AN IDENTITY IN COMBINATORIAL ANALYSIS

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In a very recent paper [1], Basil Gordon discusses generalizations of Jacobi's identity

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z)(1+x^{2n-1}z^{-1}) = \sum_{n=-\infty}^{\infty} x^{n^2}z^n, \tag{1}$$

where x and z are complex numbers and $|x| < 1$. He notes that some of its consequences, *inter alia* Euler's formula

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{1}{2}(3n^2+n)}, \tag{2}$$

are of interest in number theory and combinatory analysis. He proves the apparently new and striking result

$$\prod_{n=1}^{\infty}(1-s^n)(1-s^nt)(1-s^{n+1}t^{-1})(1-s^{2n-1}t^2)(1-s^{2n-1}t^{-2}) = \sum_{n=-\infty}^{\infty} s^{\frac{1}{2}(3n^2+n)}(t^{3n}-t^{-3n-1}), \tag{3}$$

where $|s| < 1$, and also considers the possibility of generalizations. His methods are algebraic and quite simple, but perhaps do not make obvious what underlies such formulae. It may be worth while to do so, especially since the details become simpler and the presentation more perspicuous. The method given here assumes no more knowledge than his does, although the new proof is expressed in terms of theta-functions, in simple properties of which, formulae such as (3) have their origin. Further, (3) appears in a slightly more symmetrical form.

Let ω be a complex number with positive imaginary part and write $q = e^{\pi i \omega}$. Let $f(z) = f_{g,h}(z, \omega)$ be an integral function of z such that

$$f(z+1) = (-1)^g f(z), \quad f(z+\omega) = (-1)^h e^{-n\pi i(2z+\omega)} f(z), \tag{4}$$

where g, h are given as either 0 or 1, and n is a given positive integer. Then $f(z)$ can be expanded in Fourier series of the forms $\sum_{-\infty}^{\infty} c_m e^{2m\pi i z}$, etc, convergent for all z. An application of (4) now shows that there are exactly n such integral functions, linearly independent, say $f_1(z), f_2(z), \dots, f_n(z)$, such that every function $f(z)$ of the above type can be expressed linearly in the form

$$f(z) = \sum_{r=1}^{n} A_r f_r(z), \tag{5}$$

where the coefficients A_r are independent of z. The real difficulty is to find simple and useful expressions for the A_r.

When $n = 1$, we have the four theta-functions [2]

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$$\theta_{00}(z,\omega) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2n\pi i z} = \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z}), \tag{6}$$

$$\theta_{01}(z,\omega) = \theta_{00}(z+\tfrac{1}{2},\omega), \tag{7}$$

$$\theta_{11}(z,\omega) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)\pi i z}$$

$$= -iq^{\frac{1}{4}}(e^{\pi i z}-e^{-\pi i z}) \prod_{n=1}^{\infty} (1-q^{2n})(1-q^{2n}e^{2\pi i z})(1-q^{2n}e^{-2\pi i z}), \tag{8}$$

$$\theta_{10}(z,\omega) = \theta_{11}(z+\tfrac{1}{2},\omega). \tag{9}$$

We show now that equation (3) is a particular case of the general result with $n = 3$, applied to

$$f(z) = \theta_{01}(z,\omega)\, \theta_{11}(2z, 2\omega), \tag{10}$$

for we see easily from (4) that

$$f(z+1) = f(z), \quad f(z+\omega) = e^{-3\pi i(2z+\omega)} f(z). \tag{11}$$

Put

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z}.$$

Then, from (11),

$$\sum_{n=-\infty}^{\infty} a_n q^{2n} e^{2n\pi i z} = q^{-3} \sum_{n=-\infty}^{\infty} a_n e^{2\pi i z(n-3)},$$

and so

$$a_{n+3} = q^{2n+3} a_n.$$

Arbitrary a_0, a_1, a_2 give three linearly independent solutions. We can take $a_n = cq^{n^2/3}$ then

$$f_1(z) = \sum_{n=-\infty}^{\infty} q^{3n^2+2n} e^{(6n+2)\pi i z}, \quad f_2(z) = \sum_{n=-\infty}^{\infty} q^{3n^2-2n} e^{(6n-2)\pi i z}, \tag{12}$$

$$f_3(z) = \sum_{n=-\infty}^{\infty} q^{3n^2} e^{6n\pi i z}. \tag{13}$$

Hence

$$f(z) = A_1 f_1(z) + A_2 f_2(z) + A_3 f_3(z), \tag{14}$$

where A_1, A_2, A_3 are independent of z. Since

$$f_1(-z) = \sum_{n=-\infty}^{\infty} q^{3n^2+2n} e^{-(6n+2)\pi i z} = \sum_{n=-\infty}^{\infty} q^{3n^2-2n} e^{(6n-2)\pi i z} = f_2(z),$$

and $f(z)$ is an odd function of z and $f_3(z)$ is an even function of z, we have $A_3 = 0$, $A_2 = -A_1$. Hence

$$f(z) = A_1\{f_1(z) - f_2(z)\}. \tag{15}$$

We now show that

$$A_1 = -iq^{\frac{1}{4}} \prod_{n=1}^{\infty} (1-q^{4n}).$$

Put $z = \frac{1}{2}(1-\omega)$. Then

$$f_1(z) = \sum_{n=-\infty}^{\infty} (-1)^{n+1} q^{3n^2-n-1} = \sum_{n=-\infty}^{\infty} (-1)^{n+1} q^{3n^2+n-1},$$

on replacing n by $-n$. Next

$$f_2(z) = \sum_{n=-\infty}^{\infty} (-1)^{n+1} q^{3n^2-5n+1} = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+n-1},$$

on replacing n by $n+1$. On noticing (2), we have, from (15),

$$\theta_{01}(\tfrac{1}{2}-\tfrac{1}{2}\omega, \omega)\ \theta_{11}(1-\omega, 2\omega) = -2A_1 q^{-1} \prod_{n=1}^{\infty} (1-q^{2n}). \tag{16}$$

From (7), the first factor is

$$\prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-2})(1+q^{2n}) = 2\prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n})^2.$$

The second factor in (16) is

$$\begin{aligned} & iq^{\frac{1}{4}}(q^{-1}-q) \prod_{n=1}^{\infty} (1-q^{4n})(1-q^{4n-2})(1-q^{4n+2}) \\ &= iq^{\frac{1}{4}}(q^{-1}-q)(1-q^2)^{-1} \prod_{n=1}^{\infty} (1-q^{4n})(1-q^{4n-2})(1-q^{4n-2}) \\ &= iq^{-\frac{3}{4}} \prod_{n=1}^{\infty} (1-q^{4n})(1-q^{4n-2})^2. \end{aligned}$$

Hence

$$2iq^{-\frac{3}{4}} \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n})^2(1-q^{4n})(1-q^{4n-2})^2 = -2A_1 q^{-1} \prod_{n=1}^{\infty} (1-q^{2n})$$

and

$$\begin{aligned} A_1 &= -iq^{\frac{1}{4}} \prod_{n=1}^{\infty} (1+q^{2n})^2(1-q^{2n})(1-q^{4n-2}) \\ &= -iq^{\frac{1}{4}} \prod_{n=1}^{\infty} (1+q^{2n})(1-q^{4n})(1-q^{4n-2}) \\ &= -iq^{\frac{1}{4}} \prod_{n=1}^{\infty} (1+q^{2n})(1-q^{2n}) \\ &= -iq^{\frac{1}{4}} \prod_{n=1}^{\infty} (1-q^{4n}). \end{aligned}$$

Hence

$$\theta_{01}(z, \omega)_{11}\ \theta(2z, 2\omega) = -iq^{\frac{1}{4}} \prod_{n=1}^{\infty} (1-q^{4n}) \left\{ \sum_{n=-\infty}^{\infty} q^{3n^2+2n} e^{(6n+2)\pi iz} - \sum_{n=-\infty}^{\infty} q^{3n^2-2n} e^{(6n-2)\pi iz} \right\}. \tag{17}$$

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This can be written as

$$(e^{2\pi iz}-e^{-2\pi iz})\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1}e^{2\pi iz})(1-q^{2n-1}e^{-2\pi iz})(1-q^{4n}e^{4\pi iz})(1-q^{4n}e^{-4\pi iz})$$

$$=\sum_{n=-\infty}^{\infty}q^{3n^2+2n}e^{(6n+2)\pi iz}-\sum_{n=-\infty}^{\infty}q^{3n^2-2n}e^{(6n-2)\pi iz}. \tag{18}$$

To get Gordon's result from this, put $q^2=s$ and $s^{-\frac{1}{2}}e^{2\pi iz}=t$. The left-hand side becomes

$$(ts^{\frac{1}{2}}-t^{-1}s^{-\frac{1}{2}})\prod_{n=1}^{\infty}(1-s^n)(1-s^nt)(1-s^{n-1}t^{-1})(1-s^{2n+1}t^2)(1-s^{2n-1}t^{-2}),$$

i.e.

$$-s^{-\frac{1}{2}}t^{-1}\prod_{n=1}^{\infty}(1-s^n)(1-s^nt)(1-s^{n-1}t^{-1})(1-s^{2n-1}t^2)(1-s^{2n-1}t^{-2}). \tag{19}$$

The right-hand side becomes

$$\sum_{n=-\infty}^{\infty}s^{\frac{1}{2}(3n^2+2n)}(ts^{\frac{1}{2}})^{3n+1}-\sum_{n=-\infty}^{\infty}s^{\frac{1}{2}(3n^2-2n)}(ts^{\frac{1}{2}})^{3n-1},$$

i.e.

$$\sum_{n=-\infty}^{\infty}s^{\frac{1}{2}(3n^2+5n+1)}t^{3n+1}-\sum_{n=-\infty}^{\infty}s^{\frac{1}{2}(3n^2+n-1)}t^{3n-1}. \tag{20}$$

In the first summation change n into $-n-1$. On multiplying (19) and (20) by $-s^{\frac{1}{2}}t$, we have the result (3).

REFERENCES

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