Suppose that x fails to lie in the closed modular maximal right ideal M of R. Let j be a left identity for R modulo M. There exists an integer n, $y \in R$, and $z \in M$ such that $j = nx + xy + z$. Then $j - z = nx + xy$ is r.q.r. in R_0. Take $w \in R_0$ such that $(j - z) \circ w = 0$. Then $j = z - zw + jw - w$.

This shows that j lies in the closure of M in R_0. But then j lies in the closure of M in R, which is impossible.

2.14. LEMMA. *In any normed algebra B an element which is* l.q.r. (r.q.r.) *and the limit of* q.r. *elements is also* r.q.r. (l.q.r.).

Proof. Consider $x \in B$ where x is l.q.r., $y \circ x = 0$, and x is the limit of q.r. elements. If we show that x is r.q.r. in B_c, the completion of B, then $x \circ z = 0$, for some $z \in B_c$, and $y = z$ and x is q.r. Suppose x is not r.q.r. in B_c. Then by **(11**, p. 24**)** there exists a sequence $\{u_n\}$ in B_c bounded away from zero such that $(1 - x)u_n \to 0$. Then such a sequence $\{u_n\}$ clearly exists in B. This shows **(11**, p. 23**)** that x is not l.q.r in B, which is impossible.

2.15. DEFINITION. *We call a quasi-Q-ring R a strongly quasi-Q-ring if the ring R_0 of Definition* 2.2. *has the property of Lemma* 2.14.

Clearly any normed algebra has this property. We have no example at hand of a quasi-Q-ring without this property.

2.16. THEOREM. *Let R be a strongly quasi-Q-ring. Then*

(1) *if $\mathfrak{D}_r = \mathfrak{P}_r$ and $\mathfrak{D}_l = \mathfrak{P}_l$, all four sets are identical;*

(2) *if $\mathfrak{D}_r = \{0\}$, then $\mathfrak{P}_r = \mathfrak{P}_l = \{0\}$.*

Proof. Consider first a right ideal I of R all of whose elements are r.q.r. in R_0. Let $x \in I$, $y \in R_0$ with $x \circ y = 0$. Then $y = xy - x$ is l.q.r. in R_0 and is the limit of elements r.q.r. in R_0. There exists a neighbourhood U of y in R_0 containing only l.q.r. elements, for a Q-ring is also a Q_l-ring **(6**, p. 155**)**. Thus y is the limit of q.r. elements and so is q.r. in R_0. Then so is x.

Consider the collection $\mathfrak{W}_r$ of all $x \in R$ such that the right ideal in R generated by x contains only elements r.q.r. in R_0. Likewise we define $\mathfrak{W}_l$. We show that $\mathfrak{W}_r = \mathfrak{W}_l$. Let $x \in \mathfrak{W}_r$, m an integer and $w \in R$. To see that $x \in \mathfrak{W}_l$ we must show that $(m + w)x$ is l.q.r. As shown above, $x(m + w)$ is q.r. in R_0; let v be its quasi-inverse there. By a straightforward computation

$$[(m + w)vx - (m + w)x] \circ (m + w)x = (m + w)[v \circ x(m + w)]x = 0.$$

Thus, $x \in \mathfrak{W}_l$. Similarly $\mathfrak{W}_l \subset \mathfrak{W}_r$.

From Lemma 2.12, $\mathfrak{P}_r \subset \mathfrak{W}_r$, $\mathfrak{P}_l \subset \mathfrak{W}_l$. By Lemma 2.13, $\mathfrak{W}_r \subset \mathfrak{D}_r$ and $\mathfrak{W}_l \subset \mathfrak{D}_l$. Then (1) and (2) follow immediately from these relations and $\mathfrak{W}_r = \mathfrak{W}_l$.

In particular, if every maximal-closed right (left) ideal of R is a maximal right (left) ideal of R, then $\mathfrak{P}_r = \mathfrak{P}_l = \mathfrak{D}_r = \mathfrak{D}_l$.

By a *topologically simple ring* we mean one in which $\{0\}$ is the only proper closed two-sided ideal **(11**, p. 101**)**.

2.17. COROLLARY. *Let R be a topologically simple strongly quasi-Q-ring. Then either $\mathfrak{D}_r = \mathfrak{D}_l = R$ or $\mathfrak{P}_r = \mathfrak{P}_l = \{0\}$.*

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Proof. By the proof of Theorem 2.16, xy and yx lie in $\mathfrak{W}_r = \mathfrak{W}_l$ for all $x \in \mathfrak{W}_r$ and $y \in R$. Thus the collection $\mathfrak{Z}$ of finite sums of elements in $\mathfrak{W}_r$ is a two-sided ideal. If $\mathfrak{Z}$ is dense, then, as $\mathfrak{Z} \subset \mathfrak{D}_r$, $\mathfrak{Z} \subset \mathfrak{D}_l$, we have $\mathfrak{D}_r = \mathfrak{D}_l = R$. If $\mathfrak{Z} = \{0\}$, then $\mathfrak{P}_r = \mathfrak{P}_l = \{0\}$.

3. On rings with minimal ideals. *Throughout this section we let A be a ring with no nilpotent one-sided ideals* $\neq\{0\}$. Such a ring is sometimes called semi-prime. For a subset B of A let $\mathbf{L}(B) = \{x \in A \mid xB = (0)\}$ and $\mathbf{R}(B) = \{x \in A \mid Bx = (0)\}$. We call A a *left* (*right*) *modular annihilator* ring if $\mathbf{L}(M) \neq (0)$ ($\mathbf{R}(M) \neq (0)$) for every modular maximal right (left) ideal M of A. For an idempotent e of A, eA is a minimal right ideal if and only if Ae is a minimal left ideal (**5**, p. 65) and every minimal right (left) ideal is of the form eA (Ae) (**5**, p. 57). Such an idempotent we call a *minimal idempotent* of A. The algebraic sum of the minimal right ideals of A is the same as the algebraic sum of the minimal left ideals (**5**, p. 65). This set, which we denote by S, is called the *socle* of A (we say $S = \{0\}$ if A has no minimal one-sided ideals). For any two-sided ideal I of A, the reasoning of (**3**, Theorem 7) shows that $\mathbf{L}(I) = \mathbf{R}(I)$. In particular $\mathbf{L}(S) = \mathbf{R}(S)$. This set we denote by S^a and call the *anti-socle* of A. We use J to denote the radical of A.

3.1. LEMMA. *A right* (*left*) *ideal $I \neq \{0\}$ in A contains no minimal right* (*left*) *ideal of A if and only if $I \subset S^a$.*

Proof. This is shown in (**18**, Lemma 4) as the hypothesis of semi-simplicity given there can be replaced by that of no nilpotent one-sided ideals $\neq\{0\}$.

Let e be a minimal idempotent of A. The Peirce decomposition

$$A = eA \oplus (1 - e)A$$

and the minimality of eA show that $(1 - e)A$ is a modular maximal right ideal of A.

3.2. LEMMA. *Let A be a topological ring, M a maximal-closed modular right ideal. The following are equivalent:*

(1) *$M \not\supset S$ and $\mathbf{L}(M) \neq \{0\}$;*

(2) *$\mathbf{L}(M)$ is a minimal left ideal of A;*

(3) *$M = (1 - e)A$ for a minimal idempotent e.*

If $S^a = \{0\}$, then (2) *and* (3) *are equivalent to* (1′) $\mathbf{L}(M) \neq \{0\}$.

Proof. Suppose (1). We show first that $\mathbf{L}(M)$ contains a minimal left ideal of A. For assume the contrary. By Lemma 3.1, $\mathbf{L}(M) \subset S^a$. This gives $\mathbf{RL}(M) \supset \mathbf{RL}(S) \supset S$. Since $\mathbf{RL}(M)$ is a closed modular right ideal containing M, either $\mathbf{RL}(M) = M$ or $\mathbf{RL}(M) = A$. But $\mathbf{LRL}(B) = \mathbf{L}(B)$ for any subset B of A. So if $\mathbf{RL}(M) = A$, then $\mathbf{L}(M) = \{0\}$, which is impossible. Thus $M = \mathbf{RL}(M) \supset S$, contrary to (1). Therefore there exists a minimal idempotent e such that $\mathbf{L}(M) \supset Ae$ and $M = \mathbf{RL}(M) \subset (1-e)A$.

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Clearly $(1-e)A = \mathbf{R}(Ae)$ is a closed modular right ideal so $M = (1-e)A$ or (1) implies (3).

Suppose (3). Then $\mathbf{L}(M) = Ae$, so (3) implies (2). Suppose (2). If $M \supset S$, then $\mathbf{L}(M) \subset M$ and $[\mathbf{L}(M)]^2 = \{0\}$. This makes $\mathbf{L}(M) = \{0\}$ so that (2) implies (1).

Consider the case where $S^a = \{0\}$ and suppose that $\mathbf{L}(M) \neq \{0\}$. If $M \supset S$, then $\mathbf{L}(M) \subset S^a$, which is impossible. Thus (1′) implies (1) here.

In the case of the discrete topology matters are somewhat neater.

3.3. LEMMA. *Let M be a modular maximal right ideal of A. The following statements are equivalent:*

(1) $M \not\supset S$;

(2) $\mathbf{L}(M)$ *is a minimal left ideal of* A;

(3) $\mathbf{L}(M) \neq \{0\}$;

(4) $M = (1-e)A$, *where e is a minimal idempotent of* A.

Proof. In view of Lemma 3.2, it is sufficient to show that (1) implies (4) and that (3) implies (2).

Suppose (1). Then there exists a minimal right ideal eA, $e^2 = e$, where $M \not\supset eA$. Then $eA \cap M = \{0\}$ and $A = M \oplus eA$. Consider a left identity j for A modulo M. We can write $j = u + v$, where $u \in M$ and $v \in eA$. Since $(u+v)x - x \in M$ for all $x \in A$, then $(1-v)A \subset M$. Clearly $vx = x$ for all $x \in eA$ so that $eA = vA$ and v is a minimal idempotent. By the Peirce decomposition, $A = (1-v)A \oplus vA = M \oplus vA$. As $(1-v)A \subset M$ we see that $M = (1-v)A$.

Suppose (3) and let π be the natural homomorphism of A onto A/J. Then $\pi(M)$ is a modular maximal right ideal of the semi-simple ring A/J. Now $\mathbf{L}(M) \cap J \subset \mathbf{L}(J) \cap J = \{0\}$ so that π is one-to-one on $\mathbf{L}(M)$. This makes $\pi[\mathbf{L}(M)]$ a non-zero left ideal lying in the left ideal $\mathbf{L}[\pi(M)]$ of A/J. Thus $\mathbf{L}[\pi(M)] \neq \{0\}$. Since A/J is semi-simple, it follows (**16**, p. 96) that $\mathbf{L}[\pi(M)]$ is a minimal left ideal of A/J. Therefore $\pi[\mathbf{L}(M)] = \mathbf{L}[\pi(M)]$. If $\mathbf{L}(M)$ contains a left ideal I of A, $I \neq \mathbf{L}(M)$, then $\pi(I)$ is a left ideal of A/J, $\pi(I) \neq \mathbf{L}[\pi(M)]$. Then $\pi(I) = \{0\}$ and also $I = \{0\}$, so $\mathbf{L}(M)$ is a minimal left ideal.

Clearly the analogous result to Lemma 3.3 for modular maximal left ideals is also valid. Also it is not difficult to verify that S^a is the intersection of all modular maximal right (left) ideals M of the form

$$M = (1-e)A \qquad (M = A\,(1-e))$$

so that $J \subset S^a$. Hence, in particular, A is semi-simple if $S^a = \{0\}$.

3.4. THEOREM. *The following statements are equivalent:*

(1) A *is a left modular annihilator ring*;

(2) A *is a right modular annihilator ring*;

(3) A/S *is a radical ring.*

Proof. It is clear that A/S is a radical ring if and only if no modular maximal right (left) ideal of A contains S. By Lemma 3.3. the latter is true if and only if A is a left (right) modular annihilator ring.

In view of this theorem we call a left or right modular annihilator ring (if there are no nilpotent one-sided ideals $\neq\{0\}$) simply a *modular annihilator ring.*

For a modular annihilator ring, $S^a = J$. It is easy to give examples where $S^a = \{0\}$ (so also A is semi-simple) and yet A is not a modular annihilator ring. Let A be the Banach algebra of all bounded linear operators on an infinite-dimensional Banach space. Here S is the set of all finite-dimensional operators and $S^a = \{0\}$. Since A/S contains an identity, A is not a modular annihilator ring by Theorem 3.4.

We call **(3; 16)** a topological ring R an *annihilator ring* if $\mathbf{L}(R) = \mathbf{R}(R) = \{0\}$ and $\mathbf{L}(I) \neq \{0\}$ ($\mathbf{R}(I) \neq \{0\}$) for every proper closed right (left) ideal I in R. We use the notation $\mathfrak{D}_r$ ($\mathfrak{D}_l$) of § 2.

For an annihilator Banach algebra with no nilpotent one-sided ideals $\neq\{0\}$ the socle S is dense. This is not true for topological algebras. The topological algebra L^ω of Example 2.5 is an annihilator algebra with $S = \{0\}$, which, moreover, is not a modular annihilator algebra.

3.5. Theorem. *Let A be an annihilator ring. Then $\mathfrak{D}_l = \mathfrak{D}_r$. If also S is dense in A, then every maximal-closed modular right (left) ideal in A is a maximal right (left) ideal.*

Proof. Let M be a closed modular maximal right ideal. Since $\mathbf{L}(M) \neq \{0\}$, then, by Lemma 3.3, $M = (1 - e)A$ for a minimal idempotent e. Conversely any such M is a closed modular maximal right ideal. For such $M = (1 - e)A$, $M = \mathbf{R}(Ae) \supset S^a$ so that $\mathfrak{D}_r \supset S^a$. On the other hand, if $x \in \mathfrak{D}_r$, then $x \in \mathbf{R}(Ae)$ for every minimal idempotent e so that $x \in S^a$. Therefore, $\mathfrak{D}_r = \mathfrak{D}_l = S^a$.

Suppose that S is dense in A and let N be a maximal-closed modular right ideal. Since $S^a = \{0\}$, Lemma 3.2 shows that N is a maximal right ideal.

We turn to some purely algebraic developments.

3.6. Lemma. *Any two-sided ideal $\mathfrak{J}$ of A has no nilpotent one-sided ideals $\neq\{0\}$.*

Proof. Let $\mathfrak{R}$ be a right ideal of $\mathfrak{J}$, $\mathfrak{R} \neq \{0\}$. We show that $\mathfrak{R}\mathfrak{J} \neq \{0\}$. For suppose $\mathfrak{R}\mathfrak{J} = \{0\}$. Then as $\mathfrak{R}A\mathfrak{R}A \subset \mathfrak{R}\mathfrak{J}$, we see that $\mathfrak{R}A = \{0\}$ is a nilpotent right ideal of A. This makes $\mathfrak{R} \subset \mathbf{L}(A)$. But $[\mathbf{L}(A)]^2 = \{0\}$ so that $\mathfrak{R} = \{0\}$.

If $\mathfrak{R}^n = \{0\}$ for some positive integer $n > 1$ then $(\mathfrak{R}\mathfrak{J})^n = \{0\}$ with $\mathfrak{R}\mathfrak{J}$ a right ideal of A. This forces $\mathfrak{R}\mathfrak{J} = \{0\}$, contrary to the above.

Thus the above theory pertains to $\mathfrak{J}$ as well as to A. We shall see that the connections are intimate.

3.7. THEOREM. *Let A be a modular annihilator ring and I a two-sided ideal of A. Then I is also a modular annihilator ring.*

This is an immediate consequence of the following lemma.

3.8. LEMMA. *Let I be a two-sided ideal of A and M a modular maximal right ideal of I. If $\mathbf{L}(M) \cap I = \{0\}$, then M is contained in a modular maximal right ideal N of A with $\mathbf{L}(N) = \{0\}$.*

Proof. We show first that M is also a right ideal of A. Suppose otherwise and let j be a left identity for I modulo M. There exists $x \in A$, $v \in M$, such that $vx \notin M$. Note that $vx \in I$. As M is maximal, there exists $w \in I$, $z \in M$, and an integer k such that

$$j = z + (vx)w + kvx.$$

Then

$$j^2 = zj + v(xwj + kxj).$$

We see that $j^2 \in M$ inasmuch as $xwj + kxj \in I$. Since $j^2 - j \in M$, we see that $j \in M$, which is impossible.

We are given that $\mathbf{L}(M) \cap I = \{0\} = I\,\mathbf{L}(M)$. From this we show that $\mathbf{L}(M) = \mathbf{R}(M) = \mathbf{L}(I) = \mathbf{R}(I)$. To see this we repeat arguments from **(17,** Lemma 2.7**)**. Specifically we have $\mathbf{L}(M) \subset \mathbf{R}(I) = \mathbf{L}(I)$ and, as $M \subset I$, $\mathbf{L}(I) \subset \mathbf{L}(M)$, so that $\mathbf{L}(M) = \mathbf{L}(I)$. Also $\mathbf{R}(M)M$ is, by the above, a right ideal of A and is nilpotent. Thus $(\mathbf{R}M)M = \{0\}$ so that $\mathbf{R}(M) \subset \mathbf{L}(M) = \mathbf{R}(I)$ and, as $M \subset I$, $\mathbf{R}(I) \subset \mathbf{R}(M)$.

From this we see that, if we take $x \in \mathbf{L}(M + \mathbf{R}(M))$, $x \in \mathbf{L}(M) = \mathbf{R}(M)$ and $x \in \mathbf{LR}(M)$ and $x^2 = 0$. Thus $\mathbf{L}(M + \mathbf{R}(M)) = \{0\}$.

Next set $\beta(M) = \{w \in A \mid wy \in M \text{ for all } y \in I\}$. Clearly $\beta(M) \supset M$ and is a right ideal of A. Let $x \in A$, $y \in I$, and j be a left identity for I modulo M. Then $(jx - x)y = j(xy) - (xy) \in M$ as $xy \in I$. Hence j is also a left identity for A modulo $\beta(M)$. We claim $j \notin \beta(M)$. For otherwise $j^2 \in M$, which implies that $j \in M$. This is impossible. It follows that $\beta(M)$ is contained in a modular maximal right ideal N of A. But $\mathbf{L}(M)I = \{0\}$ so that $\mathbf{L}(M) \subset \beta(M)$. This gives us $\mathbf{R}(M) \subset \beta(M) \subset N$ as well as $M \subset N$. Thus

$$\mathbf{L}(N) \subset \mathbf{L}(M + \mathbf{R}(M)) = \{0\}.$$

3.9. PROPOSITION. *A semi-simple modular annihilator ring A is the sub-direct sum of primitive modular annihilator rings.*

Proof. Let P be a primitive ideal of A. From standard ring theory **(5)** it is sufficient to show that A/P is a modular annihilator ring. Let N be a modular maximal right ideal of A/P and let π be the natural homomorphism of A onto A/P. We must show that $\mathbf{L}(N) \neq \{0\}$ in A/P. Suppose $\mathbf{L}(N) = \{0\}$.

Note that $M = \pi^{-1}(N)$ is a modular maximal right ideal of A and that $\mathbf{L}(M) \subset P$. Then $M = \mathbf{RL}(M) \supset \mathbf{R}(P)$ and $M \supset P$ so that $M \supset P + \mathbf{R}(P)$. Then $\mathbf{L}(P + \mathbf{R}(P)) \supset \mathbf{L}(M) \neq \{0\}$. But $\mathbf{L}(P + \mathbf{R}(P))$ is readily seen to be a nilpotent ideal which makes $\mathbf{L}(P + \mathbf{R}(P)) = \{0\}$, which is impossible.

3.10. LEMMA. *Let I be a two-sided ideal of A and let S_0 (T_0) be the socle (anti-socle) of I. Then $S_0 = S \cap I = SI = IS$ and $T_0 = S^a \cap I$.*

Proof. Let eA, $e^2 = e$, be a minimal right ideal of A. We show first that either eA is a minimal right ideal of I or $eA \subset \mathbf{L}(I) = \mathbf{R}(I)$. We have $eA \cap I = \{0\}$ or $eA \cap I = eA$. If $eA \cap I = eA$, then $e \in I$ and $eI = eA$. Then $eIe = eAe$ so that eIe is a division ring **(6**, p. 65**)**. This, by Lemma 3.6. and **(6**, p. 65**)**, makes eI a minimal right ideal of I. If $eA \cap I = \{0\}$, then $eI = \{0\}$ and $e \in \mathbf{L}(I) = \mathbf{R}(I)$.

It is clear that a minimal right ideal of I, being of the form eI, is also a right ideal of A and so a minimal right ideal of A. Thus $S_0 \subset SI \subset S \cap I$. Let $y = e_1x_1 + \ldots + e_nx_n$ be an arbitrary element of $S \cap I$ where each e_k is a minimal idempotent of A, $x_k \in A$ and $e_kx_k \neq 0$. As seen above, $e_kA \subset I$ or $e_kA \subset \mathbf{R}(I)$. As $I \cap \mathbf{R}(I) = \{0\}$, both cannot happen. We can write $y = u + v$, where u is the sum of the e_kx_k contained in I, v the sum of those in $\mathbf{R}(I)$. Since $y - u \in I$, we see that $v = 0$. Thus we may suppose that each $e_kA \subset I$ so that $y \in S_0$. Therefore, $S_0 = SI = S \cap I$.

Since $S \supset S_0$, it is clear that $T_0 \supset S^a \cap I$. Let $x \in T_0$ and let N be a minimal right ideal of A. If N is a minimal right ideal of I, then surely $xN = \{0\}$, whereas otherwise $N \subset \mathbf{R}(I)$ and again $xN = \{0\}$. Therefore $x \in S^a \cap I$.

We return to the theory of topological rings.

3.11. LEMMA. *Let A be a topological ring. The following are equivalent:*

(1) *S is dense in A;*

(2) *A is the topological direct sum of its minimal closed two-sided ideals and $S^a = \{0\}$.*

Also if every modular maximal right ideal of A is closed and (1) *holds, then A is a modular annihilator ring.*

Proof. By the topological direct sum is meant the closure of the algebraic direct sum **(11**, p. 46**)**. The arguments of **(3**, Theorem 5**)** show that each minimal right ideal of A is contained in a minimal closed two-sided ideal of A. Then S is contained in the direct sum of these ideals and (1) implies (2). Suppose (2) and let P be a minimal closed two-sided ideal of A. Clearly $PS \neq \{0\}$. Then surely $P = \overline{SP} \subset \bar{S}$. It follows that $A = \bar{S}$.

Example 2.6 can be shown to have S dense but possessing a modular maximal left ideal with zero right annihilator. If the modular maximal right ideal M of A is closed and S is dense, then, of course, $M \not\supset S$ so that, by Lemma 3.3, $\mathbf{L}(M) \neq \{0\}$.

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As a consequence we derive the following structure theorem.

3.12. THEOREM. *Let A be a modular annihilator ring which is a topological ring with dense socle S. Then A is the topological direct sum of topologically simple modular annihilator rings with dense socle.*

Proof. By Lemma 3.11, A is the topological sum of its minimal closed two-sided ideals. Consider such an ideal I. By Theorem 3.7, I is a modular annihilator ring. From Lemma 3.10 we see that SI is the socle of I. But $SI \neq \{0\}$ so that $I = \overline{SI}$. That I is topologically simple follows from Lemma 3.6 and **(5**, p. 65**)**.

4. Applications.

We apply these results to the theory of normed algebras. For the notions used see **(11)**.

4.1. THEOREM. *A B^*-algebra B which is a modular annihilator algebra is a dual algebra.*

Proof. This is a refinement of the result **(3**, p. 157**)** that any B^*-algebra which is an annihilator algebra is a dual algebra. First B is semi-simple **(11**, p. 244**)**. Consider its socle S. By Theorem 3.4, B/S is a radical algebra and so is $B/\bar{S}$. But **(11**, p. 249**)** $B/\bar{S}$ is a B^*-algebra and so is semi-simple. Therefore $B = \bar{S}$. A theorem of Kaplansky **(9**, Theorem 2.1**)** now asserts that A is a dual algebra.

4.2. THEOREM. *Let A be a semi-simple complex Banach algebra with dense socle and with an involution $x \to x^*$, where $xx^* = 0$ implies $x = 0$. Then A has a faithful $*$-representation by completely continuous operators on a Hilbert space.*

Proof. This is an improvement on part of **(9**, Theorem 2.1**)** which asserts that any B^*-algebra with dense socle has such a $*$-representation.

Since $S^a = \{0\}$, **(11**, Lemma 4.10.4**)** and **(18**, Theorem 5.2**)** show that A has a faithful $*$-representation $x \to \gamma(x)$ as bounded linear operators on a Hilbert space H. The closure B of $\gamma(A)$ in the Banach algebra of all bounded linear operators on H is a B^*-algebra. Since γ is continuous **(11**, p. 188**)** we see that $\gamma(S)$ is dense in B. Let eA, $e^2 = e$, be a minimal right ideal of A. Since $\gamma(e)\gamma(A)\gamma(e)$ is, by the Gelfand-Mazur theorem, just the set of scalar multiples of $\gamma(e)$, so also is $\gamma(e)B\gamma(e)$. The semi-simplicity of B **(11**, p. 244 and **5**, p. 65**)** implies that $\gamma(e)B$ is a minimal right ideal of B. Therefore $\gamma(S)$ lies in the socle of B. Then B is a B^*-algebra with dense socle so that **(9**, Theorem 2.1**)** it and a fortiori A has the desired $*$-representation.

4.3. *Example.* We describe a primitive modular annihilator Banach algebra which is not an annihilator algebra. Let $\mathfrak{X}$ be a non-reflexive Banach space and let $\mathfrak{F}(\mathfrak{X})$ be the closure, in the uniform norm of the set $\mathfrak{F}_0(\mathfrak{X})$ of all bounded linear operators on $\mathfrak{X}$ with finite-dimensional range. By the work of Arnold **(2)**, $\mathfrak{F}_0(\mathfrak{X})$ is the socle of $\mathfrak{F}(\mathfrak{X})$. It is clear that $\mathfrak{F}(\mathfrak{X})$ is primitive and, by Lemma 3.11,

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is a modular annihilator algebra, whereas, by (**3**, Theorem 13), $\mathfrak{F}(\mathfrak{X})$ is not an annihilator algebra as its minimal left ideals are isomorphic to $\mathfrak{X}$; cf. (**2**). This example was used by Smiley (**15**) as an instance of a Banach algebra with the annihilator property for proper closed left ideals but not for closed right ideals. (Our conventions on operator multiplication are the opposite of those of Smiley so that our right (left) ideals are his left (right) ideals.)

The new information contained in the fact that $\mathfrak{F}(\mathfrak{X})$ is a modular annihilator algebra enables us to prove the following theorem.

4.4. THEOREM. *The modular maximal right ideals of* $\mathfrak{F} = \mathfrak{F}(\mathfrak{X})$ *are the ideals of the form* $\{T \in \mathfrak{F} \mid T(x_0) = 0\}$ *where* $x_0 \neq 0$ *in* $\mathfrak{X}$. *The modular maximal right ideals of* $\mathfrak{F}$ *are the ideals consisting of all* $T \in \mathfrak{F}$ *whose ranges lie in a subspace of* $\mathfrak{X}$ *of deficiency one.*

Proof. It is clear that the ideals of the form in question are modular maximal left and right ideals respectively. Let M be a modular maximal left ideal. Since $\mathfrak{F}$ is a modular annihilator ring, by Lemma 3.3, there exists a minimal idempotent E such that $M = \mathfrak{F}(1 - E)$. But by (**2**) E is of the form $E(x) = x^*(x)y$ where $x^* \in X^*$, $y \in X$, and $x^*(y) = 1$. Since $(U - UE)(y) = 0$ for all $U \in \mathfrak{F}$, we see that $M \subset \{V \in \mathfrak{F} \mid V(y) = 0\}$. Since M is maximal, $M = \{V \in \mathfrak{F} \mid V(y) = 0\}$.

Now let N be a modular maximal right ideal. We can write $N = (1 - E)\mathfrak{F}$ using the above notation. Since $x^* \{U(x) - x^*[U(x)]y\} = 0$, it follows that N is the set of all V whose ranges lie in the null space of x^*.

Let B be a Banach algebra which is also a Hilbert space. Saworotnow (**12**) calls B a *right-complemented algebra* (r.c. *algebra*) if the orthogonal complement $I^\perp = \{x \in B \mid (x, I) = (0)\}$ of every right ideal I is again a right ideal. There are important examples in analysis of (incomplete) normed algebras which are pre-Hilbert spaces with this property satisfied by the right ideals. A case in point is the algebra B of all continuous complex-valued functions on a compact group G made into a pre-Hilbert space by taking as the inner product

$$(f, g) = \int_G f(t)\overline{g(t)}\, dt,$$

where the integration is with respect to Haar measure and the norm used is $|f| = (f, f)^{1/2}$. If the multiplication is taken as convolution,

$$f*g(s) = \int_G f(st^{-1})\, g(t)\, dt,$$

one obtains a non-commutative normed algebra in terms of $|f|$ which is, in general, not a Banach algebra.

Moreover, as the following result shows, the definition of Saworotnow is redundant in the semi-simple case since the defining property holds for all right ideals if it holds for all modular maximal right ideals.

4.5. THEOREM. *Let B be a semi-simple normed algebra which is a pre-Hilbert space where*

(1) *$M^{\perp}$ is a right ideal $\neq \{0\}$ for each modular maximal right ideal M,*

(2) *a right or left ideal I is dense if $I^{\perp} = \{0\}$.*

Then B is the topological direct sum of its minimal closed two-sided ideals and $I^{\perp}$ is a right (left) ideal for all right (left) ideals I.

Proof. Note that (2) is automatically true in the Hilbert space case. Let M be a modular maximal right ideal in B with j a left identity for B modulo M. Write $j = u + v$, where $u \in M$ and $v \in M^{\perp}$. Then $vx - x \in M$ for all $x \in B$. Consequently $vx = x$ for all $x \in M^{\perp}$ and $M^{\perp} = vB$, $v^2 = v$. By the Peirce decomposition, $B = (1 - v)B \oplus vB$. Since $(1 - v)B \subset M$, we see that $(1 - B)v = M$. Then $\mathbf{L}(M) = vB \neq \{0\}$. Theorem 3.4 shows that B is a modular annihilator algebra.

Consider the element v of the preceding paragraph. A computation of Saworotnow **(12**, p. 50**)** shows that $(vx, y) = (x, vy)$ for all $x, y \in B$. For let $x = x_1 + x_2$, $y = y_1 + y_2$, where $x_1, y_1 \in M^{\perp}$ and $x_2, y_2 \in M$. Then $vx_2 = vy_2 = 0$, $vx_1 = x_1$, $vy_1 = y_1$, so that $(vx, y) = (x_1, y) = (x_1, y_1) = (x, y_1) = (x, vy)$. Suppose now that $w \in M^{\perp\perp}$. Then $(w, vB) = \{0\}$ and thus $(vw, B) = \{0\}$. Therefore $M^{\perp\perp} \subset \mathbf{R}(Bv) = (1 - v)B = M$ and $M = M^{\perp\perp}$.

Let T be the algebraic sum of the $M^{\perp}$ as M ranges over the set of modular maximal right ideals of B. For each such M, $T^{\perp} \subset M^{\perp\perp} = M$. By semi-simplicity, $T^{\perp} = \{0\}$. Since $S \supset T$, where S is the socle of B, we see from (2) that S is dense. By Theorem 3.12, B is the topological direct sum of its minimal closed two-sided ideals. In the complete case this conclusion is shown in **(12**, Theorem 1**)**.

We say that w is a *left adjoint* for u if $(ux, y) = (x, wy)$ for all $x, y \in B$ and write $w = u'$. It is readily seen that u' is unique if it exists. Consider once again the element v treated above. As shown, $v' = v$. We assert that each element of vB has a left adjoint. This follows by aid of the arguments of **(13**, Theorem 1**)**. Inasmuch as it is awkward to see this by inspection we give the details. Let $a \in vB$, $a = va$. To see that a' exists, we may suppose that $av \neq 0$; otherwise consider $b = a + v$, where $bv \neq 0$. Then if b' exists, so does $a' = b' - v$. By the Gelfand–Mazur theorem there exists a scalar $\lambda \neq 0$ such that $av = vav = \lambda v$. Then $a^2 = vava = \lambda a$ and $f = \lambda^{-1}a$ is an idempotent. Also $vB = vaB = aB = fB$ so that $N = (1 - f)B$ is a modular maximal right ideal. Consider the decomposition $f = z + v_1$ where $z \in N$ and $v_1 \in N^{\perp}$. As before, we see that $v_1 B = N^{\perp}$, $v_1{}^2 = v_1$, $v_1{}' = v_1$, and $v_1 z = 0$. Then $0 \neq v_1 = v_1{}^2 = v_1(z + v_1) = v_1 f = v_1 vf$. Thus $vv_1 = (v_1 v)' \neq 0$ and $vv_1 = vv_1 f = vv_1 vf = \beta f$ for some scalar $\beta \neq 0$. Thus f' and hence a' exists.

It follows that u' exists for all $u \in T$, where T is dense in B. Let K be a left ideal in B, $x \in K$, $y \in K^{\perp}$, and $z \in T$. Then $0 = (z'x, y) = (x, zy)$ so that $TK^{\perp} \subset K^{\perp}$. Hence $K^{\perp}$ is a left ideal. In particular $P^{\perp}$ is a left ideal for each modular maximal left ideal P. But $\mathbf{R}(P) \neq \{0\}$ so that P is not dense.

Therefore $P^{\perp} \neq \{0\}$ by (2). By the interchange of right and left in the above reasoning we can now see that $I^{\perp}$ is a right ideal for any right ideal I.

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