Compositio Mathematica **136:** 61–67, 2003.

On Certain Isomorphisms between Absolute Galois Groups

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(Received: 27 November 2000; accepted in final form: 30 November 2001)

Abstract. Let k be an algebraically closed field of characteristic zero, F be an algebraically closed extension of k of transcendence degree one, and G be the group of automorphisms over k of the field F. The purpose of this note is to calculate the group of continuous automorphisms of G.

Mathematics Subject Classifications (2000). 12F20, 22D45.

Key words. field automorphisms, absolute Galois groups.

1. Introduction

Let k be an algebraically closed field of characteristic zero, L its finitely generated extension of transcendence degree > 1, and L' another finitely generated extension of k. It is a result of Bogomolov [3] that any isomorphism between $\mathrm{Gal}(\bar{L}/L)$ and $\mathrm{Gal}(\bar{L}'/L')$ is induced by an isomorphism of fields $\bar{L} \longrightarrow \bar{L}'$ identifying L with L'.

If the transcendence degree of L over k is one, the group $\mathrm{Gal}(\bar{L}/L)$ is free, and therefore, its structure tells nothing about the field L.

Let F be an algebraically closed extension of k of transcendence degree one, and $G = G_{F/k}$ be the group of automorphisms over k of the field F. Let the set of subgroups $U_L := \mathrm{Aut}(F/L)$ for all subfields L finitely generated over k be the basis of neighborhoods of the unity in G.

Let λ be a continuous automorphism of G. The purpose of this note is to show that if λ induces an isomorphism $\mathrm{Gal}(F/L) \xrightarrow{\sim} \mathrm{Gal}(F/L')$ then the fields L and L' are isomorphic (see Theorem 4.2 below for a more precise statement).

1.1. NOTATIONS

For a field F_1 and its subfield F_2 we denote by G_{F_1/F_2} the group of automorphisms of the field F_1 over F_2. Throughout the note k is an algebraically closed field of characteristic zero, F its algebraically closed extension of transcendence degree $1 \leqslant n < \infty$ and $G = G_{F/k}$. If K is a subfield of F then $\overline{K}$ denotes its algebraic closure in F.

★Supported in part by RFBR grant 99-01-01204.

For a topological group H we denote by H° its subgroup generated by the compact subgroups, and by H^{ab} the quotient of H by the closure of its commutant.

For a smooth projective curve C over a field, $\mathrm{Pic}^{\geqslant m}(C)$ is the submonoid in $\mathrm{Pic}(C)$ of sheaves of degree $\geqslant m$.

2. A Galois-Type Correspondence

We consider a topology on G with the basis of neighborhood of an automorphism $\sigma\colon F \xrightarrow{\sim} F$ over k given by the cosets of the form σU_L for all subfields L of F finitely generated over k, where $U_L = \mathrm{Aut}(F/L)$. This topology was introduced in [4]. One checks that the group G endowed with such topology is Hausdorff, locally compact, and totally disconnected.

PROPOSITION 2.1 ([4], Lemma 1, Section 3). *The map*

$$\{\text{subfields in } F \text{ over } k\} \longrightarrow \{\text{closed subgroups in } G\} \quad \textit{given by}$$

$K \mapsto \mathrm{Aut}(F/K)$ *is injective and restricts to bijections*

– $\{\text{subfields } K \text{ with } F = \bar{K}\} \leftrightarrow \{\text{compact subgroups of } G\}$*;*

– $\left\{\begin{matrix}\text{subfields } K \text{ of } F \text{ finitely} \\ \text{generated over } k \text{ with } F = \bar{K}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{compact open} \\ \text{subgroups of } G\end{matrix}\right\}.$

The inverse correspondences are given by $G \supset H \mapsto F^H \subseteq F$. □

Denote by G° the subgroup of G generated by the compact subgroups. Obviously, G° is an open normal subgroup in G.

3. Decomposition Subgroups in Abelian Quotients

Let $n = 1$. We are going to show that for any continuous automorphism λ of G and any L of finite type over k one has $\lambda(U_L) = U_{L'}$ for some L' isomorphic to L.

To do that we first need to construct decomposition subgroups in the Abelian quotients U_L^{ab}.

Set $\Phi_L = \mathrm{Hom}(\mathrm{Div}^0(C), \widehat{\mathbb{Z}}(1))$ for a smooth projective model C of L over k. By Kummer theory, $U_L^{\mathrm{ab}} = \mathrm{Hom}(L^{\times}, \widehat{\mathbb{Z}}(1))$, so, as the groups $k^{\times}$ and $\mathrm{Pic}^0(C)$ are divisible, but there are no divisible elements in $\widehat{\mathbb{Z}}(1)$ except 0, the short exact sequence $1 \longrightarrow L^{\times}/k^{\times} \longrightarrow \mathrm{Div}^0(C) \longrightarrow \mathrm{Pic}^0(C) \longrightarrow 0$ induces an embedding $\Phi_L \hookrightarrow U_L^{\mathrm{ab}}$. One identifies Φ_L with the $\widehat{\mathbb{Z}}$-module of the $\widehat{\mathbb{Z}}(1)$-valued functions on $C(k)$ modulo the constants.

The next step is to get a description of Φ_L in terms of the Galois groups. Clearly, $U_{k(x)}^{\mathrm{ab}} = \Phi_{k(x)}$.

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LEMMA 3.1.

(1) *If U is an open compact subgroup in G then $N_G(U) = N_{G^\circ}(U)$. If, moreover, $N_G(U)/U$ is infinite and has no Abelian subgroups of finite index then $U = U_{k(x)}$ for some $x \in F - k$.*
(2) *For any $x \in L - k$ the transfer $U^{\mathrm{ab}}_{k(x)} \longrightarrow U^{\mathrm{ab}}_L$ factors through Φ_L.*
(3) *The span of images of the transfers $U^{\mathrm{ab}}_{k(x)} \longrightarrow U^{\mathrm{ab}}_L$ for all $x \in L - k$ is dense in Φ_L.*

Proof. (1) By Proposition 2.1, $U = U_L$ for a field L finitely generated over k. Then the group $N_G(U_L)/U_L$ coincides with the group of automorphisms of the field L over k. As the automorphism groups of projective curves of genus > 1 are finite, if L is isomorphic to the function field of such a curve, then the normalizer of U in G is compact. As the automorphism groups of elliptic curves are generated by elements of order $\leqslant 4$ and contain Abelian subgroups of index $\leqslant 6$, if L is isomorphic to the function field of such a curve, then the normalizer of U in G is generated by its compact subgroups. This implies that if $N_G(U)/U$ has no Abelian subgroups of finite index then L should be the function field of a rational curve. As the automorphism group of the rational curve is generated by involutions, the normalizer of U in G is generated by its compact subgroups.

(2) The transfer is induced by the norm $L^\times/k^\times \xrightarrow{\mathrm{Nm}_{L/k(x)}} k(x)^\times/k^\times$, which is the restriction of the push-forward $\mathrm{Div}^0(C) \xrightarrow{x_*} \mathrm{Div}^0(\mathbb{P}^1)$. Since $k(x)^\times/k^\times = \mathrm{Div}^0(\mathbb{P}^1)$, the transfer factors through Φ_L.

(3) Each point p of a smooth projective model C of L over k is a difference of very ample effective divisors on C. These divisors themselves are zero-divisors of some rational functions, i.e., there are surjective morphisms $x, y\colon C \longrightarrow \mathbb{P}^1$ and a point $0 \in \mathbb{P}^1$ such that $x^{-1}(0) - y^{-1}(0) = p$. Then $\delta_p = x^*\delta_0 - y^*\delta_0\colon C(k) \longrightarrow \widehat{\mathbb{Z}}(1)$ is a δ-function of the point p of C. As the span of δ-functions is dense in the group Φ_L, we are done. □

For a point of $C(k)$ its *decomposition subgroup* in $\Phi_L \subset U^{\mathrm{ab}}_L$ consists of all functions supported on it. In the case $L = k(x)$ the decomposition subgroups in $U^{\mathrm{ab}}_{k(x)}$ are parametrized by the set (which is isomorphic to $\mathbb{P}^1(k)$) of parabolic subgroups P in $N_G U_{k(x)}/U_{k(x)}$. The subgroup D_P consists of elements in $U^{\mathrm{ab}}_{k(x)}$ fixed under the adjoint action of P. Clearly, $D_P \cong \widehat{\mathbb{Z}}(1)$.

Each inclusion of subgroups $U_L \subset U_{k(x)}$ induces a homomorphism $U^{\mathrm{ab}}_L \longrightarrow U^{\mathrm{ab}}_{k(x)}$. For any nonzero element h of the group U^{ab}_L, considered as a homomorphism from the group $L^\times$, there is an element $x \in L^\times$ with $h(x) \neq 0$, so the image of h in $U^{\mathrm{ab}}_{k(x)}$ is nonzero, and thus, the homomorphism $U^{\mathrm{ab}}_L \xrightarrow{\varphi_L} \prod_{x \in L-k} U^{\mathrm{ab}}_{k(x)}$ is injective.

To construct decomposition subgroups for an arbitrary L, consider such a subgroup $D \cong \widehat{\mathbb{Z}}$ in the target of φ_L that its projection to each of $U^{\mathrm{ab}}_{k(x)}$ is of finite index in some decomposition subgroup. Then our next goal is to show that the set of decomposition subgroups in U^{ab}_L coincides with the set of maximal subgroups among $\Phi_L \cap \varphi_L^{-1}(D)$.

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LEMMA 3.2 (= Lemma 5.2 of [2] = Lemma 3.4′ of [3]). *Let f be such a function on a projective space* $\mathbb{P}$ *over an infinite field that the restriction of f to each projective line in* $\mathbb{P}$ *is constant on the complement to a point on it. Then f is a flag function, i.e., there is a filtration* $P_0 \subset P_1 \subset P_2 \subset \ldots$ *of* $\mathbb{P}$ *by projective subspaces such that f is constant on* P_0 *and on all strata* $P_{j+1} - P_j$. □

The present form of the following lemma as well as its proof are suggested by the referee.

LEMMA 3.3. *For any smooth projective curve C and any* $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}^{\geqslant 4g-1}(C)$ *the natural map* $\Gamma(C, \mathcal{L}) \otimes \Gamma(C, \mathcal{L}') \longrightarrow \Gamma(C, \mathcal{L} \otimes \mathcal{L}')$ *is surjective.*

Proof. We may assume that $\deg \mathcal{L} \leqslant \deg \mathcal{L}'$. Fix an effective divisor D on C of degree $2g-1$ and a base point free pencil in $|\mathcal{L}(-D)|$ corresponding to a subspace V in $\Gamma(C, \mathcal{L}(-D))$ of dimension 2, and a subspace $V \subset W \subset \Gamma(C, \mathcal{L})$ of dimension 3 such that $W \otimes_k \mathcal{O}_C \longrightarrow \mathcal{L}$ is surjective. If R is its kernel, it fits into a natural short exact sequence

$$0 \longrightarrow \det V \otimes_k \mathcal{L}^\vee(D) \longrightarrow R \longrightarrow (W/V) \otimes_k \mathcal{O}_C(-D) \longrightarrow 0.$$

This shows that $H^1(C, R \otimes \mathcal{L}') = 0$, hence $W \otimes_k \Gamma(C, \mathcal{L}') \longrightarrow \Gamma(C, \mathcal{L} \otimes \mathcal{L}')$ is surjective. □

LEMMA 3.4. *If* $\varphi_L^{-1}(D)$ *is in* Φ_L *then it is a subgroup in a decomposition subgroup in* U_L^{ab}.

Proof. Let $f \in \varphi_L^{-1}(D) \cap \Phi_L$, i.e., $f\colon C(k) \longrightarrow \widehat{\mathbb{Z}}(1)$ for a smooth projective model C of L over k, and for any very ample invertible sheaf $\mathcal{L}$ on C restrictions of the induced function $f\colon |\mathcal{L}| \longrightarrow \widehat{\mathbb{Z}}(1)$ to projective lines in $|\mathcal{L}|$ are 'δ-functions' on them. Then, by Lemma 3.2, f is a flag function. Therefore, the function $\widehat{f}\colon |\mathcal{L}|^\vee \longrightarrow \widehat{\mathbb{Z}}(1)$ given by $H \longmapsto f$(general point of H) is a 'δ-function'.

Let g be the genus of C. Consider the composition $\widehat{f}_{\mathcal{L}}\colon C(k) \longrightarrow |\mathcal{L}|^\vee \xrightarrow{\widehat{f}} \widehat{\mathbb{Z}}(1)$. It takes x to $f(x) + f$(general point of $|\mathcal{L}(-x)|$). Since it is a 'δ-function', and all the hyperplanes $x + |\mathcal{L}(-x)|$ in $\mathcal{L}$ are pairwise distinct, there are such functions $b_0\colon \mathrm{Pic}^{\geqslant 2g+1}(C) \longrightarrow \widehat{\mathbb{Z}}(1)$ and $a\colon \mathrm{Pic}^{\geqslant 2g+1}(C) \longrightarrow C(k)$ that

$$f(x) + f(\text{general point of } |\mathcal{L}(-\mathrm{x})|) = b_0(\mathcal{L})\delta_{x,a(\mathcal{L})} + b_1(\mathcal{L}),$$

where $b_1\colon \mathrm{Pic}^{\geqslant 2g}(C) \longrightarrow \widehat{\mathbb{Z}}(1)$ is the function sending $\mathcal{L}$ to the general value of f on $|\mathcal{L}|$. Then $f(x) = b_0(\mathcal{L})\delta_{x,a(\mathcal{L})} + b_1(\mathcal{L}) - b_1(\mathcal{L}(-x))$.

By Lemma 3.3, for any $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}^{\geqslant 4g-1}(C)$ the image of the map $|\mathcal{L}| \times |\mathcal{L}'| \longrightarrow |\mathcal{L} \otimes \mathcal{L}'|$ of summation of divisors is not contained in any hyperplane in $|\mathcal{L} \otimes \mathcal{L}'|$. Then a sum of a general divisor in $|\mathcal{L}|$ and a general divisor in $|\mathcal{L}'|$ is a general divisor in the linear system $|\mathcal{L} \otimes \mathcal{L}'|$, so one has

$$b_1(\mathcal{L} \otimes \mathcal{L}') = b_1(\mathcal{L}) + b_1(\mathcal{L}'),$$

and therefore, for any sheaf $\mathcal{L}_0$ of degree zero one has

$$b_1(\mathcal{L}') + b_1(\mathcal{L}_0 \otimes \mathcal{L}) = b_1(\mathcal{L}) + b_1(\mathcal{L}_0 \otimes \mathcal{L}'),$$

so $b_2(\mathcal{L}_0) := b_1(\mathcal{L}_0 \otimes \mathcal{L}) - b_1(\mathcal{L}) \colon \mathrm{Pic}^0(C) \longrightarrow \widehat{\mathbb{Z}}(1)$ does not depend on $\mathcal{L}$. It is easy to see that b_2 is a homomorphism, which therefore should be zero, since $\mathrm{Pic}^0(C)$ is a divisible group. From this we conclude that $b_1(\mathcal{L}) = b_1(\deg \mathcal{L})$, and finally, $f(x) = b_0(\mathcal{L})\delta_{x,a(\mathcal{L})} + b_3(\mathcal{L})$ is a δ-function on $C(k)$, i.e., corresponds to a point of C, or to a decomposition subgroup in U_L^{ab}. □

4. Automorphisms of Subgroups between G° and G

LEMMA 4.1.

(1) *Suppose that for a subgroup H in G containing G° (the restriction to G° of) a homomorphism $\lambda \colon H \longrightarrow G$ induces the identity map of the set $\mathfrak{F}$ of compact open subgroups in G. Then $\lambda = id$.*
(2) *The centralizer of G° in $G_{F/\mathbb{Q}}$ is trivial.*

Proof. For any $\sigma \in H$ and any open compact subgroup U one has

$$\sigma U \sigma^{-1} = \lambda(\sigma U \sigma^{-1}) = \lambda(\sigma)\lambda(U)\lambda(\sigma)^{-1} = \lambda(\sigma)U\lambda(\sigma)^{-1},$$

so $\sigma^{-1}\lambda(\sigma)$ belongs to the normalizer of each U.

For a variety X of dimension n over k without birational automorphisms and any $x \in F - k$ there is a subfield $L_x \subset F$ containing x isomorphic to the function field of X. Then the normalizer of U_{L_x} coincides with U_{L_x}, and the intersection of all U_{L_x} is $\{1\}$, so $\sigma^{-1}\lambda(\sigma) = 1$. On the other hand, if $\tau \in G_{F/\mathbb{Q}}$ normalizes $U_{k(x,P(x)^{1/2})}$ for all polynomials P over k, then $\tau \in G_{F/k}$ and therefore, $\tau = 1$. □

Let $\mathfrak{F}$ be the set of compact open subgroups in G°, and let $\mathbb{Q}(\chi)$ be the quotient of the free Abelian group generated by $\mathfrak{F}$ by the relations $[U] = [U : U'] \cdot [U']$ for all $U' \subset U$. As the intersection of a pair of a compact open subgroups in G is a subgroup of finite index in both of them, $\mathbb{Q}(\chi)$ is a one-dimensional $\mathbb{Q}$-vector space. The group G acts on it by the conjugations. Let χ be the character of this representation of G.

One can get an explicit formula for χ as follows. Fix a subfield L of F finitely generated and of transcendence degree n over k. Then for any $\sigma \in G$ one has

$$[U_L] = [L\sigma(L) : L] \cdot [U_{L\sigma(L)}] \quad \text{and} \quad [U_{\sigma(L)}] = [L\sigma(L) : \sigma(L)] \cdot [U_{L\sigma(L)}],$$

and therefore, $\chi(\sigma) = [L\sigma(L) : \sigma(L)]/[L\sigma(L) : L]$. This implies that $\chi \colon G \longrightarrow \mathbb{Q}_+^\times$ is surjective, and its restriction to G° is trivial.

For a subgroup H in G let $N_{G_{F/\mathbb{Q}}}(H)$ be its normalizer in $G_{F/\mathbb{Q}}$.

THEOREM 4.2. *Let $n = 1$, H be a subgroup in G containing G°. Then* $N_{G_{F/\mathbb{Q}}}(H) \subseteq N_{G_{F/\mathbb{Q}}}(G) = \{\text{automorphisms of } F \text{ preserving } k\}$*, and the adjoint action of*

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$N_{G_{F/\mathbb{Q}}}(H)$ on H induces an isomorphism from $N_{G_{F/\mathbb{Q}}}(H)$ to the group of continuous open automorphisms of H.

If $H \supseteq \ker\chi$ then $N_{G_{F/\mathbb{Q}}}(H) = N_{G_{F/\mathbb{Q}}}(G^\circ)$.

Proof. For each $U \in \mathfrak{F}$ let Div_U^+ be the free Abelian semi-group, whose generators are decomposition subgroups in U^{ab}, and for each integer $d \geqslant 2$ let

$$\mathfrak{Gr}_U^{(d)} = \{U_L \supset U \,|\, [U_L : U] = d, L \cong k(t)\} \subset \mathfrak{F}.$$

For a smooth projective model C of F^U the set $\mathfrak{Gr}_U^{(d)}$ is in bijection with the disjoint union of Zariski-open subsets in Grassmannians:

$$\coprod_{\mathcal{L}\in\mathrm{Pic}^d(C)} \left(\mathrm{Gr}(1, |\mathcal{L}|) - \bigcup_{x\in C(k)} \mathrm{Gr}(1, x + |\mathcal{L}(-x)|)\right).$$

One can define

- an '*invertible sheaf of degree d without base points*' $\mathcal{L}$, as a subset of $\mathfrak{Gr}_U^{(d)} \subset \mathfrak{F}$ consisting of elements equivalent under the relation generated by $U_1 \sim_U U_2$ if there are decomposition subgroups $D_a \subset U_1^{\mathrm{ab}}$ and $D_b \subset U_2^{\mathrm{ab}}$ such that their pre-images in U^{ab} contain the same collections of decomposition subgroups with the same indices of their images in D_a and D_b;
- the '*linear system*' $|\mathcal{L}|$, as the set of maximal collections of elements of $\mathcal{L}$ 'intersecting at a single point', i.e., as the subset of the free Abelian semi-group Div_U^+;
- a '*line presented in* $\mathcal{L}$' in $|\mathcal{L}|$, as an element of $\mathcal{L} \subset \mathfrak{Gr}_U^{(d)}$, considered as a subset in $|\mathcal{L}|$;
- an arbitrary '*line*' in $|\mathcal{L}|$, as a subset in $|\mathcal{L}|$ of type $D + l$, where $D \in \mathrm{Div}_U^+$ and l is a line presented in the sheaf $\mathcal{L}(-D)$ without base points;
- an '*s-subspace*' in $|\mathcal{L}|$, as the union of all lines passing through a given point in $|\mathcal{L}|$ and intersecting a given '$(s-1)$-subspace' in $|\mathcal{L}|$.

Now we remark that for any sufficiently big d and any sheaf $\mathcal{L} \subset \mathfrak{Gr}_U^{(d)}$ the set C_U of decomposition subgroups in U^{ab} can be *canonically* identified with the subset of $|\mathcal{L}|^\vee$ consisting of those hyperplanes in $|\mathcal{L}|$ that each line on each of them is 'absent in $\mathcal{L}$'. As $|\mathcal{L}|^\vee$ has a canonical structure of a projective space (but not of a projective space over k), this gives us a *canonical* structure of a scheme on C_U. Let κ_U be the function field of C_U.

Clearly, $\lambda(G^\circ) = G^\circ$ and the restriction of λ to G° induces a bijection $\mathfrak{Gr}_U^{(d)} \xrightarrow{\sim} \mathfrak{Gr}_{\lambda(U)}^{(d)}$ for each $d \geqslant 2$, and for any sheaf $\mathcal{L} \subset \mathfrak{Gr}_U^{(d)}$ it induces a map $|\mathcal{L}| \longrightarrow |\lambda(\mathcal{L})|$ which transforms subspaces into subspaces (of the same dimension), i.e., a collineation. As λ induces a collineation $|\mathcal{L}|^\vee \xrightarrow{\sim} |\lambda(\mathcal{L})|^\vee$, the fundamental theorem of projective geometry (see, e.g., [1]) implies that such λ induces an isomorphism $C_U \xrightarrow{\sim} C_{\lambda(U)}$ of schemes over $\mathbb{Q}$. This isomorphism does not depend on d and $\mathcal{L}$, since it determines the collineations $|\mathcal{L}'| \xrightarrow{\sim} |\lambda(\mathcal{L}')|$ for all $\mathcal{L}' \subset \mathfrak{Gr}_U^{(d')}$. Denote by σ_U the induced isomorphism $\kappa_{\lambda(U)} \xrightarrow{\sim} \kappa_U$.

For each subgroup U' of finite index in U the natural map $C_{U'} \longrightarrow C_U$ is a morphism of schemes, and in particular, κ_U is naturally embedded into $\kappa_{U'}$. The group G° acts on the field $\lim_{U\longrightarrow} \kappa_U$. By Lemma 4.1 (2), the centralizer if G° in $G_{F/\mathbb{Q}}$ is trivial, and therefore, there is a unique isomorphism $\lim_{U\longrightarrow} \kappa_U \stackrel{\sim}{\longrightarrow} F$ commuting with the G°-action. Since the diagram

$$\begin{array}{ccc} C_{U'} & \longrightarrow & C_{\lambda(U')} \\ \downarrow & & \downarrow \\ C_U & \longrightarrow & C_{\lambda(U)} \end{array}$$

commutes, the restriction of $\sigma_{U'}$ to κ_U coincides with σ_U, and finally, we get an automorphism σ of F induced by λ. As k is the only maximal algebraically closed subfield in its arbitrary finitely generated extension, σ induces an automorphism of k, and therefore, normalizes G°.

Then the restriction to G° of $\mathrm{ad}(\sigma) \circ \lambda$ acts trivially on all of $\mathfrak{Gr}^{(d)}_{U'}$. As any open compact subgroup is an intersection of elements of $\mathfrak{Gr}^{(d)}_{U'}$ for d big enough and U' small enough, $\mathrm{ad}(\sigma) \circ \lambda$ acts on $\mathfrak{F}$ also trivially. By Lemma 4.1 (1), this implies that $\lambda = \mathrm{ad}(\sigma^{-1})$. □

Remark. If k is countable then the inverse of any continuous automorphism as in the statement of Theorem 4.2 is automatically continuous:

LEMMA 4.3. *If k is countable, and $U \stackrel{\lambda}{\longrightarrow} U'$ is a continuous surjective homomorphism of open subgroups in $G_{F/k}$ and $G_{F'/k'}$ then the image in U' of an open subset in U is open.*

Proof. Let $U_L \subset U$ be an open compact subgroup. Then U/U_L is a countable set surjecting onto the set $U'/\lambda(U_L)$. By Proposition 2.1, for the subfield $L' = F^{\lambda(U_L)}$ one has $\bar{L}' = F$. If $\lambda(U_L)$ is not open then L' is not finitely generated over k', and therefore, $U'/\lambda(U_L)$ is not countable. □

References

1. Artin, E.: *Geometric Algebra*, Interscience, New York, 1957.
2. Bogomolov, F. A.: Abelian subgroups of Galois groups, *Izv. AN SSSR, ser. matem.* **55**(1) (1991), 32–67.
3. Bogomolov, F. A.: On two conjectures in birational algebraic geometry, In: *Algebraic Geometry and Analytic Geometry* (*Tokyo, 1990*), ICM-90 Satell. Conf. Proc., Springer, Tokyo, 1991, pp. 26–52.
4. Piatetskii-Shapiro, I. I. and Shafarevich, I. R.: Galois theory of transcendental extensions and uniformization, *Izv. AN SSSR, ser. matem.* **30** (1966), 671–704.