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Abstract

Extending our previous work on eigenvalues of closed surfaces and work of Otal and Rosas, we show that a complete Riemannian surface S of finite type and Euler characteristic $\chi(S) < 0$ has at most $-\chi(S)$ small eigenvalues.

1. Introduction

For Riemannian metrics on the closed surface $S = S_g$ of genus $g \geqslant 2$, the eigenvalue $\lambda_{2g-2} = \lambda_{-\chi(S)}$ plays a specific role. Buser gave examples of hyperbolic metrics on S such that the first $2g-2$ eigenvalues

$$0 = \lambda_0 < \lambda_1 \leqslant \cdots \leqslant \lambda_{2g-3}$$

are arbitrarily small [Bus77, Satz 1]. On the other hand, Schoen *et al.* proved that there is a constant $c = c(g) > 0$ such that $\lambda_{2g-2} > c$ for any Riemannian metric on S with curvature $K \leqslant -1$ [SWY80]. Buser then showed that, for hyperbolic metrics, the constant c can be chosen to be independent of the genus [Bus10, Theorem 8.1.4]. This development culminated in the work of Otal and Rosas, who showed that $\lambda_{2g-2} > \lambda_0(\tilde{S})$ for any analytic Riemannian metric on S with curvature $K \leqslant -1$, where $\lambda_0(\tilde{S})$ denotes the bottom of the spectrum of the universal covering surface $\tilde{S}$ of S, endowed with the lifted Riemannian metric [OR09, Théorème 1].

Recall that the bottom of the spectrum of the hyperbolic plane is $1/4$ and that we have $\lambda_0(\tilde{S}) \geqslant 1/4$ if $K \leqslant -1$; see also (1.1) below.

Dodziuk *et al.* extended the work of Schoen *et al.* to the non-compact surfaces $S_{g,p}$ of genus g with $p > 0$ punctures (where $2g + p > 2$). They showed that there is a constant $c = c(2g + p)$ such that complete hyperbolic metrics on $S_{g,p}$, of finite or infinite area, have at most $2g + p - 2$ eigenvalues λ, counted with multiplicity, with $\lambda \leqslant c$ [DPRS87, Corollary 1.3]. In [OR09, Théorème 2], Otal and Rosas improve this for complete hyperbolic metrics of finite area to $c = 1/4$.

At the end of their article, Otal and Rosas discuss the question whether their results also hold for smooth Riemannian metrics. In our previous article [BMM16a], we showed this for closed surfaces and sharpened their bound $\lambda_0(\tilde{S})$. In the present article, we generalize their results to the case of complete Riemannian metrics on surfaces of finite type with Euler characteristic $\chi(S) < 0$. We allow the surfaces S to be compact or non-compact and the metrics to have finite or infinite area. We also allow for non-empty boundary ∂S. In that case, our results refer to the Dirichlet spectrum.

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1.1 Statement of main results

We say that a surface S with boundary (possibly empty) is of *finite type* if it is diffeomorphic to a closed surface with $p \geqslant 0$ points and $q \geqslant 0$ open discs removed. Then S has p ends, represented by the punctures, and q boundary circles, the boundaries of the deleted open discs. Note that we are only concerned with the diffeomorphism type of S. Thus a puncture has the same effect as the removal of a closed disc.

A basis of the neighborhoods of an end of S consists of punctured discs around the corresponding deleted point. We call these punctured discs *funnels* and visualize the surface as a steamboat with the funnels pointing upwards and the rest of the surface below them. As already emphasized above, we do not distinguish between different conformal types. For example, in our terminology, a hyperbolic cusp is a funnel.

We assume that S is endowed with a Riemannian metric which is complete with respect to the associated distance function. The area of the metric may be finite or infinite. We view the Laplacian Δ of S as an unbounded operator on the space $L^2(S)$ of square-integrable functions on S with domain the space of smooth functions on S with compact support in the interior $\mathring{S}$ of S. Our concern is the spectrum of the Friedrichs extension of Δ, which we call the *spectrum* of S. If the boundary of S is empty, a case which we include in our discussion, this is the usual spectrum of S. Otherwise it is the Dirichlet spectrum of S.

For any Riemannian manifold M, with or without boundary, denote by $\lambda_0(M)$ *the bottom of the spectrum* of the Laplacian Δ on M; that is,

$$\lambda_0(M) = \inf R(\varphi), \tag{1.1}$$

where φ runs over all non-zero smooth functions on M with compact support in the interior of M and where $R(\varphi)$ denotes the *Rayleigh quotient* of φ,

$$R(\varphi) = \frac{\int_M |\nabla\varphi|^2}{\int_M \varphi^2}. \tag{1.2}$$

As we mentioned above, the bottom of the spectrum of the hyperbolic plane is $1/4$. The bottom of the spectrum of the Euclidean plane is 0.

To state the main result of the present article, we need to introduce one more notion. Let S be a surface of finite type, with or without boundary, endowed with a complete Riemannian metric. Set

$$\Lambda(S) = \inf_{\Omega} \lambda_0(\Omega), \tag{1.3}$$

where the infimum is taken over all domains Ω in S which are diffeomorphic to an open disc, annulus, or cross cap. The work of Brooks [Bro85, Theorem 1] implies that we always have

$$\Lambda(S) \geqslant \lambda_0(\tilde{S}) \tag{1.4}$$

as we will see in Proposition 3.8. The inequality is strict if S is closed and hyperbolic [Mon14]. More generally, it is strict for any Riemannian metric on a compact surface with non-positive Euler characteristic; see [BMM16b]. However, the strict inequality $\Lambda(S) > \lambda_0(\tilde{S})$ does not hold in general. For example, for a non-compact complete hyperbolic surface S of finite type without boundary, we have $\Lambda(S) = \lambda_0(\tilde{S}) = 1/4$ by the special geometry of the ends of S.

We say that an eigenvalue λ of a complete Riemannian metric on a surface S is *small* if $\lambda \leqslant \Lambda(S)$. In [BMM16a] we showed that, on a closed surface S with $\chi(S) < 0$, a Riemannian metric has at most $-\chi(S)$ small eigenvalues, counted with multiplicity. The main result of this article is an extension of the latter result to surfaces of finite type.

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Theorem 1.5. *A complete Riemannian metric on a surface S of finite type with $\chi(S) < 0$ has at most $-\chi(S)$ small eigenvalues, counted with multiplicity.*

With p, q as further up, the case $p = q = 0$ corresponds to closed surfaces, treated in [BMM16a]. The case $p > 0$, $q = 0$ (with orientable S) extends [DPRS87, Corollary 1.3] of Dodziuk *et al.* and [OR09, Théorème 2] of Otal and Rosas to arbitrary complete Riemannian metrics on such surfaces. The case $p = 0$, $q > 0$ corresponds to the Dirichlet spectrum of compact surfaces with non-empty boundary.

Using the work of Lax and Phillips on so-called embedded eigenvalues [LP82], we will show that Theorem 1.5 has the following consequence.

Theorem 1.6. *A complete hyperbolic metric with infinite area on a surface S of finite type with $\chi(S) < 0$ has at most $-\chi(S)$ eigenvalues, counted with multiplicity, and all of them are contained in $(0, 1/4)$.*

In the case where the boundary of S is empty, Theorem 1.6 is an almost immediate consequence of Theorem 1.5 and the work of Lax and Phillips.

The assertion of Theorem 1.6 also holds if the metric is assumed to be complete with curvature $K \leqslant -1$ and to be asymptotically hyperbolic in the sense of Mazzeo [Maz91] on at least one of its funnels; see our discussion in § 5.

The situation for complete hyperbolic metrics of finite area is much more complicated; see e.g. [Sar03, § 2 and Conjecture 1].

1.2 Remarks and examples

(i) The bound $-\chi(S)$ in Theorems 1.5 and 1.6 is optimal. Indeed, the construction of Buser in [Bus77] applies to surfaces S of finite type with $\chi(S) < 0$ and shows that, for any $\varepsilon > 0$, there is a complete hyperbolic metric on any such S (with closed hyperbolic geodesics as boundary circles if the boundary of S is non-empty) such that S has (at least) $-\chi(S)$ eigenvalues λ, counted with multiplicity, with $\lambda < \varepsilon$. Furthermore, if S is non-compact, the metric can be chosen to have finite or infinite area.

(ii) Besides $\lambda_0(\tilde{S})$ and $\Lambda(S)$, there is another constant which is of interest in our context. Recall that the spectrum of S is the disjoint union of the discrete and the essential spectrum of S, where $\lambda \in \mathbb{R}$ belongs to the essential spectrum of S if $\Delta - \lambda$ is not a Fredholm operator. The essential spectrum of S is a closed subset of $[0, \infty)$ with bottom denoted by $\lambda_{\mathrm{ess}}(S)$. In Proposition 3.8, we will see that $\lambda_{\mathrm{ess}}(S) \geqslant \Lambda(S)$ if S is of finite type.

(iii) In [BCD93, Example 4.1], Buser *et al.* construct examples of complete hyperbolic surfaces S of infinite type with arbitrarily small $\lambda_{\mathrm{ess}}(S) > 0$ which have infinitely many eigenvalues less than $\lambda_{\mathrm{ess}}(S)$. Since the eigenvalues are smaller than $\lambda_{\mathrm{ess}}(S)$, they do not belong to the essential spectrum of S, and therefore they are of finite multiplicity. Note that zero is not an eigenvalue of S since the area of S is infinite so that non-zero constant functions are not square integrable. By Proposition 3.8 and since $\tilde{S}$ is the hyperbolic plane, we also have that $\Lambda(S) \geqslant \lambda_0(\tilde{S}) = 1/4$.

(iv) In Examples 3.7(i) and 3.7(ii), we construct complete surfaces of finite type with curvature $K \leqslant -1$, of finite and infinite area, and with empty essential spectrum. Such surfaces have infinitely many eigenvalues, and all of them are of finite multiplicity. In Example 3.7(iii), we construct complete non-compact surfaces S of finite type with curvature $K \leqslant 0$, of finite and infinite area, with $\Lambda(S)$ arbitrarily small, and $\lambda_{\mathrm{ess}}(S) = 1/4$.

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(v) Our results refer to the standard spectrum of the surface S if $\partial S = \emptyset$ and to the Dirichlet spectrum of S if $\partial S \neq \emptyset$. In the latter case, one might also ask for other boundary conditions and, in particular, for the Neumann condition. In fact, our arguments in the proof of Theorem 1.5 still apply. However, for the Neumann condition, the constant $\Lambda(S)$ would have to be replaced by a corresponding constant $\Lambda_N(S)$, where we would have to consider mixed boundary conditions for the Laplacian on embedded discs, annuli, and cross caps Ω in S with piecewise smooth boundary (Neumann condition on $\partial\Omega \cap \partial S$, Dirichlet condition on $\partial\Omega \cap \mathring{S}$). As of now, we do not see interesting estimates for that constant.

In the case where the surface is hyperbolic with periodic geodesics as boundary circles, doubling S along ∂S would lead to the estimate that the number of Dirichlet eigenvalues plus the number of Neumann eigenvalues of S below $\Lambda(2S)$ is bounded by $-2\chi(S)$ since the eigenfunctions on the double $2S$ split into even and odd ones with respect to the isometry of $2S$ which interchanges the two summands of $2S$. Via restriction, the first type corresponds to Neumann eigenfunctions, the second to Dirichlet eigenfunctions on the original S. (This observation led to [Sch91, Conjecture 2].)

1.3 Structure of the article

In the proof of our main result, Theorem 1.5, our line of arguments is different from the classical one of Buser [Bus77], Schoen *et al.* [SWY80], and Dodziuk *et al.* [DPRS87], who rely on decompositions of the surface into appropriate pieces and monotonicity properties of eigenvalues. As in our previous article [BMM16a], we follow the strategy of Otal and Rosas in [OR09], which involves a careful examination of topological properties of nodal sets and domains of finite linear combinations of eigenfunctions.

In the cases discussed by Otal and Rosas, the underlying Riemannian metrics are analytic. Then the corresponding eigenfunctions are analytic functions, and hence also finite linear combinations of them. It follows that the nodal set $Z_\varphi = \{\varphi = 0\}$ of such a linear combination φ is a locally finite graph with analytic edges. In our situation of smooth Riemannian metrics, nodal sets of eigenfunctions are still locally finite graphs. However, for a finite linear combination φ of eigenfunctions, the regularity of the nodal set Z_φ is not clear anymore. Therefore, we investigate approximate nodal sets $Z_\varphi(\varepsilon) = \{|\varphi| \leqslant \varepsilon\}$ instead and, for that reason, have to face a number of additional problems before we get the main argument of Otal and Rosas to work.

Our line of proof requires extending the corresponding arguments in [BMM16a] from closed surfaces to surfaces of finite type. In particular, we do not assume that S is compact. Hence approximate nodal sets $Z_\varphi(\varepsilon)$ and approximate nodal domains, that is, connected components of $S \backslash Z_\varphi(\varepsilon)$, might not be compact, and we have to face the problem of getting a hand on the asymptotic structure of approximate nodal sets and domains at infinity. To that end, we discuss a simultaneous twofold asymptotic behavior, namely the asymptotics of the intersection of approximate nodal sets and domains with compact subsets K of S for $\varepsilon \to 0$ and $K \to S$. We actually do not consider approximate nodal sets and domains themselves, but need to study appropriate modifications of them. One of the new technical problems in the non-compact case is that $\pm\varepsilon$ might be a regular value of φ, but not for a function ψ which is C^2-close to φ on a compact subset. Note also that the rather special behavior of nodal lines along hyperbolic cusps, as used by Otal and Rosas, is not at our disposal.

In addition to the above, we need the following generalization of Cheng's structure theorem for nodal lines of eigenfunctions of Schrödinger operators.

Theorem 1.7. *Let S be a surface with smooth boundary (possibly empty), endowed with a Riemannian metric. Let φ, V be smooth functions on S and suppose that φ vanishes along*

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the boundary of S and solves the Schrödinger equation $(\Delta + V)\varphi = 0$. Then the nodal set $Z_\varphi = \{x \in S \mid \varphi(x) = 0\}$ of φ is a locally finite graph in S. Moreover:

(i) *$z \in Z_\varphi \cap \mathring{S}$ has valence $2n$ if and only if φ vanishes to order n at z;*
(ii) *$z \in Z_\varphi \cap \partial S$ has valence $n+1$ if and only if φ vanishes to order n at z.*

In both cases, the opening angles between the edges at z are equal to π/n.

For points $z \in Z_\varphi \cap \mathring{S}$, this is [Che76, Theorem 2.5]. The case of boundary points is new. We also note the following consequence of Theorem 1.7.

Corollary 1.8. *In the situation of Theorem 1.7, Z_φ is a locally finite union of immersed circles, line segments with both end points on ∂S, rays with one end point on ∂S, and lines.*

The main part of the proof of Theorem 1.5 is contained in § 4 and is concerned with topological properties of approximate nodal domains and their asymptotic behavior. The analytical part of the proof is concentrated in Lemma 4.10. To prepare the proof of Theorem 1.5, we collect some prerequisites from topology in § 2, with Theorem 2.3 and Lemma 2.5 as our main later tools, and from analysis in § 3. In particular, the latter section contains the proof of Theorem 1.7. The proof of Theorem 1.6 is contained in § 5.

Mutatis mutandis, our arguments in the proof of Theorem 1.5 remain valid for Schrödinger operators $\Delta + V$, where the potential V is non-negative or, more or less equivalently, bounded from below. In particular, the analog of Theorem 1.5 holds also for such operators.

2. Prerequisites from topology

In this section, we collect some results about the topology of surfaces. We assume throughout that the concerned surfaces have empty or piecewise smooth boundaries.

Proposition 2.1. *The interior of a surface S is of finite type if and only if the fundamental group of S is finitely generated.*

Among the surfaces with boundary (possibly empty) whose interior is of finite type, we singled out those with compact boundary in the introduction.

For $n \geqslant 2$, denote by F_n the free group in n generators and recall that the commutator subgroup of F_2 is isomorphic to F_∞.

Proposition 2.2. *For a non-closed surface S, the following are equivalent.*

(i) *The fundamental group of S is cyclic.*
(ii) *The fundamental group of S is amenable.*
(iii) *The fundamental group of S does not contain F_2 as a subgroup.*
(iv) *The interior of S is an open disc, annulus, or cross cap.*

We say that a curve in a manifold is a *Jordan curve* if it is properly embedded. Note that Jordan curves are closed as subsets of the ambient manifold. We will need the following version of the Schoenflies theorem (compare with [Bus10, Corollary A.7]).

Theorem 2.3. *Any null-homotopic Jordan loop in a surface S bounds an embedded disc in S.*

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Corollary 2.4. *Let c_0 and c_1 be Jordan loops in an annulus A which represent the generator of the fundamental group of A (up to orientation) and which do not intersect. Then c_0 and c_1 are the boundary circles of an embedded annulus A' in A.*

A subsurface $C \subseteq S$ is called *incompressible* in S if any closed curve in C, which is homotopic to zero in S, is already homotopic to zero in C.

Lemma 2.5. *Let R be a compact and connected surface (with piecewise smooth boundary ∂R, possibly empty) which is not homeomorphic to the sphere. Let X be a non-empty incompressible closed subsurface of R with piecewise smooth boundary ∂X. Assume that $\partial X \cap \partial R$ is a union of piecewise smooth segments and circles (possibly empty) and that ∂X and ∂R are transversal, where they meet. Then*

$$\chi(R) \leqslant \chi(X).$$

In the case of equality, the components of $R \backslash \mathring{X}$ are annuli, cross caps, and lunes. More precisely, if C is a component of $R \backslash \mathring{X}$ that intersects the boundary of R, then C is an annulus attached to a boundary circle of X or is a lune attached to a part of a boundary circle of X. Otherwise C is an annulus attached to two boundary circles of X or a cross cap attached to a boundary circle of X.

Here a *lune* is a closed disc D whose boundary is subdivided into two subarcs. Attaching a lune D to X along ∂X means to glue one of the subarcs of the boundary of D to an arc in ∂X. Then X is isotopic to $X \cup D$.

Proof of Lemma 2.5. We may assume $X \subsetneq R$. Now by the assumptions on the boundaries of R and X, there is a closed collar U about ∂R in R such that $Y = X \backslash \mathring{U}$ is a deformation retract of X in R. Observe that Y does not intersect ∂R and that the boundaries of R and Y each are disjoint unions of circles.

If a component D of $R \backslash \mathring{Y}$ were a closed disc, set $c = \partial D$, a circle in ∂Y. If c were homotopic to zero in Y, then there would be a closed disc D' in Y with $\partial D' = c$. Thus $D \cup D'$ would be an embedded sphere in R. This is not possible since R is connected and would have to be equal to that sphere. Thus c is not homotopic to zero in Y, hence neither in R since Y is incompressible in R. This is a contradiction, and hence no component of $R \backslash \mathring{Y}$ is a disc. Note also that no component of $R \backslash \mathring{Y}$ is a closed surface since R is connected and Y is non-empty.

From the Mayer–Vietoris sequence, we obtain $\chi(R) = \chi(Y) + \chi(R \backslash \mathring{Y})$. Since no component of $R \backslash \mathring{Y}$ is a disc or a closed surface, we have $\chi(R \backslash \mathring{Y}) \leqslant 0$ and hence

$$\chi(R) \leqslant \chi(Y) = \chi(X).$$

If $\chi(R) = \chi(Y)$, we have $\chi(C) = 0$ for each component C of $R \backslash \mathring{Y}$. Hence each such C is an annulus or a cross cap. If C does not intersect U, then C is also a component of $R \backslash X$.

If C intersects U, it contains the corresponding parts of the boundary of R. Since R is connected and Y is non-empty, C also contains a part of the boundary of Y. Hence the boundary of C has more than one component, and hence C is an annulus. Therefore C contains precisely one boundary circle of R and intersects only the corresponding part of U.

Let C' be a component of $R \backslash \mathring{X}$ that is contained in C. If C' contains a component of ∂R, then $C = C'$ and C' is an annulus. If C' intersects a component of ∂R but does not contain it, then $C \backslash C'$ is a subdomain of C whose boundary components intersects both boundary circles of C. This is possible only if $C \backslash \mathring{X}$ consists of attached lunes. □

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3. Prerequisites from analysis

We let M be a Riemannian manifold, complete or not complete, connected or not connected, with or without (piecewise smooth) boundary. We denote by $C^k(M)$ the space of C^k-functions on M, by $C^k_c(M) \subseteq C^k(M)$ the space of C^k-functions on M with compact support, and by $C^k_{cc}(M) \subseteq C^k_c(M)$ the space of C^k-functions on M with compact support in the *interior* $\mathring{M}$ of M, respectively. In the case where the *boundary* ∂M of M is empty, we have $C^k_{cc}(M) = C^k_c(M)$. We use the term *smooth* to indicate C^∞.

We let $L^2(M)$ be the space of square-integrable functions on M and recall that $C^\infty_{cc}(M)$ is a dense subspace of $L^2(M)$. We denote by $H^1(M)$ the space of functions in $L^2(M)$ which have a square-integrable gradient ∇f in the sense of distributions. By the latter we mean that we test ∇f against smooth one-forms on M with compact support in $\mathring{M}$. Recall that $H^1(M)$ is a Hilbert space with respect to the H^1-norm and denote by $H^1_0(M)$ the closure of $C^\infty_{cc}(M)$ in $H^1(M)$.

Proposition 3.1 (Friedrichs extension). *The Laplacian Δ is self-adjoint as an unbounded operator on $L^2(M)$ with domain the space of $\varphi \in H^1_0(M)$ such that $\Delta\varphi \in L^2(M)$ in the sense of distributions.*

Proof. Since $H^1_0(M)$ is dense in $L^2(M)$, this follows immediately from the construction of the Friedrichs extension; compare with [Tay96, § A.8]. □

We call the spectrum of the Friedrichs extension of Δ as in Proposition 3.1 the *spectrum* of M. Note that, in the case where M has no boundary, this is the usual spectrum of S whereas it is the Dirichlet spectrum in the case where the boundary of M is non-empty and piecewise smooth.

Set $C^k_0(M) = \{\varphi \in C^k(M) \mid \varphi|_{\partial M} = 0\}$, and let $C^k_{c,0}(M)$ be the space of C^k-functions in $C^k_0(M)$ with compact support. We use a corresponding notation for the space $C^{0,1}(M)$ of Lipschitz functions on M.

Lemma 3.2. *If M is complete as a metric space and the boundary of M is piecewise smooth (possibly empty), then*

$$C^{0,1}_0(M) \cap H^1(M) \subseteq H^1_0(M).$$

Proof. Since the boundary of M is piecewise smooth, there is a sequence of functions χ_n in $C^1_c(M)$ such that $0 \leqslant \chi_n \leqslant 1$ and $|\nabla\chi_n| \leqslant 1/n$, such that $\{\chi_n = 1\}$ contains the support of χ_{n-1} in its interior, and such that $\cup\{\chi_n = 1\} = M$. With such a sequence, we can reduce the assertion of Lemma 3.2 to the case of functions in $C^{0,1}_{c,0}(M) \cap H^1(M)$.

Given a compact set $K \subseteq M$, there is a sequence of functions χ_n in $C^1_c(M)$ such that $0 \leqslant \chi_n \leqslant 1$ and $|\nabla\chi_n| \leqslant Cn$ for some constant $C = C(K)$, such that $\chi_n = 1$ on the set of $x \in K$ with $d(x, \partial M) \geqslant 2/n$, and such that $\chi_n = 0$ on the set of $x \in K$ with $d(x, \partial M) \leqslant 1/n$.

Now let $\varphi \in C^{0,1}_{c,0}(M) \cap H^1(M)$ and $K = \operatorname{supp}\varphi$. Choose a sequence of functions χ_n for K as above. Then $\chi_n\varphi \to \varphi$ in $H^1(M)$ since the area of the set of $x \in K$ with $1/n \leqslant d(x, \partial M) \leqslant 2/n$, which contains $K \cap \operatorname{supp}\nabla\chi_n$, is bounded by A/n for some constant A and since $\varphi \leqslant 2B/n$ on this set, where B is a Lipschitz constant for φ. This reduces the assertion of Lemma 3.2 to the case where the support of φ is contained in $\mathring{M}$. In this case, the assertion follows from smoothing. □

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As in the introduction, let $\lambda_0(M) = \inf R(\varphi)$, where the infimum is taken over all non-zero $\varphi \in C^\infty_{cc}(M)$. Since R is continuous on $H^1_0(M)\backslash\{0\}$ and $C^\infty_{cc}(M)$ is dense in $H^1_0(M)$, we have

$$\lambda_0(M) = \inf\{R(\varphi) \mid \varphi \in H^1_0(M)\backslash\{0\}\}. \tag{3.3}$$

Hence $\lambda_0(M)$ is the *bottom of the spectrum* of the Laplacian. By the definition of λ_0, we also have *domain monotonicity*,

$$\lambda_0(M) \geqslant \lambda_0(M') \tag{3.4}$$

for any Riemannian manifold M' containing M.

Lemma 3.5. *A non-zero $\varphi \in H^1_0(M)$ satisfies $R(\varphi) = \lambda_0(M)$ if and only if φ is an eigenfunction of the Laplacian with eigenvalue $\lambda_0(M)$.*

Proof. By the spectral theorem, we may represent $L^2(M)$ as the space $L^2(X)$ of square-integrable functions on a measured space X such that Δ corresponds to multiplication by a measurable function f on X. By the definition of $\lambda_0(M)$, we have $f \geqslant \lambda_0(M) \geqslant 0$ almost everywhere on X. □

For a self-adjoint operator A on a Hilbert space H, the spectrum $\operatorname{spec} A$ of A can be decomposed in several ways. By definition, the *essential spectrum* $\operatorname{spec}_{\text{ess}} A \subseteq \operatorname{spec} A$ consists of all $\lambda \in \mathbb{R}$ such that $A - \lambda\,\mathrm{id}$ is not a Fredholm operator. The *discrete spectrum* $\operatorname{spec}_d A$ is the complement,

$$\operatorname{spec}_d A = \operatorname{spec} A \backslash \operatorname{spec}_{\text{ess}} A.$$

The discrete spectrum consists of eigenvalues of finite multiplicity of A which are isolated points of $\operatorname{spec} A$. The essential spectrum is a closed subset of $\mathbb{R}$.

The following result shows that the essential spectrum of the Laplacian only depends on the geometry of the underlying manifold at infinity and that the essential spectrum of the Laplacian is empty if M is compact.

Proposition 3.6. *For a complete Riemannian manifold M with compact boundary (possibly empty), $\lambda \in \mathbb{R}$ belongs to the essential spectrum of Δ if and only if there is a Weyl sequence for λ, that is, a sequence of functions φ_n in $C^\infty_{cc}(M)$ such that:*

(i) *for any compact $K \subseteq M$, $\operatorname{supp}\varphi_n \cap K = \emptyset$ for all sufficiently large n;*
(ii) $\limsup_{n\to\infty} \|\varphi_n\|_2 > 0$ *and* $\lim_{n\to\infty} \|\Delta\varphi_n - \lambda\varphi_n\|_2 = 0$.

Proof. See the elementary argument in [Bär00, proof of Proposition 1]. □

Examples 3.7. (i) Let $F = \{(x, y) \mid x \geqslant 0, y \in \mathbb{R}/L\mathbb{Z}\}$ be a funnel with the expanding hyperbolic metric $dx^2 + \cosh(x)^2\, dy^2$. Let $\kappa : \mathbb{R} \to \mathbb{R}$ be a monotonic smooth function with $\kappa(x) = -1$ for $x \leqslant 1$ and $\kappa(x) \to -\infty$ as $x \to \infty$. Suppose that $j : \mathbb{R} \to \mathbb{R}$ solves $j'' + \kappa j = 0$ with initial condition $j(0) = 1$ and $j'(0) = 0$. Then $j(x) > \cosh x$ for all $x > 1$. Furthermore, the funnel F with Riemannian metric $g = dx^2 + j(x)^2\, dy^2$ has curvature $K(x, y) = \kappa(x) \leqslant -1$ and infinite area. By comparison, the Rayleigh quotient with respect to g of any smooth function φ with compact support in the part $\{x \geqslant x_0\}$ of the funnel is at least $-\kappa(x_0)/4$.

Let S be a non-compact surface of finite type. Endow S with a hyperbolic metric which is expanding along its funnels as above. Replace the hyperbolic metric on the funnels by the above Riemannian metric g. Then the new Riemannian metric on S is complete with curvature $K \leqslant -1$

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and infinite area. By Proposition 3.6 and by what we said above about the Rayleigh quotients, the essential spectrum of the new Riemannian metric is empty. Hence the eigenspaces of the new metric are finite dimensional and span $L^2(S)$. Therefore S has infinitely many eigenvalues, and all of them have finite multiplicity.

(ii) As a variation of example (i), suppose now that j is the unique solution of $j'' + \kappa j = 0$ which satisfies the boundary condition $j(0) = 1$ and $j(\infty) = 0$. Then $j'(0) < -1$ and $j(x) < \exp(-x)$ for all $x > 0$. The funnel F with Riemannian metric $g = dx^2 + j(x)^2\, dy^2$ has curvature $K(x, y) = \kappa(x)$ and finite area. Again by comparison, the Rayleigh quotient with respect to g of any smooth function φ with compact support in the part $\{x \geqslant x_0\}$ of the funnel is at least $-\kappa(x_0)/4$.

Let S be a non-compact surface of finite type, and choose $r > 0$ such that $\coth(r) = -j'(0)$. It is not hard to see that S minus the parts $\{x \geqslant r\}$ of its funnels carries hyperbolic metrics which are equal to $dx^2 + j_0(x)^2\, dy^2$ along the parts $\{x < r\}$ of its funnels, where $j_0(x) = \sinh(r-x)/\sinh(r)$. Then $j_0(x) = j(x)$ for $x < \min\{1, r\}$. Hence any such hyperbolic metric, restricted to S minus the parts $\{x \geqslant \min\{1, r\}\}$ of its funnels, when combined with g along the funnels, defines a smooth and complete Riemannian metric on S which has curvature $K \leqslant -1$ and finite area. Again, its essential spectrum is empty, by Proposition 3.6 and by what we said above about the Rayleigh quotients. Again we obtain that S has infinitely many eigenvalues, and all of them have finite multiplicity.

(iii) Let S be a complete non-compact hyperbolic surface without boundary. Replace a simple closed geodesic c in S by a Euclidean cylinder $C = \{(x, y) \mid 0 \leqslant x \leqslant h, y \in \mathbb{R}/L\mathbb{Z}\}$ of height h and circumference $L = L(c)$ and smooth out the resulting Riemannian metric appropriately. Let $\varphi = \varphi(x)$ be a non-vanishing smooth function on $\mathbb{R}$ with support in $[-1, 0]$. Then the support of $\varphi_{k,i} = \varphi(x/k - i)$ is in $[(i-1)k, ik]$ and the Rayleigh quotient of $\varphi_{k,i}$ is $R(\varphi)/k^2$. Hence, given $\varepsilon > 0$, we have $R(\varphi_{k,i}) < \varepsilon$ if $k^2 > R(\varphi)/\varepsilon$. We may also view $\varphi_{k,i}$ as a smooth function on the cylinder C and the surface S if h is sufficiently large. More specifically, given n, choose $h > nk$. Then the functions $\varphi_{k,1}, \ldots, \varphi_{k,n}$ have disjoint supports in C and Rayleigh quotients less than ε. Hence S has at least n eigenvalues which are less than ε. Since C is a cylinder, we also have $\Lambda(S) < \varepsilon$. On the other hand, the essential spectrum of S is still contained in $[1/4, \infty)$, by Proposition 3.6 and the special geometry of the ends of S.

Proposition 3.8. *For a complete Riemannian metric on a surface S, we have $\Lambda(S) \geqslant \lambda_0(\tilde{S})$. If S is of finite type, then $\Lambda(S) \leqslant \lambda_{\mathrm{ess}}(S)$.*

Proof. Since the ends of surfaces of finite type are funnels and funnels are diffeomorphic to annuli, we have $\Lambda(S) \leqslant \lambda_{\mathrm{ess}}(S)$ if S is of finite type.

Under a Riemannian covering of complete and connected Riemannian manifolds, the bottom of the spectrum of the covered manifold is at most the bottom of the spectrum of the covering manifold; see e.g. [Bro85, p. 101]. Brooks showed that, under a normal Riemannian covering of complete and connected Riemannian manifolds with an amenable group of covering transformations, the bottom of the spectrum does not change, provided that the covered manifold is of finite topological type [Bro85, Theorem 1]. The relevant arguments of Brooks in [Bro85, proof of Theorem 1] and of Sullivan in [Sul87, proof of Theorem 2.1] remain valid in the more general case of complete Riemannian manifolds of finite topological type with boundary.

Now in our setting, embedded discs in S lift to embedded discs in $\tilde{S}$ and embedded annuli and cross caps in S lift to embedded annuli and cross caps in cyclic subcoverings $\bar{S}$ of $\tilde{S}$. Moreover, $\bar{S}$ is either an annulus or a cross cap, in particular of finite topological type. For any such $\bar{S}$,

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we have $\lambda_0(\bar{S}) = \lambda_0(\tilde{S})$ since cyclic groups are amenable. Hence $\Lambda(S) \geqslant \lambda_0(\tilde{S})$ by the definition of $\Lambda(S)$. □

For a discussion concerning the strictness of the inequality $\Lambda(S) \geqslant \lambda_0(\tilde{S})$, we refer to [BMM16b].

We finish this section by discussing the proof of Theorem 1.7.

Proof of Theorem 1.7. Recall that non-zero eigenfunctions of the Laplacian cannot vanish of infinite order at any point; see e.g. [Aro57]. Hence by the main result of [Ber55], at any critical point $z \in Z_\varphi \cap \mathring{S}$ of φ, there are Riemannian normal coordinates (x, y) about z, a spherical harmonic $p = p(x, y) \neq 0$ of some order $n \geqslant 2$, and a constant $\alpha \in (0, 1)$ such that

$$\varphi(x, y) = p(x, y) + O(r^{n+\alpha}),$$

where we write $(x, y) = (r\cos\theta, r\sin\theta)$. By [Che76, Lemma 2.4], there is a local C^1-diffeomorphism Φ about $0 \in \mathbb{R}^2$ fixing 0 such that $\varphi = p \circ \Phi$. Note that, up to a rotation of the (x, y)-plane, we have

$$p = p(x, y) = cr^n \cos n\theta$$

for some constant $c \neq 0$. It follows that the interior nodal set $Z_\varphi \cap \mathring{S}$ of φ is a locally finite graph with critical points of φ as vertices and that the valence of points on Z_φ is as asserted.

It remains to discuss points $z \in Z_\varphi \cap \partial S$. Since $\dim S = 2$, there are isothermal coordinates around z, that is, coordinates (x, y) about z in which the Riemannian metric g of S is conformal to the Euclidean metric g_0: $g = fg_0$ with $f = f(x, y) > 0$. Then, again since $\dim S = 2$, the associated Laplacians satisfy $f\Delta = \Delta_0$, and hence φ solves the Schrödinger equation $(\Delta_0 + fV)\varphi = 0$ in the domain of the coordinates.

After an appropriate further conformal change of the coordinates, we can assume that the domain of the coordinates is $B_\varepsilon(0) \cap \{y \geqslant 0\}$ such that ∂S corresponds to $B_\varepsilon(0) \cap \{y = 0\}$. We consider φ and $W = fV$ as functions on $B^+ = B_\varepsilon(0) \cap \{y \geqslant 0\}$, where $\varphi(x, 0) = 0$, and extend them to functions on $B_\varepsilon(0)$ by setting $\varphi(x, y) = -\varphi(x, -y)$ and $W(x, -y) = W(x, y)$. Then φ and W are $C^{1,1}$ and $C^{0,1}$ on $B_\varepsilon(0)$, respectively, and φ solves $(\Delta_0 + W)\varphi = 0$ in B^+. Since the reflection about the x-axis is an isometry of the Euclidean plane, we also have $(\Delta_0\varphi)(x, y) = -(\Delta_0\varphi)(x, -y)$. Hence

$$(\Delta_0 + W)\varphi(x, y) = -(\Delta_0\varphi)(x, -y) - W(x, y)\varphi(x, -y) = 0$$

in $B^- = B_\varepsilon(0) \cap \{y \leqslant 0\}$. Since $\varphi = 0$ along the x-axis, all x-derivatives of φ vanish along the x-axis. Since φ solves $(\Delta_0 + W)\varphi = 0$, the second derivative of φ in the y-direction vanishes along the x-axis as well, and hence φ is $C^{2,1}$. We conclude that φ is a strong solution of $(\Delta_0 + W)\varphi = 0$ on $B_\varepsilon(0)$, and hence the main result of [Ber55] and [Che76, proof of Lemma 2.4] applies. The remaining assertions follow as in the case of $z \in Z_\varphi \cap \mathring{S}$ above. □

We learned from [HHT09, proof of Theorem 2.3] that the reflection about the x-axis in the Euclidean plane, which we use in the second part of the above proof, might be helpful in the discussion of the boundary regularity of solutions of Schrödinger equations.

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4. Estimating the number of small eigenvalues

Throughout this section, let φ be a non-vanishing square-integrable smooth function on S which is a finite linear combination of eigenfunctions with eigenvalues less than or equal to $\Lambda(S)$. The set of zeros of φ,

$$Z_\varphi := \{x \in S \mid \varphi(x) = 0\}, \tag{4.1}$$

is called the *nodal set* of φ. The connected components of the complement $S\backslash Z_\varphi$ are called *nodal domains* of φ.

We say that $\varepsilon > 0$ is φ-*regular*, if ε and $-\varepsilon$ are regular values of φ. For any $\varepsilon > 0$, we call

$$Z_\varphi(\varepsilon) := \{x \in S \mid |\varphi(x)| \leqslant \varepsilon\} \tag{4.2}$$

the ε-*nodal set* of φ. We are only interested in the case where ε is φ-regular. Then $Z_\varphi(\varepsilon)$ is a subsurface of S with smooth boundary, may consist of more than one component, and the boundary components of $Z_\varphi(\varepsilon)$ are embedded smooth circles and lines along which φ is constant $\pm\varepsilon$.

Lemma 4.3. *For any φ-regular $\varepsilon > 0$, consider the function φ_ε*

$$\varphi_\varepsilon(x) = \begin{cases} \varphi(x) - \varepsilon & \text{if } \varphi(x) \geqslant \varepsilon, \\ \varphi(x) + \varepsilon & \text{if } \varphi(x) \leqslant -\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\varphi_\varepsilon \in H^1(M)$ and $\lim_{\varepsilon\to 0} \varphi_\varepsilon = \varphi$ in $H^1(M)$.

Proof. For all $x \in S$, we have $|\varphi_\varepsilon(x)| \leqslant |\varphi(x)|$. Hence φ_ε is in $L^2(M)$. Moreover, $\varphi_\varepsilon(x) \to \varphi(x)$, and hence $\lim_{\varepsilon\to 0} \varphi_\varepsilon = \varphi$ in $L^2(M)$. Furthermore, φ_ε has weak gradient

$$\nabla\varphi_\varepsilon(x) = \begin{cases} \nabla\varphi(x) & \text{if } |\varphi(x)| \geqslant \varepsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

It follows that φ_ε is in $H^1(M)$.

As for the claim about the H^1-convergence $\varphi_\varepsilon \to \varphi$, we note that $\varphi_\varepsilon = \varphi$ on $S\backslash Z_\varphi(\varepsilon)$ and that the family of sets $Z_\varphi(\varepsilon)$ is nested with $\bigcap_{\varepsilon>0} Z_\varphi(\varepsilon) = Z_\varphi$. Moreover, $\nabla\varphi_\varepsilon$ vanishes on $Z_\varphi(\varepsilon)$. On the other hand, with respect to the area element of S, the set of points of density of Z_φ has full measure in Z_φ and $\nabla\varphi(x) = 0$ in any such point. It follows that $\lim_{\varepsilon\to 0} \nabla\varphi_\varepsilon = \nabla\varphi$ in $L^2(M)$ and hence that $\lim_{\varepsilon\to 0} \varphi_\varepsilon = \varphi$ in $H^1(M)$. □

In what follows, we assume throughout that ε is φ-regular. We say that a disc D in S is an ε-*disc* if D is closed in S and

$$\varphi = +\varepsilon \quad \text{and} \quad \nu(\varphi) > 0 \quad \text{or} \quad \varphi = -\varepsilon \quad \text{and} \quad \nu(\varphi) < 0 \tag{4.5}$$

along the boundary circle ∂D of D, where ν denotes the outer normal of D along ∂D. Note that, for an ε-disc D, a neighborhood of ∂D inside D is contained in $Z_\varphi(\varepsilon)$, whereas a neighborhood of ∂D outside D belongs to $\{\varphi \geqslant \varepsilon\}$ in the first case in (4.5) and $\{\varphi \leqslant -\varepsilon\}$ in the second.

The boundary circles of ε-discs are components of $\{\varphi = \pm\varepsilon\}$. Since ε is φ-regular, the normal derivative of φ has to be non-zero along $\{\varphi = \pm\varepsilon\}$. The requirements on the normal derivative in (4.5) fix its sign. As an example where these requirements do not hold, we note that components

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of $\{\varphi \geqslant \varepsilon\}$ or $\{\varphi \leqslant -\varepsilon\}$ might be discs, but never ε-discs. On the other hand, any component of $Z_\varphi(\varepsilon)$, which is a disc, is also an ε-disc.

By Theorem 2.3, any component C of $Z_\varphi(\varepsilon)$, which is contained in the interior of a closed disc, is also contained in an ε-disc. More precisely, there is an ε-disc D such that $\partial D \subseteq \partial C$ and such that C is a neighborhood of ∂D inside D. We let $Y_\varphi(\varepsilon)$ be the union of $S \backslash Z_\varphi(\varepsilon)$ with all ε-discs. Note that the union might not be disjoint since ε-discs might contain components of $\{\varphi \geqslant \varepsilon\}$ and $\{\varphi \leqslant -\varepsilon\}$.

Lemma 4.6. (i) *The subsurface $Y_\varphi(\varepsilon)$ is the union of $S \backslash \mathring{Z}_\varphi(\varepsilon)$ with all components of $Z_\varphi(\varepsilon)$ which are contained in the interior of closed discs in S.*

(ii) *The components of $Y_\varphi(\varepsilon)$ are incompressible in S.*

Proof. Part (i) follows immediately from what we said above. As for part (ii), suppose that there is a loop in a component C of $Y_\varphi(\varepsilon)$ which is not contractible in C, but is contractible in S. Then there is an embedded circle c in the interior of C with that property. By Theorem 2.3, there is a closed disc D in S with $\partial D = c$. Since c is not contractible in C, the interior of D contains a component of $S \backslash Y_\varphi(\varepsilon) \subseteq Z_\varphi(\varepsilon)$. This contradicts part (i). □

The set of ε-discs is ordered by inclusion. It is important that we have maximal elements in this ordered set.

Lemma 4.7. *If two ε-discs intersect, then they are either identical or one is contained in the interior of the other. Moreover, any ε-disc is contained in a unique maximal ε-disc and maximal ε-discs are either identical or disjoint.*

Proof. The first statement is clear since ε is φ-regular.

Fix an exhaustion of S by compact subsurfaces S_n such that $S \backslash S_n$ consists of cylindrical neighborhoods of the ends of S and such that ∂S_n intersects the set $\{\varphi = \pm\varepsilon\}$ transversally. Then the boundary components of any S_n are labeled by the ends of S they belong to, any S_n meets only finitely many components of the set $\{\varphi = \pm\varepsilon\}$, and the sets $\{\varphi = \pm\varepsilon\} \cap \partial S_n$ are finite.

Let $D_1 \subseteq D_2 \subseteq \cdots$ be an ascending chain of pairwise distinct ε-discs. For n sufficiently large, we have $D_1 \subseteq S_n$. Then $D_l \cap S_n \neq \emptyset$ and $\partial D_l \cap \partial S_n \subseteq \{\varphi = \pm\varepsilon\} \cap \partial S_n$ for all $l \geqslant 1$. Moreover, if $\partial D_l \cap \partial S_n = \emptyset$, then $D_l \subseteq S_n$. Since ε is φ-regular, it follows that the chain of discs is finite.

Let D and D' be maximal ε-discs and suppose that $D \cap D' \neq \emptyset$. Note that D and D' each have only one boundary circle, c and c'. If $c = c'$, then $D = D'$ by maximality. If c is contained in the interior of D', then c' is contained in the interior of D, since otherwise D would be contained in the interior of D', contradicting maximality. But then $D \cup D'$ is a subsurface of S without boundary which is closed as a subset, and hence $D \cup D' = S$. This is impossible since then $S = D \cup D'$ would be a sphere. □

Lemma 4.8. *Each component C of $Y_\varphi(\varepsilon)$ is the union of a component C_0 of $\{\varphi \geqslant +\varepsilon\}$ or of $\{\varphi \leqslant -\varepsilon\}$ together with maximal ε-discs attached to them along common boundary circles. In particular, $\partial C \subseteq \partial C_0$ and*

$$\begin{aligned} \varphi|_{\partial C} &= +\varepsilon \quad \text{and} \quad \nu(\varphi) < 0 \quad \text{if } C_0 \text{ is contained in } \{\varphi \geqslant +\varepsilon\}, \\ \varphi|_{\partial C} &= -\varepsilon \quad \text{and} \quad \nu(\varphi) > 0 \quad \text{if } C_0 \text{ is contained in } \{\varphi \leqslant -\varepsilon\}, \end{aligned}$$

where ν denotes the outer normal field of C.

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Proof. By Lemma 4.7, $Y_\varphi(\varepsilon)$ is the union of $S\backslash Z_\varphi(\varepsilon)$ with all maximal ε-discs. By the requirement on the normal derivative in (4.5), the boundary circle c of each maximal ε-disc D is attached to a component of $\{\varphi \geqslant \varepsilon\}$ if $\varphi|_c = \varepsilon$ and a component of $\{\varphi \leqslant -\varepsilon\}$ if $\varphi|_c = -\varepsilon$, respectively. □

By Lemma 4.8, we may write $Y_\varphi(\varepsilon)$ as the disjoint union,

$$Y_\varphi(\varepsilon) = Y_\varphi^+(\varepsilon) \,\dot\cup\, Y_\varphi^-(\varepsilon), \tag{4.9}$$

where $Y_\varphi^+(\varepsilon)$ and $Y_\varphi^-(\varepsilon)$ consist of the components C of $Y_\varphi(\varepsilon)$ such that the corresponding C_0 is contained in $\{\varphi \geqslant +\varepsilon\}$ and $\{\varphi \leqslant -\varepsilon\}$, respectively.

For the statement of the following lemma, recall Proposition 2.2.

LEMMA 4.10. *For any sufficiently small φ-regular $\varepsilon > 0$, the fundamental group of at least one component of $Y_\varphi(\varepsilon)$ contains the free group F_2 in two generators. Moreover, if φ is an eigenfunction, then each nodal domain C of φ is incompressible and the fundamental group of C contains F_2.*

Proof. We may assume that $Y_\varphi(\varepsilon) \neq S$. We suppose first that the Rayleigh quotient $R(\varphi) < \Lambda(S)$ and choose $\delta > 0$ such that

$$R(\varphi) \leqslant \Lambda(S) - 2\delta. \tag{4.11}$$

By Lemma 4.3 and since $S\backslash Y_\varphi(\varepsilon) \subseteq Z_\varphi(\varepsilon)$, we have, for any sufficiently small φ-regular $\varepsilon > 0$,

$$\frac{\sum_C \int_C |\nabla\varphi_\varepsilon|^2}{\sum_C \int_C \varphi_\varepsilon^2} \leqslant \frac{\int_S |\nabla\varphi|^2\, dv}{\int_S \varphi^2\, dv} + \delta = R(\varphi) + \delta \leqslant \Lambda(S) - \delta, \tag{4.12}$$

where the sums run over the components C of $Y_\varphi(\varepsilon)$. We conclude that there is a component C of $Y_\varphi(\varepsilon)$ such that

$$R(\varphi_\varepsilon|_C) = \frac{\int_C |\nabla\varphi_\varepsilon|^2}{\int_C \varphi_\varepsilon^2} \leqslant \Lambda(S) - \delta. \tag{4.13}$$

Now φ is smooth on S, and hence $\varphi_\varepsilon|_C$ is smooth on C and vanishes along ∂C. Therefore $\varphi_\varepsilon|_C \in H_0^1(C)$, by Lemma 3.2. Now it follows from the definition of $\Lambda(S)$ that the interior of C cannot be diffeomorphic to an open disc, an open annulus, or an open cross cap. Thus the fundamental group of C contains F_2.

Assume now that $R(\varphi) = \Lambda(S)$. Recall that φ is a finite linear combination of eigenfunctions of S, $\varphi = \sum c_i\varphi_i$, where $\varphi_i \in E$ is a λ_i-eigenfunction with $\lambda_i \leqslant \Lambda(S)$. If there would be an i with $c_i \neq 0$ and $\lambda_i < \Lambda(S)$, then we would have $R(\varphi) < \Lambda(S)$, a contradiction. It follows that all λ_i with $c_i \neq 0$ are equal to $\Lambda(S)$, and hence that φ is a $\Lambda(S)$-eigenfunction.

Suppose now, more generally, that φ is an eigenfunction with corresponding eigenvalue $\lambda \leqslant \Lambda(S)$. Then φ is smooth on S. By Theorem 1.7, the nodal domains of φ have piecewise smooth boundary. Hence Lemma 3.2 implies that, for any nodal domain C of φ, we have $\varphi|_C \in H_0^1(C)$ with $R(\varphi|_C) = \lambda$. In particular, $\lambda_0(C) \leqslant \lambda$.

Let C' be a thickening of C, that is, C' is a domain in S with piecewise smooth boundary which contains C in its interior and such that C is a deformation retract of C'. If the fundamental group of C does not contain F_2, then neither does the fundamental group of C', and then

$$\Lambda(S) \leqslant \lambda_0(C') \leqslant \lambda_0(C).$$

The extension φ' of $\varphi|_{\partial C}$ to C', setting $\varphi'|_{C'\backslash C} = 0$, is in $H_0^1(M)$ and in the domain of the Laplacian of C'. Moreover, it has Rayleigh quotient $R(\varphi') = \lambda$. Hence Lemma 3.5 applies and

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shows that $\lambda_0(C') = \Lambda(S)$ and $\Delta\varphi' = \Lambda(S)\varphi'$. Now φ' does not vanish identically on C, but vanishes on $C'\backslash C$. This is in contradiction to the unique continuation property for Laplace operators. Hence the fundamental group of any nodal domain of φ contains F_2.

Let C be a nodal domain of φ and suppose that C is not incompressible in S. Then there is a loop c in C which is not homotopic to zero in C, but is homotopic to zero in S. Without loss of generality, we may assume that c is a Jordan curve in $\mathring{C}$. Then c bounds a disc in S, which is not contained in C, by Theorem 2.3. Again by Theorem 2.3, there would be a nodal domain D of φ whose closure is a closed disc with piecewise smooth boundary and with $\lambda_0(D) = \Lambda(S)$. This is impossible, since $\Lambda(S)$ is not attained on (embedded) closed discs. Hence all nodal domains of φ are incompressible in S.

Let C be a nodal domain and $c_1, c_2 : [0,1] \to C$ be two loops at a point $x \in C$ which generate a free subgroup $F_2 \in \pi_1(C, x)$. By Theorem 1.7, we may assume that the images of c_1 and c_2 are contained in $\mathring{C}$. Without loss of generality, we may also assume that φ is positive on $\mathring{C}$. Then

$$\mathring{C} = \bigcup_{\varepsilon>0} \{y \in C \mid \varphi(y) > \varepsilon\}.$$

Therefore the image of c_0 and c_1 is contained in $\{y \in C \mid \varphi(y) > \varepsilon\}$ for all sufficiently small $\varepsilon > 0$. Hence the fundamental group of the component $C_0 \subseteq C$ of $Y_\varphi(\varepsilon)$ which contains c_0 and c_1 also contains F_2. From Lemma 4.8 we conclude that the component of $Y_\varphi(\varepsilon)$ containing C_0 contains F_2. □

Say that a component C of $Y_\varphi(\varepsilon)$ is an *F_2-component* if the fundamental group of C contains F_2. By Lemma 4.8, a component C of $Y_\varphi^+(\varepsilon)$ is an F_2-component if and only if there is a point x in the interior of the corresponding component C_0 of $\{\varphi \geqslant \varepsilon\}$ and a pair of loops c_0 and c_1 at x which generate an F_2 in $\pi_1(C, x)$ such that c_0 and c_1 are contained in the interior of C_0; that is, such that $\varphi > \varepsilon$ along them. The characterization of F_2-components of $Y_\varphi^-(\varepsilon)$ is analogous.

Lemma 4.14. *If K is a compact subsurface of S such that $S\backslash K$ is a cylindrical neighborhood of the ends of S and C is a component of $Y_\varphi(\varepsilon)$ which is contained in $S\backslash K$, then the interior of C is diffeomorphic to an open disc or an open annulus. In particular, we have the following.*

(i) *Any F_2-component of $Y_\varphi(\varepsilon)$ intersects K.*
(ii) *The subsurface $Y_\varphi(\varepsilon)$ contains only finitely many F_2-components.*

Components of $Y_\varphi(\varepsilon)$ might be non-compact and might have infinitely many boundary components. But since they are incompressible in S, their interiors are surfaces of finite type.

Proof of Lemma 4.14. The components of $S\backslash K$ are diffeomorphic to open annuli, hence their fundamental group is infinite cyclic. Moreover, C is incompressible in S, hence also in the component of $S\backslash K$ containing it. Therefore the fundamental group of C is either trivial or infinite cyclic. Now, as a domain in a cylinder, C is orientable. Hence the interior of C is diffeomorphic to an open disc or an open annulus. □

We denote by $X_\varphi(\varepsilon)$ the union of the F_2-components of $Y_\varphi(\varepsilon)$ and by $X_\varphi^\pm(\varepsilon)$ the ones among them which belong to $Y_\varphi^\pm(\varepsilon)$. Then $X_\varphi(\varepsilon)$ is the disjoint union of $X_\varphi^+(\varepsilon)$ and $X_\varphi^-(\varepsilon)$.

Lemma 4.15. *If $\varepsilon' \leqslant \varepsilon$ are φ-regular, then $X_\varphi^\pm(\varepsilon) \subseteq X_\varphi^\pm(\varepsilon')$.*

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Proof. By definition, $Z_\varphi(\varepsilon') \subseteq Z_\varphi(\varepsilon)$. If a component of $Z_\varphi(\varepsilon)$ is contained in the interior of an embedded closed disc, then also all the components of $Z_\varphi(\varepsilon')$ it contains. It follows that $Y_\varphi(\varepsilon) \subseteq Y_\varphi(\varepsilon')$. By Lemma 4.6(ii), if C is an F_2-component of $X^\pm_\varphi(\varepsilon)$, then also the component C' of $X^\pm_\varphi(\varepsilon')$ containing it. Hence $X_\varphi(\varepsilon) \subseteq X_\varphi(\varepsilon')$.

Now let C be a component of $X^+_\varphi(\varepsilon)$ and suppose that the component C' of $X_\varphi(\varepsilon')$ containing C belongs to $X^-_\varphi(\varepsilon')$. By Lemma 4.8, C is the union of a component C_0 of $\{\varphi \geqslant \varepsilon\}$ with maximal ε-discs. Now let x be a point in the interior of C_0 and choose loops c_0 and c_1 in the interior of C_0 generating an F_2 in $\pi_1(C, x)$; compare with our discussion further up. In particular, $\varphi > \varepsilon$ along c_0 and c_1. Under the inclusion $C \to C'$, c_0 and c_1 cannot be contained in the maximal ε'-discs belonging to C' because they would be homotopic to zero in S otherwise. But then they must meet $\{\varphi \leqslant -\varepsilon'\}$, a contradiction. We conclude that $X^+_\varphi(\varepsilon) \subseteq X^+_\varphi(\varepsilon')$; similarly $X^-_\varphi(\varepsilon) \subseteq X^-_\varphi(\varepsilon')$. □

We view the funnels of S as vertical and pointing upwards. In this picture, a Jordan curve c in a funnel F, which is a generator of the fundamental group of F, cuts $S \backslash c$ in two open pieces, the set F_c of *points above* c and the set of remaining points, sometimes called the *points below* c. The set of points above c is contained in F and is a funnel around the same end as F, the set of points below c is not contained in F.

We call Jordan curves in F, which generate the fundamental group of F, *cross sections of* F. We say that a cross section c of F is (φ, ε)-regular if it meets the curves $\{\varphi = \pm\varepsilon\}$ transversally. By transversality theory, any cross section of F can be approximated by smooth (φ, ε)-regular cross sections of F in any reasonable topology.

Our aim is now to describe the structure of $X^\pm_\varphi(\varepsilon)$ with respect to F. Let c be a (φ, ε)-regular cross section of F. Then c intersects $\{\varphi = \varepsilon\}$ transversally. We emphasize the following three cases:

(i) $F_c \subseteq X^\pm_\varphi(\varepsilon)$;
(ii) $c \cap X^\pm_\varphi(\varepsilon) = \emptyset$;
(iii) $c \cap \partial X^\pm_\varphi(\varepsilon) \neq \emptyset$.

We now want to normalize the position of a (φ, ε)-regular cross section c of F in such a way that the part of $X^\pm_\varphi(\varepsilon)$ below c is homotopy equivalent to $X^\pm_\varphi(\varepsilon)$. If it is possible to choose c such that case (i) or case (ii) holds, then any such choice will be a normalization. In the remaining case, $c \cap \partial X^\pm_\varphi(\varepsilon) \neq \emptyset$ for any choice of cross section of F. Since ε is φ-regular, $c \cap \{\varphi = \pm\varepsilon\}$ is finite. By Lemma 4.8, $c \cap \partial X^\pm_\varphi(\varepsilon) \subseteq c \cap \{\varphi = \pm\varepsilon\}$.

Since $\{\varphi = \pm\varepsilon\}$ is a properly embedded submanifold (of dimension 1) of S, the components of $F_c \cap \{\varphi = \pm\varepsilon\}$ above c are of the following two types: either they are Jordan segments with endpoints on c or they are Jordan rays with one end on c and escaping to infinity along the other. We call these components *recurrent* and *escaping*, respectively. Since $\{\varphi = \pm\varepsilon\}$ is properly embedded, escaping components in $F_c \cap \{\varphi = \pm\varepsilon\}$ extend continuously as Jordan curves to the one point compactification of F_c at infinity.

If a is a recurrent component, then there is a segment b in c such that $a \cup b$ is a null homotopic Jordan loop in F. The disc bounded by $a \cup b$ will be called the part of F_c below a. Since $c \cap \{\varphi = \pm\varepsilon\}$ is finite, there are only finitely many such discs, and they are ordered by inclusion. The components a above maximal such discs will be called *uppermost*. We replace the segments b of c below such maximal discs by the corresponding uppermost components a and obtain a piecewise smooth cross section of F. Pushing this cross section upwards and smoothing it appropriately, we arrive at the normalized third case: c is (φ, ε)-regular and the interior of $F_c \cap X^\pm_\varphi(\varepsilon)$ is a finite union of open discs, bounded by segments of c, escaping components

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of $F_c \cap \partial X_\varphi^+(\varepsilon)$, and, possibly, boundary lines of $X_\varphi^\pm(\varepsilon)$ which start and end at infinity in F. Note that boundary circles of $X_\varphi^\pm(\varepsilon)$ cannot occur, since they would not be null homotopic and we would be in the second case above.

Remark 4.16. In all three cases, after normalization, the part of $X_\varphi^\pm(\varepsilon)$ below c is homotopy equivalent to $X_\varphi^\pm(\varepsilon)$. With a bit of more work, it would be possible to show that the part of $X_\varphi^\pm(\varepsilon)$ below c is a deformation retract of $X_\varphi^\pm(\varepsilon)$. The technical problem consists in handling the components of $F_c \cap \partial X_\varphi^+(\varepsilon)$ above c which contain boundary lines which come from and return back to infinity in F_c. These boundary lines cut out *infinite peninsulas* which are hanging down from infinity in our picture of F_c. Since we do not need more than homotopy equivalence, we leave it with these remarks.

Consider a pair (ε, K), where $\varepsilon > 0$ and K is a smooth and compact subdomain of S such that $S \backslash K$ consists of funnels. Say that the pair (ε, K) is *φ-regular* if ε is φ-regular and ∂K consists of normalized (φ, ε)-regular cross sections as above. For any such pair (ε, K), define

$$X_\varphi(\varepsilon, K) = X_\varphi(\varepsilon) \cap K \quad \text{and} \quad X_\varphi^\pm(\varepsilon, K) = X_\varphi^\pm(\varepsilon) \cap K. \tag{4.17}$$

Note that $X_\varphi(\varepsilon, K)$ is the disjoint union $X_\varphi^+(\varepsilon, K) \dot\cup X_\varphi^-(\varepsilon, K)$.

By what we said above, the inclusions $X_\varphi^\pm(\varepsilon, K) \to X_\varphi^\pm(\varepsilon)$ are homotopy equivalences. Since K is a deformation retraction of S, Lemma 2.5 implies

$$\chi(S) = \chi(K) \leqslant \chi(X_\varphi(\varepsilon, K)) = \chi(X_\varphi^+(\varepsilon, K)) + \chi(X_\varphi^-(\varepsilon, K)) < 0. \tag{4.18}$$

By Lemma 4.6(ii), the components of $X_\varphi(\varepsilon, K)$ are incompressible in S. The following result is an immediate consequence of Lemmas 2.5 and 4.15.

Lemma 4.19. *If (ε, K) and (ε', K') are φ-regular with $\varepsilon' \leqslant \varepsilon$ and $K \subseteq K'$, then*

$$X_\varphi^\pm(\varepsilon, K) \subseteq X_\varphi^\pm(\varepsilon', K') \quad \text{and} \quad \chi(X_\varphi^\pm(\varepsilon', K')) \leqslant \chi(X_\varphi^\pm(\varepsilon, K)).$$

Moreover, if $\chi(X_\varphi(\varepsilon, K)) = \chi(X_\varphi(\varepsilon', K'))$, then $X_\varphi^\pm(\varepsilon', K')$ arises from $X_\varphi^\pm(\varepsilon, K)$ by attaching annuli, cross caps, and lunes along boundary curves of $X_\varphi^\pm(\varepsilon, K)$.

As a direct application of (4.18) and Lemma 4.19, we get the following.

Corollary 4.20. *There exists a φ-regular pair $(\varepsilon_\varphi, K_\varphi)$ such that*

$$\chi(X_\varphi^\pm(\varepsilon, K)) = \chi(X_\varphi^\pm(\varepsilon_\varphi, K_\varphi))$$

for all φ-regular pairs (ε, K) with $\varepsilon \leqslant \varepsilon_\varphi$ and $K_\varphi \subseteq K$.

Now we assume throughout that we are in the *stable range*, that is, we consider φ-regular pairs (ε, K) with $\varepsilon \leqslant \varepsilon_\varphi$ and $K_\varphi \subseteq K$. For such a pair (ε, K), we study the isotopy type of the triples $(S, X_\varphi^+(\varepsilon, K), X_\varphi^-(\varepsilon, K))$. Here and below we mean *compactly supported topological isotopy* when speaking of isotopy. By the definition of φ-regularity and the discussion leading to it, the isotopy type of $(S, X_\varphi^+(\varepsilon, K), X_\varphi^-(\varepsilon, K))$ does not depend on K. Hence to compare it with the isotopy type of another such triple $(S, X_\varphi^+(\varepsilon', K'), X_\varphi^-(\varepsilon', K'))$, we may assume that $\varepsilon' < \varepsilon$ and that K is contained in the interior of K'.

Now $X_\varphi^\pm(\varepsilon, K)$ has two kinds of boundary circles: the first kind consists of boundary circles in the interior of K, the second kind consists of segments of boundary circles of ∂K concatenated

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with segments of $\partial X^{\pm}_{\varphi}(\varepsilon) \subseteq \{\varphi = \pm\varepsilon\}$ that run inside K from ∂K to ∂K. By the definition of φ-regularity, boundary circles of K do not occur as boundary circles of $X^{\pm}_{\varphi}(\varepsilon, K)$. The first kind of boundary circles of $X^{\pm}_{\varphi}(\varepsilon, K)$ is smooth, the second kind is piecewise smooth with vertices in the points, where the circle enters or respectively leaves ∂K. The boundary of $X^{\pm}_{\varphi}(\varepsilon', K')$ consists of the corresponding two kinds of boundary circles.

We start with a closer look at the gluings required to obtain $X^{\pm}_{\varphi}(\varepsilon', K')$ from $X^{\pm}_{\varphi}(\varepsilon, K)$. Since their Euler characteristics coincide, only annuli, cross caps, and lunes are concerned; compare with Lemma 2.5. Now $\varepsilon' < \varepsilon$ and K is contained in the interior of K'. Hence the boundaries of $X^{\pm}_{\varphi}(\varepsilon, K)$ and $X^{\pm}_{\varphi}(\varepsilon', K')$ are disjoint, and therefore no lunes occur.

Suppose that a boundary circle c of $X^{\pm}_{\varphi}(\varepsilon, K)$ bounds a cross cap C in the complement (of the interior) of $X^{\pm}_{\varphi}(\varepsilon, K)$ in S. Now C decomposes S into two connected regions, the points inside C and the points outside C. Since ∂C is contained in $\partial X^{\pm}_{\varphi}(\varepsilon, K)$, we conclude that a curve from a point inside C to a point outside of $C \cup X^{\pm}_{\varphi}(\varepsilon, K)$ has to pass through $X^{\pm}_{\varphi}(\varepsilon, K)$. In particular, C cannot contain points on or beyond boundary circles of K since otherwise it would also contain the corresponding funnels, a contradiction to the compactness of C. We conclude that in the gluing required to obtain $X^{\pm}_{\varphi}(\varepsilon', K')$ from $X^{\pm}_{\varphi}(\varepsilon, K)$, the cross caps, including their boundary circles, are contained in the interior of K. In particular, $\varphi = \pm\varepsilon$ along their boundary circles.

Since an annulus has two boundary circles, there are different ways of attaching them and we need to distinguish two cases. Suppose first that two boundary circles c_0 and c_1 of $X^{\pm}_{\varphi}(\varepsilon, K)$ bound a closed annulus A in the complement (of the interior) of $X^{\pm}_{\varphi}(\varepsilon, K)$ in S. Then any curve from a point inside A to a point outside of $A \cup X^{\pm}_{\varphi}(\varepsilon, K)$ has to pass through $X^{\pm}_{\varphi}(\varepsilon, K)$. As in the case of cross caps, we get that A cannot contain points on or beyond boundary circles of K since otherwise it would also contain the corresponding funnels, a contradiction to the compactness of A. We conclude that in the gluing required to obtain $X^{\pm}_{\varphi}(\varepsilon', K')$ from $X^{\pm}_{\varphi}(\varepsilon, K)$, the annuli A with $\partial A \subseteq \partial X^{\pm}_{\varphi}(\varepsilon, K)$ are contained in the interior of K. In particular, $\varphi = \pm\varepsilon$ along their boundary circles.

Finally, an annulus might be glued to $X^{\pm}_{\varphi}(\varepsilon, K)$ along one boundary circle such that the second boundary circle belongs to the boundary of $X^{\pm}_{\varphi}(\varepsilon', K')$. Such gluings do not change the isotopy type of $X^{\pm}_{\varphi}(\varepsilon', K')$ in S, but gluings of cross caps and annuli as above do. To remedy this, attach all annuli and cross caps to $X^{\pm}_{\varphi}(\varepsilon, K)$ which are contained in the interior of K and have their boundary in $X^{+}_{\varphi}(\varepsilon, K)$ and call the resulting subsurface $S^{\pm}_{\varphi}(\varepsilon, K)$.

Note that no component of $X_{\varphi}(\varepsilon, K)$ is contained in any of the attached cross caps and annuli since the components of $X_{\varphi}(\varepsilon, K)$ are incompressible in S and their fundamental groups contain an F_2. Hence

$$S^{+}_{\varphi}(\varepsilon, K) \cap S^{-}_{\varphi}(\varepsilon, K) = \emptyset.$$

Note also that attaching annuli and cross caps does not change the Euler characteristic.

Lemma 4.21. *If $(\varepsilon, K), (\varepsilon', K')$ are φ-regular with $\varepsilon' \leqslant \varepsilon \leqslant \varepsilon_{\varphi}$ and $K_{\varphi} \subseteq K \subseteq K'$, then*

$$(S, S^{+}_{\varphi}(\varepsilon, K), S^{-}_{\varphi}(\varepsilon, K)) \quad \text{and} \quad (S, S^{+}_{\varphi}(\varepsilon', K'), S^{-}_{\varphi}(\varepsilon', K'))$$

are isotopic in S.

Proof. After the above discussion leading to the definition of $S^{\pm}_{\varphi}(\varepsilon, K)$, we have the following remaining issues.

If a boundary circle c of $X^{\pm}_{\varphi}(\varepsilon, K)$ bounds a cross cap C in S, then either already $C \subseteq X^{\pm}_{\varphi}(\varepsilon', K')$ or else an annulus $A \subseteq C$ is attached to c along one of its boundary circles

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and the other boundary circle c' belongs to the boundary of $X^{\pm}_{\varphi}(\varepsilon', K')$. Then c' bound a cross cap C' in the complement (of the interior) of A in C and $C = A \cup C'$.

Conversely, if a boundary circle c of $X^{\pm}_{\varphi}(\varepsilon', K')$ bounds a cross cap C' in S, then c is contained in the interior of K' and thus $\varphi = \pm\varepsilon'$ along c. We conclude that c is a boundary circle of an annulus A attached to $X^{\pm}_{\varphi}(\varepsilon, K)$ along the other boundary circle of A. Thus $C = A \cup C'$ is a cross cap in S with ∂C a boundary circle of $X^{\pm}_{\varphi}(\varepsilon, K)$. By the discussion further up we obtain that C is in the interior of K.

If boundary circles c_0 and c_1 of $X^{\pm}_{\varphi}(\varepsilon, K)$ bound an annulus A in S, then either already $A \subseteq X^{\pm}_{\varphi}(\varepsilon', K')$ or else disjoint annuli $A_0, A_1 \subseteq A$ are attached to c_0 and c_1, each along one of its boundary circles, and the other boundary circles c_0' and c_1' bound an annulus $A' \subseteq A$ between A_0 and A_1. Then $A = A_0 \cup A' \cup A_1$.

Conversely, if boundary circles c_0 and c_1 of $X^{\pm}_{\varphi}(\varepsilon', K')$ bound an annulus A' in S, then A' is contained in the interior of K' and thus $\varphi = \pm\varepsilon'$ along $\partial A'$. Arguing as in the case of cross caps, we get annuli A_0 and A_1 with one boundary circle in $X^{\pm}_{\varphi}(\varepsilon, K)$ and the other equal to c_0 and c_1, respectively. Thus $A = A_0 \cup A' \cup A_1$ is an annulus in S such that ∂A lies in $X^{\pm}_{\varphi}(\varepsilon, K)$. By the discussion further up we obtain that A is in the interior of K. □

We call the isotopy type of the triple $(S, S^{+}_{\varphi}(\varepsilon, K), S^{-}_{\varphi}(\varepsilon, K))$ the *type of* φ and the Euler characteristic of $S_{\varphi}(\varepsilon, K)$ the *characteristic of* φ.

Lemma 4.22. *Let L be a compact neighborhood of K which contains all the ε-discs with respect to φ which intersect K and such that ∂L is smooth and transversal to $\{\varphi = \pm\varepsilon\}$. Then a non-trivial finite linear combination ψ of eigenfunctions of S with corresponding eigenvalues less than or equal to $\Lambda(S)$, with the same characteristic as φ, and sufficiently C^2-close to φ on L, has the same type as φ.*

Proof. If ψ is sufficiently C^2-close to φ on L, then $\pm\varepsilon$ are regular values of $\psi|_L$, the curves $\psi|_L = \pm\varepsilon$ intersect ∂K transversally, and there is a small isotopy of S which leaves K and ∂K invariant which deforms the configuration of curves $\{\psi = \varepsilon\} \cap L$ and $\{\psi = -\varepsilon\} \cap L$ to the configuration of curves $\{\varphi = \varepsilon\} \cap L$ respectively $\{\varphi = -\varepsilon\} \cap L$ and, therefore, also the subsurfaces $\{\psi \geqslant \varepsilon\} \cap K$ and $\{\psi \leqslant -\varepsilon\} \cap K$ to the subsurfaces $\{\varphi \geqslant \varepsilon\} \cap K$ respectively $\{\varphi \leqslant -\varepsilon\} \cap K$.

Clearly, if a boundary segment of the latter intersects an ε-disc D of φ, then D is contained in L and corresponds under the isotopy to an ε-disc B of ψ. Attaching the parts $B \cap K$ of such discs, we get a surface $T^{\pm}$ such that the above isotopy deforms $T^{\pm}$ to $X^{\pm}_{\varphi}(\varepsilon, K)$. In particular, the fundamental group of $T^{\pm}$ contains an F_2, $T^{\pm}$ is incompressible in S and

$$\chi(T^{\pm}) = \chi(X^{\pm}_{\varphi}(\varepsilon, K)).$$

By changing ε slightly, we can achieve that ε is also ψ-regular. Then, by what we said, $T^{\pm}$ is a component of $X^{\pm}_{\psi}(\varepsilon, K)$. Moreover, choosing a ψ-regular (ε, K') with K in the interior of K', we have $T^{\pm} \subseteq X^{\pm}_{\psi}(\varepsilon, K')$. Hence $X^{\pm}_{\psi}(\varepsilon, K')$ is obtained from $T^{\pm}$ by attaching annuli, cross caps, and lunes. Now annuli where both boundary curves are attached to $T^{\pm}$ and cross caps attached to $T^{\pm}$ are contained in the interior of K and belong to $S^{\pm}_{\psi}(\varepsilon, K)$. We (finally) conclude that $(S, S^{+}_{\psi}(\varepsilon, K'), S^{-}_{\psi}(\varepsilon, K'))$ is isotopic to the triple $(S, S^{+}_{\varphi}(\varepsilon, K), S^{-}_{\varphi}(\varepsilon, K))$. □

End of proof of Theorem 1.5. Let $\mathbb{E}$ be a subspace of $L^2(M)$ which is generated by finitely many eigenfunctions with corresponding eigenvalues less than or equal to $\Lambda(S)$ and denote by $\mathbb{S}$ the

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unit sphere in $\mathbb{E}$ and by $\mathbb{P}$ the projective space of $\mathbb{E}$. Theorem 1.5 follows if any such $\mathbb{E}$ has dimension at most $-\chi(S)$.

Since $\chi(S^{\pm}_{\varphi}(\varepsilon, K)) = \chi(X^{\pm}_{\varphi}(\varepsilon, K))$, (4.18) and Lemma 4.21 imply that we obtain a partition of $\mathbb{S}$ into the subsets $\mathbb{A}_i$ consisting of functions φ with characteristic $i \in \{-\chi(S), \dots, -1\}$. By definition, $\varphi \in \mathbb{A}_i$ if and only if $-\varphi \in \mathbb{A}_i$. Hence the partition of $\mathbb{S}$ into the sets $\mathbb{A}_i$ is the preimage of a partition of $\mathbb{P}$ into subsets $\mathbb{B}_i$ under the covering projection $\pi : \mathbb{B} \to \mathbb{P}$.

Now at least one of the subsurfaces $S^{+}_{\varphi}(\varepsilon, K)$ or $S^{-}_{\varphi}(\varepsilon, K)$ is non-empty and has negative Euler characteristic. Therefore, it follows from [Iva92, Theorem 1.2] that there is no isotopy of S which interchanges the disjoint subsurfaces $S^{+}_{\varphi}(\varepsilon, K)$ and $S^{-}_{\varphi}(\varepsilon, K)$. Hence the type of $\varphi \in \mathbb{S}$ is different from the type of $-\varphi$. Hence by Lemma 4.22, the covering π is trivial over the subsets $\mathbb{A}_i$. Now $\mathbb{P}$ cannot be covered by less than $\dim \mathbb{E}$ subsets over which π is trivial, by [Sév02, Lemma 8]. We conclude that $\dim \mathbb{E} \leqslant -\chi(S)$. □

5. Applying the absence of large eigenvalues

We say that a subset U of the hyperbolic plane is an infinite hinge if it is the region in the hyperbolic plane bounded by two geodesic rays which issue from a common point. Besides Theorem 1.5, the other crucial point of our argument in the proof of Theorem 1.6 is the result of Lax and Phillips that an infinite hinge cannot carry a non-trivial square-integrable solution φ of the equation $\Delta\varphi = \lambda\varphi$ with $\lambda \geqslant 1/4$; see [LP82, Theorem 4.8]. Notice that Theorem 4.8 also applies in dimension two; see the last sentence in [LP82, § 4].

Proof of Theorem 1.6. The proof is straightforward in the case where the boundary of S is empty. In that case, since the area of S is infinite, S has an end $F = \{(x, y) \mid x \geqslant 0, y \in \mathbb{R}/L\mathbb{Z}\}$ with metric $dx^2 + \cosh^2(x)\, dy^2$. Clearly, any such metric contains an infinite hinge U as above and hence the restriction of any eigenfunction of S to U is a square-integrable solution of $\Delta\varphi = \lambda\varphi$ on U. By the unique continuation principle, it is non-trivial if φ is no-trivial, and then $\lambda < 1/4$ by the work of Lax and Phillips. On the other hand, we have $\Lambda(S) \geqslant \lambda_0(\tilde{S})$ by (1.4) and $\lambda_0(\tilde{S}) = 1/4$ since $\tilde{S}$ is the hyperbolic plane. Hence any eigenvalue of S is small. Therefore Theorem 1.5 applies and shows that S has at most $-\chi(S)$ eigenvalues.

The case where the boundary of S is non-empty and the boundary circles of S are closed geodesics is easy, too. In that case, S also has an end F as above and the same arguments apply. The difficult part of the proof is the case where the boundary of S is non-empty, but no restrictions on the geometry of the boundary are assumed. Then S contains at least one funnel F with infinite area.

Let c be the boundary circle of F. Consider a sequence of points q_n in F which diverge to infinity and let p_n be a point in c with distance $d(q_n, p_n) = d(q_n, c)$. Then the minimal geodesic segment γ_n from p_n to q_n is perpendicular to c and $\gamma_n \cap c = p_n$. Since c separates F from $S \backslash F$, we conclude that γ_n is contained in F. Passing to a subsequence if necessary, the sequence γ_n converges to a minimizing geodesic ray γ in F, starting from a point p in c and perpendicular to c at p. Parametrizing γ by unit speed such that $\gamma(0) = p$, we get that $d(\gamma(t), c) = t$ for all $t \geqslant 0$. Now cut F along γ to obtain an infinite strip P, bounded by two copies γ_0 and γ_1 of γ on the left and right and by c at the bottom, now a smooth curve from $p_0 = \gamma_0(0)$ to $\gamma_1(0)$.

For any $t \geqslant 1$, let σ_t be the unit speed geodesic segment in P starting from $\gamma_0(t)$ and perpendicular to γ_0, parametrized on the maximal possible interval. Since σ_t never hits γ_0 again, there are three alternatives.

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(i) If σ_t neither hits γ_1 nor c, then γ_0 and σ_t bound an infinite hinge, isometric to a corresponding hinge in the hyperbolic plane. Then the result of Lax and Phillips and Theorem 1.5 apply to conclude Theorem 1.6.

(ii) Suppose that $\sigma_t(r) = \gamma_1(s)$ for some $r > 0$. Then $\sigma_t|_{[0,r]}$ is a Jordan segment in P which decomposes P into a bounded part containing c and an unbounded part Q. Since the boundary of Q consists of the three geodesics $\gamma_1|_{[t,\infty)}$, $\sigma_t|_{[0,r]}$, and $\gamma_1|_{[s,\infty)}$, we may think of Q as a part of the hyperbolic plane. Since the area of Q is infinite, the geodesic rays $\gamma_1|_{[t,\infty)}$ and $\gamma_1|_{[s,\infty)}$ cannot be asymptotic. Hence Q contains infinite hinges, and then the result of Lax and Phillips and Theorem 1.5 apply again to conclude Theorem 1.6.

(iii) Let L be the length of c and parametrize c by unit speed on $[0, L]$. Suppose that $\sigma_1(r_1) = c(u)$ for some $u \in (0, L)$. With $t = L + 1$, we claim that σ_t does not hit c. Otherwise we would have $\sigma_t(r_t) = c(v)$ for some $v \in (u, L)$. Furthermore, since γ_0 is minimizing to c, we have $r_t \geqslant t$. Now the geodesic segments $\gamma_0|_{[1,t]}$, $\sigma_1|_{[0,r_1]}$, and $\sigma_t|_{[0,r_t]}$ together with $c|_{[u,v]}$ bound a disc which we may view as a part of the hyperbolic plane. Since σ_1 and σ_t are perpendicular to γ_0, we conclude that

$$d(\sigma_1(r_1), \sigma_t(r_t)) > d(\sigma_1(0), \sigma_t(0)) = L.$$

Since $c|_{[u,v]}$ connects $\sigma_1(r_1)$ with $\sigma_t(r_t)$, this contradicts $L(c) = L$. Hence σ_t satisfies one of the first two alternatives. □

In [Maz91], Mazzeo introduced the notion of asymptotically hyperbolic Riemannian metrics. In our setting, we now view a funnel F of S as a cylinder $[0, 1) \times \mathbb{R}/\mathbb{Z}$ with a boundary circle $z = \{1\} \times \mathbb{R}/\mathbb{Z}$ at infinity. We consider Riemannian metrics on F of the form $g = \rho^{-2}h$, where ρ is a smooth function on $Z = [0, 1] \times \mathbb{R}/\mathbb{Z}$ with $\{\rho = 0\} = z$ and $d\rho \neq 0$ along z and where h is a Riemannian metric on Z. Note that because of the factor ρ^{-2}, such a metric is complete on F with the boundary circle z located at infinity. By the proposition of [Maz88, p. 311], the curvature of g tends to $-(\partial\rho/\partial\nu)^2$ when approaching z, where $\partial\rho/\partial\nu$ denotes the h-normal derivative of ρ along z. Now it follows from [Maz91, Theorem 16] (see also the remark on locality in the first paragraph of [Maz91, § IV]) that F cannot carry a non-trivial L^2-solution φ of $\Delta\varphi = \lambda\varphi$ with $\lambda \geqslant 1/4$ if $|\partial\rho/\partial\nu| = 1$ along z (or, more generally, if $|\partial\rho/\partial\nu| \geqslant 1$ along z such that $\{|\partial\rho/\partial\nu| = 1\}$ contains a non-empty open subarc of z).

Theorem 5.1. *Let S be a surface of finite type with a complete Riemannian metric with $K \leqslant -1$ and with at least one asymptotically hyperbolic funnel F as above such that $|\partial\rho/\partial\nu| = 1$ along z. Then S has at most $-\chi(S)$ eigenvalues, counted with multiplicity, and all of them are contained in $(0, 1/4)$.*

Proof. By what we said above, S does not have eigenvalues greater than or equal to $1/4$. On the other hand, since $K \leqslant -1$, we have $\Lambda(S) \geqslant \lambda_0(\tilde{S}) \geqslant 1/4$. Hence all eigenvalues of S are small. By Theorem 1.5, there are at most $-\chi(S)$ of them. Since the area of S is infinite, 0 is not an eigenvalue, and hence all eigenvalues of S are contained in $(0, 1/4)$. □

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