

WHEN IS A COMPLETION OF THE UNIVERSAL ENVELOPING ALGEBRA A BANACH PI-ALGEBRA?

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Abstract

We prove that a Banach algebra B that is a completion of the universal enveloping algebra of a finite-dimensional complex Lie algebra \mathfrak{g} satisfies a polynomial identity if and only if the nilpotent radical \mathfrak{n} of \mathfrak{g} is associatively nilpotent in B . Furthermore, this holds if and only if a certain polynomial growth condition is satisfied on \mathfrak{n} .

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Introduction

The theory of representations of a finite-dimensional complex Lie algebra \mathfrak{g} on a Banach space and, more generally, Lie-algebra homomorphisms from \mathfrak{g} to a Banach algebra, unexpectedly turned out to be not as trivial as it might seem at first glance (see the book [7]). Indeed, in the solvable case, all irreducible representations of \mathfrak{g} are one-dimensional. Moreover, it was proved by Taylor that in the semisimple case, all (not only irreducible) representations of \mathfrak{g} are finite dimensional [23]. Nevertheless, a general representation on a Banach space can be quite complicated even when \mathfrak{g} is nilpotent.

Here we consider arbitrary finite-dimensional Lie algebras but restrict to representations (and homomorphisms) with range generating a Banach algebra satisfying a polynomial identity (a PI-algebra). We give an answer to the question: *When does a Banach algebra that is a completion of the universal enveloping algebra $U(\mathfrak{g})$ satisfy a PI?* We do this by providing several necessary and sufficient conditions (algebraic and analytic) in terms of the nilpotent radical \mathfrak{n} of \mathfrak{g} (Theorems 1 and 2). We also include some examples, which show how the criterion works in concrete cases.

According to the nature of the conditions, the argument can be divided in two parts, algebraic and analytic. The proof of the algebraic part is based on two results in algebraic PI-theory, a theorem of Bahturin [6] on PI-quotients in the semisimple

case and the Braun–Kemer–Razmyslov theorem on the nilpotency of the Jacobson radical. In the proof of the analytic part, we use a result of Turovskii [24] on topological nilpotency, the theorem of Taylor on semisimple Lie algebras mentioned above and a small piece of the theory of generalised scalar operators.

This study was initially motivated by noncommutative spectral theory. In [13], Dosi considered a Fréchet algebra $\mathfrak{F}_{\mathfrak{g}}$, ‘the algebra of formally-radical functions’, associated with a nilpotent Lie algebra \mathfrak{g} . His goal was applications to noncommutative spectral theory (see [10, 11, 12, 14]). (Dosi’s construction is extended to the case of a general solvable Lie algebra in the author’s article [3]. It is also shown there that $\mathfrak{F}_{\mathfrak{g}}$ is an Arens–Michael algebra, that is, it can be approximated by Banach algebras.) It is proved in [13, Lemma 4] that in the nilpotent case, an embedding \mathfrak{g} to a Banach algebra B can be extended to a continuous homomorphism $\mathfrak{F}_{\mathfrak{g}} \rightarrow B$ if and only if \mathfrak{g} is supernilpotent, that is, every element of $[\mathfrak{g}, \mathfrak{g}]$ (which is equal to \mathfrak{n} in this case) is associatively nilpotent. This paper comes from a desire to understand what form this condition can take in the general case. Our main results show that the property to satisfy a PI, which is formally weaker than the supernilpotency, is reasonable in this context.

This paper is just the beginning of work on PI-completions. As a continuation of the study, we discuss such completions of the algebra of analytic functionals on a complex Lie group in subsequent articles [4, 5], which contain the following topics: a generalisation of Theorem 1, a relationship with large-scale geometry and a decomposition into an analytic smash product.

Statement of the main results

An associative algebra A (in our case, over a field, which usually is \mathbb{C}) *satisfies a polynomial identity* (in short, a PI-algebra) if there is a nontrivial noncommutative polynomial p (that is, an element of a free algebra on n generators) such that $p(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in A$. PI-algebras can be both unital and nonunital, but we usually assume that associative algebras are unital unless otherwise stated. Banach PI-algebras are discussed in [19, 21], but we do not use the results contained there. For up-to-date information on general PI-algebras, see [1] or [18].

The *nilpotent radical* \mathfrak{n} of a Lie algebra \mathfrak{g} is the intersection of the kernels of all irreducible representations of \mathfrak{g} (see [8, Ch. I, Section 5.3, Definition 3, page 44] or [9, Section 1.7.2, page 27]).

The following two theorems are our main results.

THEOREM 1. *Suppose that \mathfrak{g} is a finite-dimensional complex Lie subalgebra of a Banach algebra B . Let $\|\cdot\|$ denote the norm on B and \mathfrak{n} the nilpotent radical of \mathfrak{g} . If B is generated by \mathfrak{g} as a Banach algebra, then the following conditions are equivalent:*

- (1) B is a PI-algebra;
- (2a) every element of \mathfrak{n} is nilpotent;
- (2b) the nonunital associative subalgebra of B generated by \mathfrak{n} is nilpotent;

- (3a) $e^b - 1$ is nilpotent for every $b \in \mathfrak{n}$;
 (3b) there is $d \in \mathbb{N}$ such that $e^b - 1$ is nilpotent of degree at most d for every $b \in \mathfrak{n}$;
 (4) there are $C > 0$ and $\alpha > 0$ such that $\|e^b\| \leq C(1 + \|b\|)^\alpha$ for every $b \in \mathfrak{n}$.

Note that the conditions (2b) and (3b) are the uniform versions of the point-wise conditions (2a) and (3a).

In Theorem 1, we assume that \mathfrak{g} is embedded into B . Is not hard to extend the result to the case of an arbitrary Lie-algebra homomorphism $\mathfrak{g} \rightarrow B$ (or equivalently, a homomorphism $U(\mathfrak{g}) \rightarrow B$). Namely, the following result holds.

THEOREM 2. *Suppose that \mathfrak{g} is a finite-dimensional complex Lie algebra, B is a Banach algebra and $\theta: U(\mathfrak{g}) \rightarrow B$ is a homomorphism of associative algebras. Let $\|\cdot\|$ be the norm on B , $|\cdot|$ a norm on \mathfrak{n} and $U(\mathfrak{n})_0$ the augmentation ideal of $U(\mathfrak{n})$, that is, the kernel of the trivial representation. If θ has dense range, then the following conditions are equivalent:*

- (1) B is a PI-algebra;
 (2a) $\theta(\eta)$ is nilpotent for every $\eta \in \mathfrak{n}$;
 (2b) the nonunital associative algebra $\theta(U(\mathfrak{n})_0)$ is nilpotent;
 (3a) $e^{\theta(\eta)} - 1$ is nilpotent for every $\eta \in \mathfrak{n}$;
 (3b) there is $d \in \mathbb{N}$ such that $e^{\theta(\eta)} - 1$ is nilpotent of degree at most d for every $\eta \in \mathfrak{n}$;
 (4) there are $C > 0$ and $\alpha > 0$ such that $\|e^{\theta(\eta)}\| \leq C(1 + |\eta|)^\alpha$ for every $\eta \in \mathfrak{n}$.

The proof of Theorem 2 is placed after the proof of Theorem 1, to which we now turn.

The algebraic argument

As mentioned in the Introduction, the proof of the theorem is divided into two parts, algebraic and analytic. The algebraic part is in the following proposition.

PROPOSITION 3. *Let \mathfrak{g} be a finite-dimensional Lie subalgebra of an associative algebra A over a field of characteristic 0, \mathfrak{s} a Levi subalgebra and \mathfrak{n} the nilpotent radical of \mathfrak{g} . Suppose that A is generated by \mathfrak{g} as an associative algebra. Then A is a PI-algebra if and only if the unital subalgebra generated by \mathfrak{s} is finite dimensional and every element of \mathfrak{n} is nilpotent.*

For the proof, we need three lemmas.

LEMMA 4. *Let A be an associative algebra over a field, \mathfrak{g} a finite-dimensional Lie subalgebra that generates A as an associative algebra and \mathfrak{n} the nilpotent radical of \mathfrak{g} .*

- (A) *Then \mathfrak{n} is contained in the Jacobson radical of A .*
 (B) *If, in addition, A is a PI-algebra, then \mathfrak{n} is associatively nilpotent, that is, there is $d \in \mathbb{N}$ such that $b_1 \dots b_d = 0$ for all $b_1, \dots, b_d \in \mathfrak{n}$.*

The proof of part (B) is based on a deep result in PI-theory, the Braun–Kemer–Razmyslov theorem on the nilpotency of the Jacobson radical.

PROOF. It is obvious that A is a quotient of $U(\mathfrak{g})$ and, in particular, finitely generated.

(A) Recall that the Jacobson radical, $\text{Rad } A$, of A is the intersection of the kernels of all irreducible representations of A . However, the nilpotent radical \mathfrak{n} of the Lie algebra \mathfrak{g} is the intersection of the kernels of all irreducible representations of \mathfrak{g} . Since irreducible representations of $U(\mathfrak{g})$ are in one-to-one correspondence with irreducible representations of \mathfrak{g} and every irreducible representation of A can be lifted to an irreducible representation of $U(\mathfrak{g})$, it follows that $\mathfrak{n} \subset \text{Rad } A$.

(B) The Braun–Kemer–Razmyslov theorem asserts that the Jacobson radical of a finitely generated PI-algebra over a commutative Jacobson ring (in particular, over a field) is nilpotent [18, Theorem 4.0.1, page 149]. This completes the proof. \square

A theorem of Wedderburn asserts that a finite-dimensional associative algebra over a field (of arbitrary characteristic) with linear basis consisting of nilpotent elements is nilpotent (see, for example, [16, Theorem 2.3.1, page 56]). We need a more general result in characteristic 0: if \mathfrak{h} is a Lie subalgebra of an associative algebra and \mathfrak{h} is generated by finitely many associatively nilpotent elements, then the nonunital associative subalgebra generated by the solvable radical of \mathfrak{h} is nilpotent [17, Theorem 8]. In particular, we have the following lemma.

LEMMA 5. *Let A_0 be a nonunital associative algebra over a field of characteristic 0. Suppose that A_0 is generated by a nilpotent Lie subalgebra \mathfrak{h} . If every element of \mathfrak{h} is nilpotent, then so is A_0 .*

REMARK 6. In [10, Lemma 2.2], Dosi gave a direct proof of the lemma in the particular case when $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}]$ for a nilpotent \mathfrak{g} .

LEMMA 7. *An extension of a PI-algebra (over an arbitrary commutative ring) by an ideal that is a PI-algebra is also a PI-algebra.*

PROOF. Let I be an ideal of an algebra A . Suppose that A/I and I satisfy polynomial identities p and q , respectively. Let n and m be the numbers of variables in p and q , respectively. If $a_{ij} \in A$ ($i = 1, \dots, n, j = 1, \dots, m$), then $p(a_{1j}, \dots, a_{nj}) \in I$ for every j because $p(a_{1j} + I, \dots, a_{nj} + I) \subset I$. Hence, $q(p(a_{11}, \dots, a_{n1}), \dots, p(a_{1m}, \dots, a_{nm})) = 0$. It is easy to see that since p and q are not trivial, this noncommutative polynomial is not trivial. Thus A is a PI-algebra. \square

PROOF OF PROPOSITION 3. Denote by S the unital subalgebra of A generated by \mathfrak{s} and suppose that A is a PI-algebra. Then S is a quotient of $U(\mathfrak{s})$ and satisfies a PI. Since we work in characteristic 0 and \mathfrak{s} is semisimple, we can apply a result of Bahturin, which asserts that every quotient of $U(\mathfrak{s})$ that is a PI-algebra is finite dimensional (see [6, Theorem 1 and Corollary]). In particular, S is finite dimensional. However, it follows from Lemma 4 that every element of \mathfrak{n} is nilpotent. The necessity is proved.

Suppose now that S is finite dimensional and every element of \mathfrak{n} is nilpotent. Denote by I the ideal of A generated by \mathfrak{n} . From Lemma 7, it suffices to show that I and A/I are PI-algebras.

Denote by A_0 the nonunital subalgebra of A generated by \mathfrak{n} and by $U(\mathfrak{n})_0$ the augmentation ideal of $U(\mathfrak{n})$. Note that \mathfrak{n} is a nilpotent Lie algebra. So by Lemma 5, there is a $d \in \mathbb{N}$ such that $A_0^d = 0$. Since \mathfrak{n} is a Lie ideal of \mathfrak{g} , it easily follows that $U(\mathfrak{g})U(\mathfrak{n})_0 = U(\mathfrak{n})_0U(\mathfrak{g})$. Then, $(U(\mathfrak{g})U(\mathfrak{n})_0)^d \subset U(\mathfrak{n})_0^dU(\mathfrak{g})$ and therefore $I^d = A_0^dA = 0$. Hence, I is nilpotent and, in particular, a PI-algebra.

On the other hand, being reductive, $\mathfrak{g}/\mathfrak{n}$ is a direct sum of a semisimple summand and an abelian summand \mathfrak{s} [9, Proposition 1.7.3, page 27]. Note that the restriction of the map $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{n}$ to \mathfrak{s} is injective and so we can identify the semisimple summand in $\mathfrak{g}/\mathfrak{n}$ with \mathfrak{s} . Consider the naturally defined homomorphisms $S \rightarrow A \rightarrow A/I$ and $U(\mathfrak{s}) \rightarrow U(\mathfrak{g}/\mathfrak{n}) \rightarrow A/I$. Their ranges commute since $U(\mathfrak{s})$ and $U(\mathfrak{a})$ commute in $U(\mathfrak{g}/\mathfrak{n})$ and the diagram

$$\begin{array}{ccc} \mathfrak{s} & \longrightarrow & \mathfrak{g}/\mathfrak{n} \\ \downarrow & & \downarrow \\ A & \longrightarrow & A/I \end{array}$$

is commutative. By the universal property of tensor products of associative algebras, we have the induced homomorphism $S \otimes U(\mathfrak{s}) \rightarrow A/I$. This homomorphism is surjective because A/I and $S \otimes U(\mathfrak{s})$ are quotients of $U(\mathfrak{g})/(U(\mathfrak{g})U(\mathfrak{n})_0)$ and $U(\mathfrak{s}) \otimes U(\mathfrak{a})$, respectively, and the last two algebras are isomorphic being isomorphic to $U(\mathfrak{g}/\mathfrak{n})$. By Regev’s theorem [18, Theorem 3.4.7, page 138], the class of PI-algebras is stable under tensor products. Thus, $S \otimes U(\mathfrak{s})$ is a PI-algebra since it is the tensor product of a finite-dimensional and abelian algebra. Being a quotient of a PI-algebra, A/I also satisfies a PI. This completes the proof of the sufficiency. \square

The analytic argument

Now we turn to auxiliary lemmas, which are needed in the analytic part of the proof of Theorem 1. The first lemma is a corollary of a result of Turovskii in [24].

LEMMA 8. *Suppose that the hypotheses in Theorem 1 hold. Then $\mathfrak{n} \subset \text{Rad } B$.*

PROOF. Note that $\mathfrak{n} = [\mathfrak{g}, \mathfrak{r}]$, where \mathfrak{r} is the solvable radical of \mathfrak{g} [9, Proposition 1.7.1, page 26]. Turovskii’s theorem [24] asserts that $[\mathfrak{g}, \mathfrak{h}] \subset \text{Rad } B$ for every solvable ideal \mathfrak{h} of \mathfrak{g} (for a proof, see [7, Section 24, Theorem 1, page 130]). By putting $\mathfrak{h} = \mathfrak{r}$, we have the result. \square

An element of a Banach algebra is *topologically nilpotent* if $\|b^n\|^{1/n} \rightarrow 0$ as $n \rightarrow \infty$.

LEMMA 9. *Let b be a topologically nilpotent element of a Banach algebra. Then b is nilpotent if and only if $e^b - 1$ is nilpotent. Moreover, b has degree of nilpotency at most d if and only if $e^b - 1$ does.*

PROOF. Put $r := e^b - 1$. If $b^d = 0$ for some $d \in \mathbb{N}$, then $r = b + \dots + b^{d-1}/(d-1)!$ and therefore $r^d = 0$.

Suppose now that there is $d \in \mathbb{N}$ such that $r^d = 0$. Since b is topologically nilpotent, its spectrum is $\{0\}$. Let f be the function on \mathbb{C} such that $f(z) = (e^z - 1)/z$ when $z \neq 0$ and $f(0) = 1$. Since f is holomorphic, it follows from the spectral mapping theorem [15, Theorem 2.2.23] that the spectrum of $f(b)$ is $\{1\}$ and therefore $f(b)$ is invertible. Since $bf(b) = e^b - 1 = r$ and $r^d = 0$, we have $b^d f(b)^d = 0$. Since $f(b)$ is invertible, $b^d = 0$. □

LEMMA 10. *Let r be a topologically nilpotent element of a Banach algebra with the norm $\|\cdot\|$. Then the following conditions are equivalent:*

- (1) *there are $C > 0$ and $\alpha > 0$ such that $\|(1 + r)^k\| \leq C(1 + |k|)^\alpha$ for all $k \in \mathbb{Z}$;*
- (2) *r is nilpotent.*

Moreover, we can assume that C and α depend only on $\|r\|$ and the degree of nilpotency of r .

Note that $(1 + r)^k$ is well defined for negative k because the spectrum of r is $\{0\}$.

PROOF. If (1) holds, then $1 + r$ is generalised scalar [20, Theorem 1.5.12, page 66], that is, admits a C^∞ -functional calculus on \mathbb{C} . (This result and the proposition cited below are stated for operators but, in fact, the arguments for them use only the Banach algebra structure.) Shifting by a number does not change the property of being generalised scalar. Thus, r is both topologically nilpotent and generalised scalar. It follows from [20, Proposition 1.5.10, page 64] that r is nilpotent.

However, if r is nilpotent, then we immediately have the desired estimate for positive k with constants $C > 0$ and $\alpha > 0$ depending only on the degree of nilpotency of r , say d , and $\|r\|$. To prove it for negative k , note that $(1 + r)^{-1} = 1 + r'$ for some nilpotent r' whose degree of nilpotency and norm depend only on d and $\|r\|$. It follows that (1) holds with C and α also depending only on d and $\|r\|$. □

PROOF OF THEOREM 1. We show that (1) \Leftrightarrow (2a) and (2a) \Rightarrow (2b) \Rightarrow (3b) \Rightarrow (4) \Rightarrow (3a) \Rightarrow (2a). Let \mathfrak{s} be a Levi subalgebra of \mathfrak{g} , A the associative unital subalgebra of B generated by \mathfrak{g} and A_0 the nonunital subalgebra of A generated by \mathfrak{n} .

(1) \Leftrightarrow (2a). Since \mathfrak{s} is semisimple, it follows from a result of Taylor [23] (see also [7] or [2]) that the image of $U(\mathfrak{s})$ in a Banach algebra is finite dimensional. Being dense in B , the subalgebra A satisfies a PI if and only if so does B . Applying Proposition 3 to A , we conclude that (1) and (2a) are equivalent.

(2a) \Rightarrow (2b). Applying Lemma 5 to A_0 shows that A is nilpotent when every element of \mathfrak{n} is nilpotent.

(2b) \Rightarrow (3b). Suppose that A_0 is nilpotent, that is, there is $d \in \mathbb{N}$ such that $A_0^d = 0$. In particular, every b in \mathfrak{n} is of degree of nilpotency at most d and so is $e^b - 1$ by Lemma 9.

(3b) \Rightarrow (4). Let d be a positive integer such that $e^b - 1$ is nilpotent of degree at most d for every $b \in \mathfrak{n}$. Applying the implication (2) \Rightarrow (1) in Lemma 10 shows that there are

constants $C > 0$ and $\alpha > 0$ such that

$$\|e^{kb}\| \leq C(1 + |k|)^\alpha \quad \text{for } b \in \mathfrak{n} \text{ and } k \in \mathbb{Z} \quad (4.1)$$

when $\|e^b - 1\| \leq 1$. Therefore, there is $\varepsilon > 0$ such that (4.1) holds when $\|b\| \leq \varepsilon$.

Now fix a nonzero element b of \mathfrak{n} . Take $k \in \mathbb{N}$ such that $(k - 1)\varepsilon \leq \|b\| \leq k\varepsilon$ and put $b' := k^{-1}b$. Then $\|b'\| \leq \varepsilon$ and

$$\|e^b\| = \|e^{kb'}\| \leq C(1 + k)^\alpha. \quad (4.2)$$

Since $k \leq 1 + \varepsilon^{-1}\|b\|$, we have

$$1 + k \leq 2 + \varepsilon^{-1}\|b\| \leq 2\left(1 + \frac{\varepsilon^{-1}}{2}\|b\|\right)$$

Combining this with (4.2) shows that there is $C' > 0$ such that $\|e^b\| \leq C'(1 + \|b\|)^\alpha$ and C' is independent in b .

(4) \Rightarrow (3a). Suppose that for some $C > 0$ and $\alpha > 0$, the inequality $\|e^b\| \leq C(1 + \|b\|)^\alpha$ holds when $b \in \mathfrak{n}$. We claim that $r := e^b - 1$ is topologically nilpotent whenever $b \in \mathfrak{n}$. Indeed, Lemma 8 implies that $b \in \text{Rad } B$. Note that $r = \sum_{n=1}^{\infty} b^n/n!$ and $\text{Rad } B$ is closed. So r is also contained in $\text{Rad } B$. Being an element of the radical of a Banach algebra, r is topologically nilpotent.

Further, it follows from the assumption that

$$\|(1 + r)^k\| = \|e^{kb}\| \leq C(1 + \|kb\|)^\alpha \leq C(1 + \|b\|)^\alpha(1 + |k|)^\alpha$$

for every $k \in \mathbb{Z}$. So by the implication (1) \Rightarrow (2) in Lemma 10, r is nilpotent.

(3a) \Rightarrow (2a). Suppose that $e^b - 1$ is nilpotent for every $b \in \mathfrak{n}$. Then it follows immediately from Lemma 9 that b is nilpotent.

This completes the proof of Theorem 1. \square

PROOF OF THEOREM 2. To deduce Theorem 2 from Theorem 1, note that every Lie-algebra homomorphism $\phi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ maps the nilpotent radical of \mathfrak{g}_1 into the nilpotent radical of \mathfrak{g}_2 . If, in addition, ϕ is surjective, then so is the Lie-algebra homomorphism $\mathfrak{r}_1 \rightarrow \mathfrak{r}_2$ of the solvable radicals (see, for example, [3, Lemma 4.10]). Since the nilpotent radicals of \mathfrak{g}_1 and \mathfrak{g}_2 coincide respectively with $[\mathfrak{g}_1, \mathfrak{r}_1]$ and $[\mathfrak{g}_2, \mathfrak{r}_1]$ (see the reference in the proof of Lemma 8), we have a surjective Lie-algebra homomorphism of the nilpotent radicals. Thus, (1), (2a), (2b), (3a) and (3b) are equivalent since so are the corresponding conditions in Theorem 1.

The proof of the implication (4) \Rightarrow (3a) is a slight modification of the argument for the corresponding implication in Theorem 1. First note that there is $K > 0$ such that

$$\|\theta(\eta)\| \leq K|\eta| \quad \text{for every } \eta \in \mathfrak{n}. \quad (4.3)$$

Then writing $\theta(\eta)$ instead of b and using (4.3), we deduce from the inequality $\|e^{\theta(\eta)}\| \leq C(1 + |\eta|)^\alpha$ for $\eta \in \mathfrak{n}$ that

$$\|(1 + r)^k\| \leq C(1 + |\eta|)^\alpha(1 + |k|)^\alpha$$

and then apply Lemma 10.

Finally, if (1) holds, then so does (4) in Theorem 1 for elements b of the form $\theta(\eta)$ with $\eta \in \mathfrak{n}$. It follows from (4.3) that (4) in Theorem 2 is also satisfied. \square

Some examples

There are many finite-dimensional Lie algebras such that every completion of the universal enveloping algebra with respect to a submultiplicative prenorm (that is, every completion that is a Banach algebra) satisfies a polynomial identity. The two simplest cases follow.

PROPOSITION 11. *If a finite-dimensional complex Lie algebra \mathfrak{g} is reductive, then every completion of $U(\mathfrak{g})$ with respect to a submultiplicative prenorm satisfies a PI.*

PROOF. Since \mathfrak{g} is reductive, the nilpotent radical is trivial. Thus by Theorem 2, every completion of $U(\mathfrak{g})$ with respect to a submultiplicative prenorm satisfies a PI. \square

PROPOSITION 12. *Let \mathfrak{g} be a two-dimensional nonabelian complex Lie algebra. Then every completion of $U(\mathfrak{g})$ with respect to a submultiplicative prenorm satisfies a PI.*

PROOF. There is a basis e_1, e_2 in \mathfrak{g} such that $[e_1, e_2] = e_2$. Note that $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}] = \mathbb{C}e_2$. Let B be a Banach-algebra completion of $U(\mathfrak{g})$ and π the corresponding Lie-algebra homomorphism from \mathfrak{g} to B . It is not hard to see that $\pi(e_2)$ is nilpotent (see, for example, [22, Example 5.1]). Then B satisfies a PI by Theorem 2. \square

However, there are Lie algebras whose universal enveloping algebra admits a completion that does not satisfy a PI. Here we give a particular case.

PROPOSITION 13. *Let \mathfrak{g} be a nonabelian nilpotent complex Lie algebra \mathfrak{g} . Then there is a submultiplicative prenorm on $U(\mathfrak{g})$ such that the completion does not satisfy a PI.*

PROOF. Since \mathfrak{g} is nonabelian and nilpotent, there is a sequence (θ_n) of representations of $U(\mathfrak{g})$ on (finite-dimensional) Hilbert spaces such that the sequence (d_n) of the corresponding degrees of nilpotency of $\theta_n(\mathfrak{n})$ is unbounded (see, for example, [3, Proposition 4.14 and Lemma 4.16]). Using renormalisation, we can assume that the set $\{|\theta_n(e_j)|\}$, where e_1, \dots, e_k are algebraic generators of \mathfrak{g} , is bounded for every j . Then the infinite sum $\theta := \bigoplus_{n=1}^{\infty} \theta_n$ is a well-defined representation of $U(\mathfrak{g})$ on a Hilbert space. Since for every $p \in \mathbb{N}_+$ there are $n \in \mathbb{N}_+$ and $\eta \in \mathfrak{n}$ such that $\theta_n(\eta)^p \neq 0$, the completion of the range of θ does not satisfy a PI by Theorem 2. \square

For a general criterion, see the follow-up paper [5].

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