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EDGEWORTH EXPANSION ON n-SPHERES AND JACOBI HYPERGROUPS

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Suitable normalisation of time–homogeneous rotation–invariant random walks on unit spheres $S^d \subset \mathbb{R}^{d+1}$ for $d \geqslant 2$ leads to a central limit theorem with a Gaussian limit measure. This paper is devoted to an associated Edgeworth expansion with respect to the total variation norm. This strong type of convergence is different from the classical case. The proof is performed in the more general setting of Jacobi–type hypergroups on an interval.

1. INTRODUCTION

The aim of the present paper is to derive an Edgeworth expansion for isotropic random walks on unit spheres $S^d \subset \mathbb{R}^{d+1}$ for $d \geqslant 2$ and, in general, on Jacobi hypergroups.

Let us consider a stationary random walk $(X_n)_{n\geqslant 0}$ on S^d starting, say, at the North Pole $x_0 \in S^d$ and having some $SO(d+1)$–invariant transition probability. The transition can be described by some probability measure $\mu \in M^1([0,\pi])$ which is the distribution of the angles

$$\sphericalangle(X_n, X_{n-1}) \in [0,\pi], \qquad n \in \mathbb{N},$$

between two successive jumps. For all $k \in \mathbb{N}$ let $\left(X_n^{(k)}\right)_{n\geqslant 0}$ be a shrinked isotropic random walk starting at $x_0 \in S^d$ and associated with the probability measure $\mu_k \in M^1\left(\left[0,\pi/\sqrt{k}\right]\right)$ defined by $\mu_k(A) := \mu\left(\sqrt{k}A\right)$ for Borel sets $A \subset \left[0,\pi/\sqrt{k}\right]$. This random walk is discussed, for instance, by Bingham [**2**] and by Voit [**9**]. From Voit [**9**], under a mild restriction on μ ('μ must not be concentrated in 0 too much'), there exists a $k_0 = k_0(d,\mu)$ such that for each $k \geqslant k_0$ the distribution $\mu_k^{(k)}$ of $X_k^{(k)}$ has a continuous bounded density f_k with respect to the uniform distribution ω_d on S^d. Moreover, it was shown in that paper that there is a 'Gaussian' measure ν_0^μ on S^d with a continuous bounded ω_d–density g_0 such that

$$\|f_k - g_0\|_\infty = O(k^{-1}), \qquad \left\|\mu_k^{(k)} - \nu_0^\mu\right\| = O(k^{-1}) \qquad \text{for } k \to \infty,$$

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where $\|\varrho\|$ denotes the total variation norm of the measure ϱ, and the order $O(k^{-1})$ is sharp.

In this paper we give bounded (signed) measures ν_j^μ $(j = 1, 2, \dots)$ on S^d with continuous bounded ω_d-densities g_j $(j = 1, 2, \dots)$ such that for all $s \in \mathbb{N}$

$$\Big\| f_k - \sum_{j=0}^{s-1} k^{-j} \cdot g_j \Big\|_\infty = O(k^{-s}), \qquad \Big\| \mu_k^{(k)} - \sum_{j=0}^{s-1} k^{-j} \cdot \nu_j^\mu \Big\| = O(k^{-s}) \qquad \text{for } k \to \infty.$$

The expansion will be derived from a more general result valid for Jacobi hypergroups covering also the cases of compact symmetric spaces of rank one (which include the projective spaces; see, for instance, Helgason [**7**]).

2. Jacobi hypergroups and Gaussian measures

For $\alpha \geqslant \beta \geqslant -1/2$ we consider the Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) := \frac{1}{\binom{n+\alpha}{n} 2^n} \sum_{j=0}^{n} \binom{n+\alpha}{j} \binom{n+\beta}{n-j} (x-1)^{n-j} (x+1)^j, \qquad x \in \mathbb{R},\ n \in \mathbb{N}_0,$$

which are orthogonal on $[-1, 1]$ with respect to the weight $(1-x)^\alpha (1+x)^\beta$ and normalised by $P_n^{(\alpha,\beta)}(1) = 1$. (We shall also use the notation P_n if it cannot cause misunderstanding.) It was shown by Gasper [**5, 6**] that for all $s, t \in [0, \pi]$ there exists a (unique) probability measure $\varepsilon_s * \varepsilon_t$ on $[0, \pi]$ such that

$$P_n^{(\alpha,\beta)}(\cos s) \cdot P_n^{(\alpha,\beta)}(\cos t) = \int_0^\pi P_n^{(\alpha,\beta)}(\cos u)\, d(\varepsilon_s * \varepsilon_t)(u), \qquad n \in \mathbb{N}.$$

The convolution $\varepsilon_s * \varepsilon_t$ can be extended uniquely to a bilinear, commutative, associative and weakly continuous convolution $*$ on the Banach space $M_b([0, \pi])$ of all (signed) Borel measures on $[0, \pi]$. Moreover, $*$ establishes a commutative hypergroup structure on $K = [0, \pi]$. Its normalised Haar measure $\omega = \omega^{(\alpha,\beta)}$ is given by

$$d\omega(\vartheta) := \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1) 2^{\alpha+\beta+1}} (\sin\vartheta)^{2\alpha+1} (1+\cos\vartheta)^{\beta-\alpha}\, d\vartheta, \qquad \vartheta \in [0, \pi].$$

The dual space $\widehat{K} = \{P_n^{(\alpha,\beta)} \circ \cos : n \in \mathbb{N}_0\}$ can be identified with $\mathbb{N}_0$, and the Plancherel measure $\pi = \pi^{(\alpha,\beta)}$ on it is given by

$$\pi(\{n\}) := h_n^{(\alpha,\beta)} := \frac{(2n+\alpha+\beta+1)(\alpha+\beta+1)_n (\alpha+1)_n}{(\alpha+\beta+1) n! (\beta+1)_n}, \qquad n \in \mathbb{N}_0,$$

where for $\gamma \in \mathbb{R}$ and $j \in \mathbb{N}$ we use the standard notation

$$(\gamma)_j := \gamma(\gamma+1)\cdots(\gamma+j-1).$$

The Jacobi–Fourier transform of a measure $\mu \in M_b([0,\pi])$ and a function $f \in L^1([0,\pi],\omega)$ are given by

$$\widehat{\mu}(n) := \int_0^\pi P_n^{(\alpha,\beta)}(\cos\vartheta)\,d\mu(\vartheta), \quad \widehat{f}(n) := \int_0^\pi P_n^{(\alpha,\beta)}(\cos\vartheta) f(\theta)\,d\omega(\vartheta), \quad n \in \mathbb{N}_0.$$

According to the inversion theorem (see [**3**, Theorem 2.2.36]), if $\mu \in M_b([0,\pi])$ is such that

$$\|\widehat{\mu}\|_1 := \sum_{n=0}^{\infty} h_n^{(\alpha,\beta)}\,|\widehat{\mu}(n)| < \infty$$

then μ has a continuous ω–density f given by

$$f(\vartheta) := \sum_{n=0}^{\infty} h_n^{(\alpha,\beta)}\widehat{\mu}(n) P_n^{(\alpha,\beta)}(\cos\vartheta), \qquad \vartheta \in [0,\pi],$$

and

$$\|\mu\| \leqslant \|f\|_\infty \leqslant \left\|\widehat{f}\right\|_1 = \|\widehat{\mu}\|_1.$$

For all $\alpha \geqslant \beta \geqslant -1/2$ and $j \in \mathbb{N}$ we introduce the functions

$$q_j(n) := q_j^{(\alpha,\beta)}(n) := \frac{n!(n+\alpha+\beta+1)_j}{2^j(n-j)!(\alpha+1)_j}, \qquad n \in \mathbb{N}_0.$$

In fact, q_j is a polynomial of degree $2j$ and in particular

$$q_1(n) = \frac{n(n+\alpha+\beta+1)}{2(\alpha+1)}.$$

For $\sigma^2 > 0$ the *Gaussian measure γ_{σ^2} with parameter σ^2 on the Jacobi hypergroup* can be defined by its Jacobi–Fourier transform

$$\widehat{\gamma}_{\sigma^2}(n) := \exp\left\{-\frac{\sigma^2 q_1(n)}{2}\right\}, \qquad n \in \mathbb{N}_0,$$

see, for instance, Bochner [4]. Clearly $\|\widehat{\gamma}_{\sigma^2}\|_1 < \infty$ implies that it has a continuous, bounded ω–density.

3. Edgeworth expansion

Let μ be a probability measure on $[0,\pi]$. We denote the moments of μ by

$$m_j := m_j^\mu := \int_0^\pi t^j \, d\mu(t), \qquad j \in \mathbb{N}.$$

Let $(X_n)_{n\geqslant 0}$ be the Markov chain on $[0,\pi]$ associated with μ and on the Jacobi hypergroup as follows. The chain starts at 0 at time 0, and the transition probabilities satisfy

$$P(X_n \in A \mid X_{n-1} = z) = (\varepsilon_z * \mu)(A)$$

for all $n \in \mathbb{N}$, $z \in [0,\pi]$ and Borel sets $A \subset [0,\pi]$. As in Voit **[9]** we obtain that the distribution of X_n is given by the n-fold Jacobi convolution power $\mu^{(n)}$.

For all $k \in \mathbb{N}$ let μ_k be the probability measure on $[0,\pi/\sqrt{k}]$ defined by $\mu_k(A) := \mu(\sqrt{k}A)$ for Borel sets $A \subset [0,\pi/\sqrt{k}]$. Let $\left(X_n^{(k)}\right)_{n\geqslant 0}$ be the Markov chain on $[0,\pi]$ associated with μ_k. Then $X_k^{(k)}$ has distribution $\mu_k^{(k)}$ with its Jacobi–Fourier transform

$$\widehat{\mu_k^{(k)}}(n) = \left(\widehat{\mu}_k(n)\right)^k = \left(\int_0^\pi P_n(\cos\theta)\, d\mu_k(\theta)\right)^k = \left(\int_0^\pi P_n\left(\cos\left(t/\sqrt{k}\right)\right) d\mu(t)\right)^k.$$

The *Edgeworth polynomials* Q_j^μ, $j \in \mathbb{N}_0$, *corresponding to the measure* μ are defined by the formal expansion

$$\left(\int_0^\pi P_n\left(\cos(t/\sqrt{k})\right) d\mu(t)\right)^k = \exp\left\{-\frac{m_2 q_1(n)}{2}\right\} \sum_{j=0}^\infty Q_j^\mu(n) k^{-j}.$$

More precisely, defining first the polynomials ψ_j^μ, $j \in \mathbb{N}$, by the power series (in the variable y)

$$\frac{1}{2} m_2 q_1(n) + \frac{1}{y^2} \log \int_0^\pi P_n^{(\alpha,\beta)}(\cos(ty))\, d\mu(t) = \sum_{j=1}^\infty \psi_j^\mu(n) y^{2j},$$

we can determine the polynomials Q_j^μ, $j \in \mathbb{N}_0$, by

$$\exp\left\{\sum_{j=1}^\infty \psi_j^\mu(n) x^j\right\} = \sum_{j=0}^\infty Q_j^\mu(n) x^j.$$

(In Section 4 we prove that the above power series has a positive radius of convergence.) We remark that the Edgeworth polynomials Q_j^μ, $j \in \mathbb{N}_0$, also depend on the parameters α and β through the functions $q_j = q_j^{(\alpha,\beta)}$, $j \in \mathbb{N}$. The degree of the polynomial Q_j^μ

is at most $2j+2$, and the coefficients are polynomials of the moments m_2, m_4, $\ldots$, $m_{2(j+1)}$. In particular,

$$Q_0^\mu(n) \equiv 1, \qquad Q_1^\mu(n) = m_4\left(\frac{q_1(n)}{24} + \frac{q_2(n)}{8}\right) - m_2^2 \frac{(q_1(n))^2}{8},$$

$$Q_2^\mu(n) = m_2 m_4\left(\frac{(q_1(n))^2}{48} + \frac{q_1(n)q_2(n)}{9}\right) - m_2^3 \frac{(q_1(n))^3}{24}$$
$$- m_6\left(\frac{q_1(n)}{720} + \frac{q_2(n)}{48} + \frac{q_3(n)}{48}\right) + \frac{1}{2}\left(Q_1^\mu(n)\right)^2.$$

(The definition of the Q_j^μ, $j \in \mathbb{N}_0$, should be compared with the usual expansion

$$\left(\widehat{\mu}\left(t/\sqrt{k}\right)\right)^k = \exp\left\{-\frac{1}{2}\sigma^2 t^2\right\} \sum_{j=0}^{\infty} k^{-j/2} \widetilde{P}_j^\mu(it)$$

of the characteristic function of a normalised convolution power of a probability measure μ on $\mathbb{R}$ with zero mean and covariance σ^2; see Bhattacharya and Ranga Rao [**1**, Section 7].) Furthermore, for all $j \in \mathbb{N}_0$ we define the signed measure ν_j^μ on $[0,\pi]$ by its Jacobi–Fourier transform

$$\widehat{\nu_j^\mu}(n) := Q_j^\mu(n) \exp\left\{-\frac{m_2 q_1(n)}{2}\right\}, \qquad n \in \mathbb{N}_0.$$

Clearly $\left\|\widehat{\nu}_j^\mu\right\|_1 < \infty$, and hence ν_j^μ exists and has a continuous, bounded ω–density g_j. Obviously ν_0^μ is just the Gaussian measure γ_{σ^2} with parameter $\sigma^2 = m_2$.

We need the following mild restriction on μ, which means that 'μ is not concentrated in 0 too much'.

CONDITION (G). We say that a probability measure μ on $[0,\pi]$ *has the growth property* (G) at the point 0 if there exist constants $c, p > 0$ such that

$$\mu([0,\delta]) \leqslant c \cdot \delta^p \qquad \text{for all } \delta \in [0,\pi].$$

Condition (G) is equivalent to

$$\int_0^\pi t^{-q}\, d\mu(t) < \infty \qquad \text{for some } q > 0,$$

see Voit [**9**, Remark 1.9].

From Voit [**9**, Theorem 1.6] if $\alpha \geqslant \beta \geqslant -1/2$, $\alpha > -1/2$, and μ is a probability measure on $[0,\pi]$ with property (G) then there is a $k_0 = k_0(d,\mu)$ such that for all $k \geqslant k_0$ the distribution $\mu_k^{(k)}$ of $X_k^{(k)}$ has a continuous, bounded ω–density f_k.

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THEOREM. *Let $\alpha \geqslant \beta \geqslant -1/2$, $\alpha > -1/2$, and let μ be a probability measure on $[0,\pi]$ with property (G). Then for all $s \in \mathbb{N}$,*

$$\Big\| f_k - \sum_{j=0}^{s-1} k^{-j} \cdot g_j \Big\|_\infty = O\big(k^{-s}\big) \qquad \text{for } k \to \infty.$$

In particular,

$$\Big\| \mu_k^{(k)} - \sum_{j=0}^{s-1} k^{-j} \cdot \nu_j^\mu \Big\| = O\big(k^{-s}\big) \qquad \text{for } k \to \infty.$$

4. Proof of the theorem

The letter c with or without indices will denote a positive constant depending only on the fixed parameters α, β and the measure μ. The same symbol may stand for different constants. A relationship $a_k = O(b_k)$ between sequences (a_k) and (b_k) of real numbers means that there exists a constant c (depending only on α, β and μ) such that $|a_k| \leqslant c b_k$ for all $k \in \mathbb{N}$.

In view of the properties of the Jacobi–Fourier transform (see Section 2) it is sufficient to show that

$$\Big\| \widehat{\mu_k^{(k)}} - \sum_{j=0}^{s-1} k^{-j} \cdot \widehat{\nu_j^\mu} \Big\|_1 = O\big(k^{-s}\big).$$

We have

$$\begin{aligned}
&\Big\| \widehat{\mu_k^{(k)}} - \sum_{j=0}^{s-1} k^{-j} \cdot \widehat{\nu_j^\mu} \Big\|_1 \\
&\quad = \sum_{n=0}^{\infty} h_n \left| \left(\int_0^\pi P_n\Big(\cos\big(t/\sqrt{k}\big)\Big)\, d\mu(t) \right)^k - \exp\Big\{-\frac{m_2 q_1(n)}{2}\Big\} \sum_{j=0}^{s-1} Q_j^\mu(n) k^{-j} \right| \\
&\quad \leqslant R_1(k) + R_2(k) + R_3(k)
\end{aligned}$$

where

$$R_1(k) := \sum_{n \leqslant A\sqrt{k}} h_n \left| \left(\int_0^\pi P_n\Big(\cos\big(t/\sqrt{k}\big)\Big)\, d\mu(t) \right)^k - \exp\Big\{-\frac{m_2 q_1(n)}{2}\Big\} \sum_{j=0}^{s-1} Q_j^\mu(n) k^{-j} \right|$$

$$R_2(k) := \sum_{n > A\sqrt{k}} h_n \left(\int_0^\pi P_n\Big(\cos\big(t/\sqrt{k}\big)\Big)\, d\mu(t) \right)^k$$

$$R_3(k) := \sum_{n > A\sqrt{k}} h_n \exp\Big\{-\frac{m_2 q_1(n)}{2}\Big\} \sum_{j=0}^{s-1} Q_j^\mu(n) k^{-j}$$

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with arbitrary $A > 0$, which will be chosen appropriately later.

Using the arguments leading to (3.10) in Voit [9] we observe that for each $A > 0$ the term $R_3(k)$ tends exponentially to 0 as $k \to \infty$, that is,

$$R_3(k) \leqslant c_1 \cdot c_2^k$$

for suitable constants $c_1 > 0$ and $0 < c_2 < 1$.

By appealing to Voit [9, Lemma 3.8] if the measure μ has the property (G), then for each $A > 0$ the term $R_2(k)$ also tends exponentially to 0 as $k \to \infty$ in the sense that

$$R_2(k) \leqslant c_1 \cdot c_2^k \qquad \text{for all } k \geqslant c_3$$

for suitable constants $c_1, c_3 > 0$ and $0 < c_2 < 1$.

The aim of the following discussion is to show that for sufficiently small $A = A(\mu, \alpha, \beta)$ we have

$$R_1(k) = O\big(k^{-s}\big).$$

In view of our normalisation $P_n^{(\alpha,\beta)}(1) = 1$, Szegö [8, (4.21.7) and (4.1.1)] lead to the recursive formula

$$\frac{dP_n^{(\alpha,\beta)}}{dx}(x) = \frac{n(n+\alpha+\beta+1)}{2(\alpha+1)} \cdot P_{n-1}^{(\alpha+1,\beta+1)}(x), \qquad x \in \mathbb{R},\ n \in \mathbb{N}.$$

Consequently we have

$$\frac{d^j P_n^{(\alpha,\beta)}}{dx^j}(1) = q_j^{(\alpha,\beta)}(n), \qquad j, n \in \mathbb{N},$$

and we obtain easily the Taylor expansion for the Jacobi polynomials: there exists a constant c such that

$$\left| P_n(x) - \sum_{j=0}^{s} \frac{(-1)^j}{j!} q_j(n)(1-x)^j \right| \leqslant c n^{2(s+1)}(1-x)^{s+1}, \qquad \text{for } x \in [-1,1],\ n \in \mathbb{N}_0.$$

Using the Taylor expansion of the cosine function, we obtain

$$\left| \int_0^\pi P_n(\cos(ty))\, d\mu(t) - \sum_{j=0}^{s} \varphi_j(n) m_{2j} y^{2j} \right| \leqslant c n^{2(s+1)} y^{2(s+1)}$$

for $y \in \mathbb{R}$, $n \in \mathbb{N}_0$ with some polynomials φ_j, $j \in \mathbb{N}_0$, where $\varphi_0(n) = 1$, $\varphi_1(n) = -q_1(n)/2$ for $n \in \mathbb{N}_0$, and φ_j is a polynomial of degree $2j$. In particular, we can choose $A_1 > 0$ such that

$$\left| \int_0^\pi P_n(\cos(ty))\, d\mu(t) - 1 \right| \leqslant \frac{1}{2} \qquad \text{for } y \in \mathbb{R} \text{ and } n \in \mathbb{N}_0 \text{ with } n\,|y| \leqslant A_1.$$

Applying the Taylor inequality

$$\left|\log(1+x) - \sum_{j=1}^{s-1} \frac{(-1)^{j-1}}{j} x^j\right| \leqslant c\,|x|^s, \qquad \text{for } |x| \leqslant \frac{1}{2},$$

we conclude

$$\left|\frac{1}{2} m_2 q_1(n) + \frac{1}{y^2} \log \int_0^\pi P_n(\cos(ty))\, d\mu(t) - \sum_{j=1}^{s-1} \psi_j^\mu(n) y^{2j}\right| \leqslant C_s n^{2(s+1)} y^{2s}$$

for $y \in \mathbb{R}$ and $n \in \mathbb{N}_0$ with $n\,|y| \leqslant A_1$ and with some polynomials ψ_j, $j \in \mathbb{N}$, where ψ_j is a polynomial of degree at most $2j+2$. In particular, we have

$$\left|\frac{1}{2} m_2 q_1(n) + \frac{1}{y^2} \log \int_0^\pi P_n(\cos(ty))\, d\mu(t)\right| \leqslant C_2 n^4 y^2 \leqslant \frac{m_2}{8(\alpha+1)} n^2$$

for $y \in \mathbb{R}$ and $n \in \mathbb{N}_0$ with $n\,|y| \leqslant A$, where

$$A := \min\left\{A_1, \sqrt{\frac{m_2}{8C_2(\alpha+1)}}\right\}.$$

With the help of the Taylor inequality

$$\left|e^x - \sum_{j=0}^{s-1} \frac{x^j}{j!}\right| \leqslant \frac{|x|^s}{s!} e^{|x|}, \qquad \text{for } x \in \mathbb{R},$$

and putting $y = 1/\sqrt{k}$ we obtain

$$\left|\left(\int_0^\pi P_n\left(\cos\left(t/\sqrt{k}\right)\right) d\mu(t)\right)^k \exp\left\{\frac{m_2 q_1(n)}{2}\right\} - \sum_{j=0}^{s-1} Q_j^\mu(n) k^{-j}\right|$$

$$\leqslant c n^{4s} k^{-s} \exp\left\{\frac{m_2}{8(\alpha+1)} n^2\right\}$$

for $n \leqslant A\sqrt{k}$. Obviously

$$q_1(n) = \frac{n(n+\alpha+\beta)}{2(\alpha+1)} \geqslant \frac{n^2}{2(\alpha+1)}$$

implies

$$\exp\left\{-\frac{m_2 q_1(n)}{2}\right\} \leqslant \exp\left\{-\frac{m_2}{4(\alpha+1)} n^2\right\},$$

and hence we have

$$\left| \left(\int_0^\pi P_n\left(\cos\left(t/\sqrt{k}\right)\right) d\mu(t) \right)^k - \exp\left\{ -\frac{m_2 q_1(n)}{2} \right\} \sum_{j=0}^{s-1} Q_j^\mu(n) k^{-j} \right|$$

$$\leqslant c_1 n^{4s} k^{-s} \exp\left\{ -\frac{m_2}{8(\alpha+1)} n^2 \right\}$$

for $n \leqslant A\sqrt{k}$. Taking into account that

$$h_n = O\left(n^{2\alpha+1}\right),$$

we conclude that

$$R_1(k) \leqslant c_1 \sum_{n \leqslant A\sqrt{k}} n^{2\alpha+1} n^{4s} k^{-s} e^{-c_2 n^2} = O\left(k^{-s}\right).$$

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