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Distal systems in topological dynamics and ergodic theory

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Abstract. We generalize a result of Lindenstrauss on the interplay between measurable and topological dynamics which shows that every separable ergodic measurably distal dynamical system has a minimal distal model. We show that such a model can, in fact, be chosen completely canonically. The construction is performed by going through the Furstenberg–Zimmer tower of a measurably distal system and showing that at each step there is a simple and canonical distal minimal model. This hinges on a new characterization of isometric extensions in topological dynamics.

Key words: ergodic theory, topological dynamics, distal systems, topological models
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1. *Introduction*

The famous Furstenberg structure theorem shows that every distal minimal topological dynamical system can be built from a trivial system only by successively performing 'structured' (that is, (pseudo)isometric) extensions; see, for example, [**Fur63**] and [**dV93**, §V.3]. As shown by Zimmer (see [**Zim76**, Theorem 8.7]), this classical result has an analogue for distal systems in ergodic theory introduced earlier by Parry in [**Par68**]: a measure-preserving system on a standard probability space is distal if and only if it can be constructed from a tower of measure-theoretically isometric extensions. The significance of these measurably distal systems is due to the fact that—by the Furstenberg–Zimmer structure theorem—any measure-preserving system can be recovered by taking a weakly mixing extension of a distal one (see, for example, [**Fur77**], [**Tao09**, Ch. 2] or [**EHK21**]). This allows one to reduce the proof of important results for measure-preserving transformations, such as Furstenberg's recurrence result used for an ergodic-theoretic proof of Szemerédi's theorem, to the case of distal systems.

Given a minimal distal system $(K;\varphi)$ and an ergodic φ-invariant probability measure μ on K, it is not hard to prove that $(K,\mu;\varphi)$ is also distal as a measure-preserving system. While this has long been known, the converse question, whether a distal measure-preserving system admits a distal topological model, was only answered much later in an paper by Lindenstrauss (see [**Lin99**] and also [**GW06**, §§5 and 13]).

THEOREM. *Every ergodic distal measure-preserving system on a standard probability space has a minimal distal metrizable topological model.*

Topological models such as these continue to prove useful since they allow us to use a range of topological results to derive results in ergodic theory. For recent examples, we refer to the proof of pointwise convergence for multiple averages for distal systems (see [**HSX19**]), new approaches to the Furstenberg–Zimmer structure theorem (see [**EHK21**, §7.4]), and a new approach to the Host–Kra factors in [**GL19**].

Lindenstrauss's proof rests on the Mackey–Zimmer representation of isometric extensions as skew-products and on measure-theoretic considerations. In this paper we pursue an operator-theoretic approach to the result and even prove that for every ergodic distal system there is a canonical minimal distal topological model. This allows to construct such models in a functorial way: every extension between ergodic distal systems induces a topological extension between their canonical models.

The key step is to show that an isometric extension of measure-preserving systems admits a completely canonical topological model that is a (pseudo)isometric extension of topological dynamical systems. The proof of this requires a new functional-analytic characterization of structured extensions in topological dynamics as established in [**EK21**], related results from ergodic theory (see [**EHK21**]), as well as a result of Derrien on approximation of measurable cocycles by continuous ones (see [**Der00**]). The functional-analytic view makes the parallels between the structure theory of topological and measure-preserving dynamical systems more apparent and leads to canonical models in a straightforward way. In addition, our methods allow us to generalize Lindenstrauss's result to measure-preserving transformations on arbitrary probability spaces, following up on a recent endeavor to drop separability assumptions from classical results of ergodic theory (see [**EHK21**, **JT20**]).

Organization of the paper. We start in §2 with the concepts of structured extensions of topological dynamical systems. Then, based on the results of [**EK21**], we prove an operator-theoretic characterization of (pseudo)isometric extensions in terms of the Koopman operator (see Theorem 2.8). In §3 we consider structured extensions in ergodic theory, that is, extensions with relative discrete spectrum, and recall an important characterization from [**EHK21**] (see Proposition 3.8). The major work is done in §4 where we construct topological models for structured extensions of measure-preserving systems (see Theorem 4.6). This result is applied in the final section of the paper to show that every ergodic distal measure-preserving system has a minimal distal topological model (see Theorem 5.9). Finally, we discuss the meaning of our result in a category-theoretical sense in Remark 5.11.

Preliminaries and notation. We now set up the notation and recall some important concepts from topological dynamics and ergodic theory. The monograph [**EFHN15**] serves as a general reference for the operator-theoretic approach followed in this paper.

In the following all vector spaces are complex and all compact spaces are assumed to be Hausdorff. If E and F are Banach spaces, then $\mathscr{L}(E, F)$ denotes the space of all bounded linear operators from E to F. We write $\mathscr{L}(E) := \mathscr{L}(E, E)$ and $E' := \mathscr{L}(E, \mathbb{C})$. If $T \in \mathscr{L}(E, F)$ is a bounded operator, then $T' \in \mathscr{L}(F', E')$ denotes its adjoint.

Given a compact space K, we write $\mathcal{U}_K$ for the unique uniformity compatible with the topology of K. Moreover, $\mathrm{C}(K)$ denotes the space of all continuous complex-valued functions which is a unital commutative C*-algebra (cf. [**EFHN15**, Ch. 4]). Using the Markov–Riesz representation theorem (see [**EFHN15**, Appendix E]), we identify its dual space $\mathrm{C}(K)'$ with the space of all complex regular Borel measures on K. Likewise, if $\mathrm{X} = (X, \Sigma, \mu)$ is a probability space, then we write $\mathrm{L}^p(\mathrm{X})$ with $1 \leqslant p \leqslant \infty$ for the associated complex L^p-spaces and identify the dual $\mathrm{L}^1(\mathrm{X})'$ with $\mathrm{L}^\infty(\mathrm{X})$.

A *topological dynamical system* $(K; \varphi)$ consists of a compact space K and a homeomorphism $\varphi : K \to K$. It is *minimal* if there are non-trivial closed subsets $M \subseteq K$ with $\varphi(M) = M$. Moreover, we call $(K; \varphi)$ *metrizable* if the underlying compact space K is metrizable. We refer to [**Aus88**] for a general introduction to such systems. Topological dynamical systems can be studied effectively via operator theory by considering the induced *Koopman operator* $T_\varphi \in \mathscr{L}(\mathrm{C}(K))$ defined by $T_\varphi f := f \circ \varphi$ for $f \in \mathrm{C}(K)$ (see [**EFHN15**, Ch. 4]). We remind the reader that T_φ is a *-automorphism of the C*-algebra $\mathrm{C}(K)$. In fact, every *-automorphism of $\mathrm{C}(K)$ is a Koopman operator associated to a uniquely determined homeomorphism of K (see [**EFHN15**, Theorem 4.13]). We write $\mathrm{P}_\varphi(K) \subseteq \mathrm{C}(K)'$ for the space of all invariant probability measures μ on K, that is, $T_\varphi'\mu = \mu$. Moreover, supp μ denotes the support of such a measure (see [**EFHN15**, pp. 82]), and we say that μ is *fully supported* if supp $\mu = K$. Moreover, we write (K, μ) for the induced probability space.

Classically, a *measure-preserving point transformation* is a pair $(\mathrm{X}; \varphi)$ of a probability space $\mathrm{X} = (X, \Sigma_X, \mu_X)$ and a measurable and measure-preserving map $\varphi : \mathrm{X} \to \mathrm{X}$ which is *essentially invertible*, that is, there is a map $\psi : \mathrm{X} \to \mathrm{X}$ such that $\psi \circ \varphi = \mathrm{id}_X = \varphi \circ \psi$ almost everywhere. We refer to [**EW11**, Gla03] for an introduction. Given any measure-preserving point transformation $(\mathrm{X}; \varphi)$, we define the *Koopman operator* on the corresponding L^1-space via $T_\varphi f := f \circ \varphi$ for $f \in \mathrm{L}^1(\mathrm{X})$. These operators are so-called *Markov lattice isomorphisms* on the Banach lattice $\mathrm{L}^1(\mathrm{X})$, that is, invertible isometries $T \in \mathscr{L}(\mathrm{L}^1(\mathrm{X}))$ satisfying:

- $|Tf| = T|f|$ for every $f \in \mathrm{L}^1(\mathrm{X})$, and
- $T\mathbb{1} = \mathbb{1}$.

We refer to [**EFHN15**, Ch. 13] for more information on such operators. If X is a standard probability space (see [**EFHN15**, Definition 6.8]), then a result of von Neumann shows that every Markov lattice isomorphism $T \in \mathscr{L}(\mathrm{L}^1(\mathrm{X}))$ is a Koopman operator of a measure-preserving point transformation (see [**EFHN15**, Proposition 7.19 and Theorem 7.20]). For a general probability space X one can only show that such operators are induced by transformations of the measure algebra of X (see [**EFHN15**, Theorem 12.10]). Here, we avoid these measure-theoretic intricacies by defining a measure-preserving system in terms

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of operators theory. A *measure-preserving system* is a pair $(\mathrm{X}; T)$ of a probability space X and a Markov lattice isomorphism $T \in \mathscr{L}(\mathrm{L}^1(\mathrm{X}))$ (cf. [**EFHN15**, Definition 12.18]). It is *ergodic* if the *fixed space*

$$\mathrm{fix}(T) := \{f \in \mathrm{L}^1(X) \mid Tf = f\}$$

is one-dimensional (cf. [**EFHN15**, Proposition 7.15]). We say that $(\mathrm{X}; T)$ is *separable* if the measure space X is separable, or equivalently, if the Banach space $\mathrm{L}^1(\mathrm{X})$ is separable.

2. *Structured extensions in topological dynamics*

In order to state and prove one of our main results, Theorem 4.6, we need to briefly recap the notions of structured extensions in topological dynamics and ergodic theory together with their different characterizations. So we start with the notion of (pseudo)isometric extensions of topological dynamical systems and their functional-analytic characterization in Theorem 2.8.

Definition 2.1. An *extension* $q : (K; \varphi) \to (L; \psi)$ between topological dynamical systems $(K; \varphi)$ and $(L; \psi)$ is a continuous surjection $q : K \to L$ such that the diagram

$$\begin{array}{ccc} K & \xrightarrow{\ \varphi\ } & K \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ L & \xrightarrow[\ \psi\ ]{} & L \end{array}$$

commutes. In this case, we call $(L; \psi)$ a *factor* of $(K; \varphi)$. We write $K_l := q^{-1}(l)$ for the *fiber of K over* $l \in L$ and define the *fiber product* $K \times_L K$ *of K over L* as

$$K \times_L K := \bigcup_{l \in L} K_l \times K_l \subseteq K \times K.$$

Remark 2.2. There is an equivalent functional-analytic perspective on extensions based on Gelfand duality. Let $(K; \varphi)$ be a topological dynamical system and call a subset $M \subseteq \mathrm{C}(K)$ *invariant* if $T_\varphi M = M$. If $q : (K; \varphi) \to (L; \psi)$ is an extension, then $T_q : \mathrm{C}(L) \to \mathrm{C}(K)$, $f \mapsto f \circ q$ is an isometric *-homomorphism intertwining the Koopman operators. Therefore, $A_q := T_q(\mathrm{C}(L)) \subseteq \mathrm{C}(K)$ is an invariant unital C*-subalgebra of $\mathrm{C}(K)$. On the other hand, if $A \subseteq \mathrm{C}(K)$ is such an invariant unital C*-subalgebra, then T_φ induces a homeomorphism ψ on the Gelfand space L of A and the embedding $A \hookrightarrow \mathrm{C}(K)$ gives rise to an extension $q : (K; \varphi) \to (L; \psi)$ with $A = A_q$ (see [**EFHN15**, Ch. 4]). Thus, instead of looking at factors of a given system $(K; \varphi)$, one can also examine the invariant unital C*-subalgebras of $\mathrm{C}(K)$.

We now look at structured extensions of topological dynamical systems. There are basically two ways to start from the notion of an isometric or equicontinuous system and relativize it to extensions: one is based on the existence of invariant (pseudo)metrics, while the other generalizes the concept of equicontinuity (cf. [**dV93**, §§V.2 and V.5] and [**EK21**, Definition 1.15]).

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Definition 2.3. An extension $q : (K; \varphi) \to (L; \psi)$ of topological dynamical systems is called:

(i) *pseudoisometric* if there is a family P of continuous mappings

$$p : K \times_L K \to \mathbb{R}_{\geqslant 0}$$

such that:

- $p|_{K_l \times K_l}$ is a pseudometric on K_l for every $l \in L$ and $p \in P$,
- $\{p|_{K_l \times K_l} \mid l \in L\}$ generates the topology on K_l for every $l \in L$, and
- $p(\varphi(x), \varphi(y)) = p(x, y)$ for all $(x, y) \in K \times_L K$ and $p \in P$;

(ii) *isometric* if it is pseudoisometric and P in (i) can be chosen to only have one element (which then defines a metric on every fiber);

(iii) *equicontinuous* if for every entourage $V \in \mathcal{U}_K$ there is an entourage $U \in \mathcal{U}_K$ such that for every pair $(x, y) \in K \times_L K$

$$(x, y) \in U \;\Rightarrow\; (\varphi^k(x), \varphi^k(y)) \in V \quad \text{for every } k \in \mathbb{Z}.$$

Remark 2.4. Every pseudoisometric extension is equicontinuous by [**EK21**, Proposition 1.17], while the converse may fail (see [**EK21**, Example 3.15]). However, if $(K; \varphi)$ (and hence also $(L; \psi)$) is minimal, then the two notions coincide (see [**dV93**, Corollary 5.10]).

The following is a standard example of an isometric extension.

Example 2.5. (Skew-rotation) Let $\mathbb{T} := \{x \in \mathbb{C} \mid |x| = 1\}$ and $a \in \mathbb{T}$. We consider $(K; \varphi)$ defined by $K := \mathbb{T}^2$ with $\varphi(x, y) := (ax, xy)$ for $(x, y) \in K$, and $(L; \psi)$ given by $L := \mathbb{T}$ with $\psi(x) = ax$ for $x \in L$. Then the projection $q : \mathbb{T}^2 \to \mathbb{T}$ onto the first component defines an isometric extension $q : (K; \varphi) \to (L; \psi)$.

Recall that a system $(K; \varphi)$ is equicontinuous (see [**Aus88**, Ch. 2]) if and only if the induced Koopman operator $T_\varphi \in \mathscr{L}(\mathrm{C}(K))$ has discrete spectrum, that is, $\mathrm{C}(K)$ is the closed linear hull of all eigenspaces of the Koopman operator (see, for example, [**Ede19**, Proposition 1.6]). Is there a more general version of this that can be used to characterize when an extension $q : (K; \varphi) \to (L; \psi)$ is pseudoisometric? If $(L; \psi)$ satisfies a mild irreducibility condition, Theorem 2.8 below provides an affirmative answer. To state it, we require the definition of *topological ergodicity* (analogous to ergodicity) and we need to recall the module structure an extension gives rise to.

Definition 2.6. A topological dynamical system $(K; \varphi)$ is *topologically ergodic* if the *fixed space*

$$\mathrm{fix}(T_\varphi) := \{f \in \mathrm{C}(K) \mid T_\varphi f = f\}$$

of the Koopman operator $T_\varphi \in \mathscr{L}(\mathrm{C}(K))$ is one-dimensional.

Every minimal system is topologically ergodic, but the class of topologically ergodic systems is considerably larger and contains, for example, all topologically transitive systems.

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Remark 2.7. One of the key tools in the study of extensions of dynamical systems is the module structure that canonically emerges from an extension and is often tacitly used. Let $q : K \to L$ be a continuous surjection between compact spaces and $T_q : \mathrm{C}(L) \to \mathrm{C}(K),\ f \mapsto f \circ q$ the induced isometric *-homomorphism. Via this embedding we can define a multiplication

$$\mathrm{C}(L) \times \mathrm{C}(K) \to \mathrm{C}(K), \quad (f, g) \mapsto T_q f \cdot g$$

that turns $\mathrm{C}(K)$ into a $\mathrm{C}(L)$-module in a canonical way.

We now obtain the following operator-theoretic characterization of pseudoisometric extensions in terms of a relative notion of discrete spectrum. Recall here that a module M over a unital commutative ring R is *projective* if there is another module N over R such that the direct sum $M \oplus N$ is free, that is, has a basis.

THEOREM 2.8. *Let $q : (K; \varphi) \to (L; \psi)$ be an extension of topological dynamical systems. Assume that $(L; \psi)$ is topologically ergodic and q is open. Then the following assertions are equivalent.*

(a) *q is pseudoisometric.*

(b) *The union of all closed, invariant, finitely generated, projective $\mathrm{C}(L)$-submodules is dense in $\mathrm{C}(K)$.*

(c) *The unital C^*-algebra generated by all closed, invariant, finitely generated, projective $\mathrm{C}(L)$-submodules is the whole space $\mathrm{C}(K)$.*

If K is even metrizable, then (a) can be replaced by

(a′) *q is isometric.*

If $(L; \psi)$ is minimal, the assumption that q is open can be dropped.

Remark 2.9. Loosely speaking, assertions (b) and (c) of Theorem 2.8 mean that $\mathrm{C}(K)$ is generated by invariant parts which are 'small' relative to $\mathrm{C}(L)$.

We remark that, except for the last statement about minimal $(L; \psi)$, Theorem 2.8 is a special case of [**EK21**, Theorem 7.2]. Thus, we only need to prove this additional statement. To do this, we show that, in case of a minimal system $(L; \psi)$, each of the assertions (a) and (c) (and consequently also the stronger conditions (b) and (a′)) imply that q is open.

We start by proving that (a) yields that the extension is open. In fact, this implication is valid for the more general class of *distal* extensions (cf. [**Bro79**, §3.12]).

Definition 2.10. An extension $q : (K; \varphi) \to (L; \psi)$ is *distal* if the following condition is satisfied: whenever $(x, y) \in K \times_L K$ and $(\varphi^{n_\alpha})_{\alpha \in A}$ is a net with $n_\alpha \in \mathbb{Z}$ for $\alpha \in A$ and $\lim_\alpha \varphi^{n_\alpha}(x) = \lim_\alpha \varphi^{n_\alpha}(y)$, then $x = y$.

LEMMA 2.11. *Let $q : (K, \varphi) \to (L, \psi)$ be a distal extension of topological dynamical systems with (L, ψ) minimal. Then q is open.*

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The result is stated as a remark in [**Aus13**, p. 2] without proof. Since we have found a proof in the literature only for the case of minimal $(K; \varphi)$, we provide a proof of Lemma 2.11 based on the arguments of [**Bro79**, Lemma 3.14.5] and [**Aus88**, Theorem 10.8].

Proof of Lemma 2.11. Consider the Ellis semigroup of the system $(K; \varphi)$ given by

$$\mathrm{E}(K; \varphi) := \overline{\{\varphi^n \mid n \in \mathbb{Z}\}} \subseteq K^K$$

where the closure is taken with respect to the topology of pointwise convergence (see [**Aus88**, Ch. 3]). For every $l \in L$ the set

$$E_l := \{\vartheta : K_l \to K_l \mid \text{there exists } \tau \in \mathrm{E}(K; \varphi) \text{ with } \vartheta = \tau|_{K_l}\} \subseteq K_l^{K_l},$$

equipped with composition of mappings and the product topology, is a compact right-topological semigroup (see [**BJM89**, §1.3] or [**EFHN15**, Ch. 16] for this concept). As a preliminary step, we show that these are actually groups. To this end, we recall from the structure theory of compact right-topological semigroups that every such semigroup contains at least one idempotent and is a group if and only if it has a unique idempotent (see [**BJM89**, Theorems 2.12 and 3.11]).

Now take $l \in L$ and an idempotent $\vartheta \in E_l$ (that is, $\vartheta^2 = \vartheta$). For $x \in K_l$ consider $y := \vartheta(x) \in K_l$. Then $\vartheta(y) = \vartheta^2(x) = \vartheta(x)$, which implies $x = y$ since q is distal. Therefore, id_{K_l} is the only idempotent in E_l and thus E_l is in fact a group.

We now prove that q is open. Take an $x \in K$ and let $l := q(x)$. Assume that $(l_\alpha)_{\alpha \in A}$ is a net in L converging to l. It suffices to show that there is a subnet $(l_\beta)_{\beta \in B}$ of $(l_\alpha)_{\alpha \in A}$ and $x_\beta \in K_{l_\beta}$ for every $\beta \in B$ such that $\lim_\beta x_\beta = x$. We recall that, since $(L; \psi)$ is minimal, for every $\alpha \in A$,

$$\mathrm{E}(L; \psi)(l_\alpha) = \overline{\{\varphi^n(l_\alpha) \mid n \in \mathbb{Z}\}} = L.$$

Moreover,

$$\mathrm{E}(K; \varphi) \to \mathrm{E}(L; \psi), \quad \tau \mapsto [q(y) \mapsto q(\tau(y))]$$

is a surjective homomorphism of compact right-topological semigroups by [**Aus88**, Theorem 3.7]. With these two observations we find $\tau_\alpha \in \mathrm{E}(K, \varphi)$ with $q(\tau_\alpha(x)) = l_\alpha$ for every $\alpha \in A$. Passing to a subnet, we may assume that $(\tau_\alpha)_{\alpha \in A}$ converges to some $\tau \in \mathrm{E}(K; \varphi)$. Moreover, $\tau(x) = \lim_\alpha \tau_\alpha(x) \in K_l$, which already implies $\tau(K_l) \subseteq K_l$ (see [**Bro79**, Lemma 3.12.10]), that is, $\vartheta := \tau|_{K_l} \in E_l$. But then $x_\alpha := (\tau_\alpha(\vartheta^{-1}(x)))_{\alpha \in A}$ converges to x and $x_\alpha \in K_{l_\alpha}$ for every $\alpha \in A$. □

Since equicontinuous (and, in particular, pseudoisometric) extensions are distal (see [**Bro79**, Lemma 3.12.5]), we now obtain that assertion (a) of Theorem 2.8 implies that the extension is open if $(L; \psi)$ is minimal. The following result combined with Lemma 2.11 shows that, for minimal $(L; \psi)$, (c) also implies openness, which proves Theorem 2.8. In the proof, we will use the canonical correspondence between Banach bundles and Banach modules; the reader can find a self-contained summary of the essentials in [**EK21**, §4].

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LEMMA 2.12. *Let $q : (K, \varphi) \to (L, \psi)$ be an extension. Suppose that the C^*-algebra generated by all closed, invariant, finitely generated, projective $C(L)$-submodules is the whole space $C(K)$. Then q is equicontinuous.*

Proof. The uniformity on K is generated by the sets $U_{f,\varepsilon}$ which, for $f \in C(K)$ and $\varepsilon > 0$, are defined as

$$U_{f,\varepsilon} := \{(x, y) \in K \times K \mid |f(x) - f(y)| < \varepsilon\}.$$

Our assumption therefore yields that q is equicontinuous if and only if the assertion following holds. For every $C(L)$-submodule $M \subseteq C(K)$ that is closed, invariant, finitely generated, and projective, for every $f \in M$ and every $\varepsilon > 0$, we find an entourage $U \in \mathcal{U}_K$ such that

$$\text{for all } (x, y) \in K \times_L K : (x, y) \in U \;\Rightarrow\; |f(\varphi^k(x)) - f(\varphi^k(y))| < \varepsilon \quad \text{for all } k \in \mathbb{Z}.$$

It suffices to show the claim only for $f \in M$ with $\|f\| \leqslant 1$. By looking at finitely many generators for M, we will in fact be able to show that the uniformity U can be chosen uniformly for all $f \in M$ with $\|f\| \leqslant 1$. We will do so by constructing a continuous pseudometric $p : K \times K \to \mathbb{R}_{\geqslant 0}$ from M such that

$$\text{for all } f \in M \cap \overline{B_1(0)}, \text{ for all } x, y \in K : |f(x) - f(y)| \leqslant p(x, y).$$

The desired uniformity is then given by $U = \{(x, y) \in K \times K \mid p(x, y) < \varepsilon\}$. To do this, we exploit the fact that there is a one-to-one correspondence between projective, finitely generated $C(L)$-modules and locally trivial vector bundles over L: by [Gie82, Theorem 8.6 and Remark 8.7] and [EK21, Example 4.5], the vector spaces

$$M_l := \{f|_{K_l} \mid l \in L\} \subseteq C(K_l)$$

for $l \in L$ define a Banach bundle over L (see [Gie82] or [DG83] for this concept). This is locally trivial by [EK21, Lemma 4.13], which—using [Gie82, Proposition 17.2 and Corollary 4.5]—can be characterized in the following way.

- There are closed subsets $L_1, \ldots, L_m \subseteq L$ with $L = \bigcup_{j=1}^{m} L_j$.
- For every $n \in \{1, \ldots, m\}$ there are $s_{n,1}, \ldots, s_{n,k_n} \in M$ such that

$$\Phi_n : C(L_n)^{k_n} \to M|_{q^{-1}(L)}, \quad (f_1, \ldots, f_{k_n}) \mapsto \sum_{j=1}^{k_n} f_j s_{n,j}|_{q^{-1}(L_n)}$$

  is a $C(L_n)$-linear (not necessarily isometric) isomorphism between the product Banach space $C(L_n)^{k_n}$ with the maximum norm and the subspace $M|_{q^{-1}(L_n)} \subseteq C(q^{-1}(L_n))$.

For every $n \in \{1, \ldots, m\}$ we now consider the continuous seminorm

$$p_n : K \times K \to \mathbb{R}_{\geqslant 0}, \quad (x, y) \mapsto \sum_{j=1}^{k_n} |s_{n,j}(x) - s_{n,j}(y)|$$

and show that there is a constant $C > 0$ such that

$$|f(x) - f(y)| \leqslant C \cdot \max_{n=1,\ldots,m} p_n(x, y)$$

for all $(x, y) \in K \times_L K$ and $f \in M$ with $\|f\| \leqslant 1$. This will finish the proof.

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It suffices to show that for every $n \in \{1, \ldots, m\}$ there is a constant $C_n > 0$ such that the inequality

$$|f(x) - f(y)| \leqslant C_n \cdot p_n(x, y)$$

holds for every pair $(x, y) \in K \times_L K$ with $q(x) = q(y) \in L_n$, and every $f \in M$ with $\|f\| \leqslant 1$. We fix $n \in \{1, \ldots, m\}$ and set $C_n := \|\Phi_n^{-1}\| > 0$. For $(x, y) \in K \times_L K$ with $l := q(x) = q(y) \in L_n$ we then obtain, for all $f_1, \ldots, f_{k_n} \in \mathrm{C}(L_n)$,

$$\begin{aligned}|\Phi(f_1, \ldots, f_{k_n})(x) - \Phi(f_1, \ldots, f_{k_n})(y)| &\leqslant \sum_{j=1}^{k_n} |f_j(l)(s_{n,j}(x) - s_{n,j}(y))| \\ &\leqslant \|(f_1, \ldots, f_{k_n})\| \cdot p_n(x, y).\end{aligned}$$

Since Φ_n is an isomorphism, we conclude that

$$|f(x) - f(y)| \leqslant C_n \cdot p_n(x, y)$$

for every pair $(x, y) \in K \times_L K$ with $q(x) = q(y) \in L_n$, and every $f \in M$ with $\|f\| \leqslant 1$. This is the desired inequality. □

3. *Structured extensions in ergodic theory*

We now turn to structured extensions of measure-preserving systems. In our operator-theoretic language the following is the natural definition of an extension in ergodic theory. Recall here that if X and Y are probability spaces, then an isometry $J \in \mathscr{L}(\mathrm{L}^1(\mathrm{Y}), \mathrm{L}^1(\mathrm{X}))$ is a *Markov embedding* (or *Markov lattice homomorphism*) if:

- $|Jf| = J|f|$ for every $f \in \mathrm{L}^1(\mathrm{Y})$;
- $J\mathbb{1} = \mathbb{1}$.

Definition 3.1. An *extension* (or *morphism*) $J : (\mathrm{Y}; S) \to (\mathrm{X}; T)$ of measure-preserving systems is a Markov embedding $J \in \mathscr{L}(\mathrm{L}^1(\mathrm{Y}), \mathrm{L}^1(\mathrm{X}))$ such that the diagram

$$\begin{array}{ccc} \mathrm{L}^1(\mathrm{X}) & \xrightarrow{T} & \mathrm{L}^1(\mathrm{X}) \\ {\scriptstyle J}\uparrow & & \uparrow{\scriptstyle J} \\ \mathrm{L}^1(\mathrm{Y}) & \xrightarrow[S]{} & \mathrm{L}^1(\mathrm{Y}) \end{array}$$

commutes. If J is also bijective, then it is an *isomorphism* of measure-preserving systems.

Remark 3.2. Given any Markov lattice homomorphism $J \in \mathscr{L}(\mathrm{L}^1(\mathrm{Y}), \mathrm{L}^1(\mathrm{X}))$ for probability spaces X and Y, the adjoint $J' \in \mathscr{L}(\mathrm{L}^\infty(\mathrm{X}), \mathrm{L}^\infty(\mathrm{Y}))$ extends uniquely to a bounded positive operator $\mathbb{E}_{\mathrm{Y}} \in \mathscr{L}(\mathrm{L}^1(\mathrm{X}), \mathrm{L}^1(\mathrm{Y}))$ satisfying $\mathbb{E}_{\mathrm{Y}}((Jf) \cdot g) = f \cdot \mathbb{E}_{\mathrm{Y}}(g)$ for all $f \in \mathrm{L}^\infty(\mathrm{Y})$ and $g \in \mathrm{L}^1(\mathrm{X})$ (see [**EFHN15**, §13.3]). We call $\mathbb{E}_{\mathrm{Y}}$ the *conditional expectation operator* associated with J. If J is an extension of measure-preserving systems, then $\mathbb{E}_{\mathrm{Y}}$ intertwines the dynamics.

Remark 3.3. As in the topological setting, we obtain a canonical module structure (cf. Remark 2.7). Indeed, if $f_1, f_2 \in \mathrm{L}^\infty(\mathrm{Y})$, then $J(f_1 \cdot f_2) = J(f_1) \cdot J(f_2)$ (see [**EFHN15**,

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§13.2]) and this implies that the multiplication

$$\mathrm{L}^\infty(\mathrm{Y}) \times \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{X}), \quad (f, g) \mapsto J(f) \cdot g$$

turns $\mathrm{L}^1(\mathrm{X})$ into a module over $\mathrm{L}^\infty(\mathrm{Y})$. For every $p \in [1, \infty]$ the space $\mathrm{L}^p(\mathrm{X})$ is a $\mathrm{L}^\infty(\mathrm{X})$-submodule of $\mathrm{L}^1(\mathrm{X})$.

As in topological dynamics, there are several notions of 'structured extensions' in ergodic theory. We use the following one, implicitly used by Ellis in [**Ell87**] and inspired by the classical notion of discrete spectrum for measure-preserving systems. Here, as above, a subset $M \subseteq \mathrm{L}^1(\mathrm{X})$ is *invariant* if $T(M) = M$.

Definition 3.4. An extension $J : (\mathrm{Y}; S) \to (\mathrm{X}; T)$ of measure-preserving systems has *relative discrete spectrum* if the union of all finitely generated invariant submodules of $\mathrm{L}^\infty(\mathrm{X})$ over $\mathrm{L}^\infty(\mathrm{Y})$ is dense in $\mathrm{L}^1(\mathrm{X})$.

Example 3.5. Consider the skew-rotation $q : (K; \varphi) \to (L; \psi)$ of Example 2.5. By equipping the torus $L = \mathbb{T}$ with the Haar measure ν and its product $K = \mathbb{T}^2$ with the product measure $\mu = \nu \times \nu$, we arrive at an extension $(L, \nu; T_\psi) \to (K, \mu; T_\varphi)$ of measure-preserving systems which has relative discrete spectrum (see [**EHK21**, Example 6.12]). More generally, homogenous skew-products are prototypical examples of extensions with relative discrete spectrum (see, for example, [**Zim76**] and [**Ell87**, §§4 and 5]).

Equivalent definitions of relative discrete spectrum are listed in [**EHK21**, Proposition 6.13]. We need one using modules with an orthonormal basis which uses the idea that

$$\mathrm{L}^\infty(\mathrm{X}) \times \mathrm{L}^\infty(\mathrm{X}) \to \mathrm{L}^\infty(\mathrm{Y}), \quad (f, g) \mapsto \mathbb{E}_\mathrm{Y}(f\overline{g})$$

can be thought of as an $\mathrm{L}^\infty(\mathrm{Y})$-valued inner product. We refer to [**EHK21**] for a systematic approach to this idea in terms of Hilbert modules.

Definition 3.6. Let $J : (\mathrm{Y}; S) \to (\mathrm{X}; T)$ be an extension of measure-preserving systems. A finite subset $\{e_1, \dots, e_n\} \subseteq \mathrm{L}^\infty(\mathrm{X})$ is Y-*orthonormal* if $\mathbb{E}_\mathrm{Y}(e_i \cdot \overline{e_j}) = \delta_{ij}\mathbb{1} \in \mathrm{L}^\infty(\mathrm{Y})$ for $i, j \in \{1, \dots, n\}$. In this case we say that $e_1, \dots, e_n$ is a Y-*orthonormal basis* of the $\mathrm{L}^\infty(\mathrm{Y})$ -module generated by $e_1, \dots, e_n$.

Remark 3.7. If $\{e_1, \dots, e_n\} \subseteq \mathrm{L}^\infty(\mathrm{X})$ is a Y-orthonormal basis of an $\mathrm{L}^\infty(\mathrm{Y})$-submodule M as in Definition 3.6, then every $f \in M$ can be written as

$$f = \sum_{j=1}^{n} \mathbb{E}_\mathrm{Y}(f \cdot \overline{e_j})e_j.$$

With this observation it is readily checked that any submodule $M \subseteq \mathrm{L}^\infty(\mathrm{X})$ having a Y-orthonormal basis is automatically a free (and, in particular, projective) module and closed in $\mathrm{L}^\infty(\mathrm{X})$.

The following result (see [**EHK21**, Proposition 8.5 and Lemmas 8.3 and 6.8] or [**Ell87**, Remark 5.16 (1)]) shows that for extensions of ergodic systems with relative

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discrete spectrum there exist ‘many’ invariant finitely generated submodules with an Y-orthonormal basis.

PROPOSITION 3.8. *Let* $J : (\mathrm{Y}; S) \to (\mathrm{X}; T)$ *be an extension of ergodic measure-preserving systems. Then the following assertions are equivalent.*

(a) *J has relative discrete spectrum.*

(b) *The union of all finitely generated invariant* $\mathrm{L}^\infty(\mathrm{Y})$*- submodules of* $\mathrm{L}^\infty(\mathrm{X})$ *having a* Y*-orthonormal basis is dense in* $\mathrm{L}^1(\mathrm{X})$.

4. *Topological models for structured extensions*

Comparing Theorem 2.8 and Proposition 3.8 (combined with Remark 3.7) makes the parallelisms between structured extensions in topological dynamics and ergodic theory apparent. We now study the relation between both worlds. Recall first that, as in Example 3.5, we can always construct extensions of measure-preserving systems from extensions of topological dynamical systems by picking an invariant measure.

Definition 4.1. Let $q : (K; \varphi) \to (L; \psi)$ be an extension of topological dynamical systems. Moreover, let $\mu \in \mathrm{P}_\varphi(K)$ be an invariant probability measure on K and $q_*\mu \in \mathrm{P}_\psi(L)$ its pushforward, that is, $q_*\mu = T_q'\mu$. Then the extension

$$T_q : (L, q_*\mu; T_\psi) \to (K, \mu; T_\varphi), \qquad f \mapsto f \circ q$$

is the *extension of measure-preserving systems induced by* (q, μ).

With the help of Theorem 2.8 we now immediately obtain the following proposition.

PROPOSITION 4.2. *Assume that* $q : (K; \varphi) \to (L; \psi)$ *is an open pseudoisometric extension with a topologically ergodic system* $(L; \psi)$*. For every* $\mu \in \mathrm{P}_\varphi(K)$ *the induced extension* $T_q : (L, q_*\mu; T_\psi) \to (K, \mu; T_\varphi)$ *has relative discrete spectrum.*

Proof. By Theorem 2.8 the union of all closed, invariant, finitely generated, projective $\mathrm{C}(L)$-submodules is dense in $\mathrm{C}(K)$ and, via the canonical map $\mathrm{C}(K) \to \mathrm{L}^1(K, \mu)$, also dense in $\mathrm{L}^1(K, \mu)$. However, if M is a finitely generated invariant $\mathrm{C}(L)$ submodule of $\mathrm{C}(K)$ with generators $e_1, \ldots, e_n$, then the $\mathrm{L}^\infty(L, q_*\mu)$-submodule of $\mathrm{L}^\infty(K, \mu)$ generated by the canonical images of $e_1, \ldots, e_n$ in $\mathrm{L}^\infty(K, \mu)$ is also invariant. This shows the claim. □

In particular, by Lemma 2.11 we can construct extensions with relative discrete spectrum from pseudoisometric extensions of minimal topological dynamical systems. In the remainder of this section we study the converse situation. Given an extension $J : (\mathrm{Y}; S) \to (\mathrm{X}; T)$ of measure-preserving systems with relative discrete spectrum, can we find a pseudoisometric *topological model*? In order to make this question precise, we recall the following definition (cf. [**Gla03**, §2.2] and [**EFHN15**, Ch. 12]).

Definition 4.3. Let $J_i : (\mathrm{Y}_i; S_i) \to (\mathrm{X}_i; T_i)$ be extensions of measure-preserving systems for $i = 1, 2$. An *isomorphism* from J_1 to J_2 is a pair (Ψ, Φ) of an isomorphism

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$\Psi : (\mathrm{Y}_1; S_1) \to (\mathrm{Y}_2; S_2)$ and an isomorphism $\Phi : (\mathrm{X}_1; T_1) \to (\mathrm{X}_2; T_2)$ such that the diagram

$$\begin{array}{ccc} \mathrm{L}^1(\mathrm{X}_1) & \xrightarrow{\ \Phi\ } & \mathrm{L}^1(\mathrm{X}_2) \\ {\scriptstyle J_1}\uparrow & & \uparrow{\scriptstyle J_2} \\ \mathrm{L}^1(\mathrm{Y}_1) & \xrightarrow[\ \Psi\ ]{} & \mathrm{L}^1(\mathrm{Y}_2) \end{array}$$

commutes.

If $J : (\mathrm{Y}; S) \to (\mathrm{X}; T)$ is an extension of measure-preserving systems, then a *topological model* for J is a pair $(q, \mu; \Psi, \Phi)$ such that

- $q : (K; \varphi) \to (L; \psi)$ is an extension of topological dynamical systems;
- $\mu \in \mathrm{P}_\varphi(K)$ is a fully supported invariant probability measure; and
- (Ψ, Φ) is an isomorphism from the extension T_q induced by (q, μ) to J.

Remark 4.4. We use the following observation to construct topological models. If $(q, \mu; \Psi, \Phi)$ is a topological model for $J : (\mathrm{Y}; S) \to (\mathrm{X}; T)$ with $q : (K; \varphi) \to (L; \psi)$, then

$$A_{q,\mu} := \Psi(\mathrm{C}(L)) \subseteq \mathrm{L}^\infty(\mathrm{Y}) \quad \text{and} \quad B_{q,\mu} := \Phi(\mathrm{C}(K)) \subseteq \mathrm{L}^\infty(\mathrm{X})$$

are invariant unital C*-subalgebras of $\mathrm{L}^\infty(\mathrm{Y})$ and $\mathrm{L}^\infty(\mathrm{X})$, respectively, being dense in the corresponding L^1-spaces. Moreover, $J(A) \subseteq B$. Conversely, by Gelfand's representation theory, for every pair (A, B) of L^1-dense invariant unital C*-subalgebras $A \subseteq \mathrm{L}^\infty(\mathrm{Y})$ and $B \subseteq \mathrm{L}^\infty(\mathrm{X})$ with $J(A) \subseteq B$, we can construct, in a canonical way, a topological model $(q, \mu; \Psi, \Phi)$ for J such that $A = A_{q,\mu}$ and $B = B_{q,\mu}$ (cf. [**EFHN15**, Ch. 12]).

Using Theorem 2.8 and Proposition 3.8 we can always find a pseudoisometric topological model for extensions of ergodic measure-preserving systems with relative discrete spectrum.

THEOREM 4.5. *Let $J : (Y; S) \to (X; T)$ be an extension of ergodic measure-preserving systems with relative discrete spectrum. Then J has a topological model $(q, \mu; \Psi; \Phi)$ such that $q : (K; \varphi) \to (L; \psi)$ is an open pseudoisometric extension with $(K; \varphi)$ topologically ergodic.*

Proof. We define $A := \mathrm{L}^\infty(\mathrm{Y})$ and take B as the unital C*-algebra generated by all invariant, finitely generated, projective $\mathrm{L}^\infty(\mathrm{Y})$-submodules of $\mathrm{L}^\infty(\mathrm{X})$ which are closed in $\mathrm{L}^\infty(\mathrm{X})$. Clearly, A is dense $\mathrm{L}^1(\mathrm{Y})$, and, in view of Remark 3.7 and Proposition 3.8, B is also dense in $\mathrm{L}^1(\mathrm{X})$. By Remark 4.4 we find a topological model $(q, \mu; \Psi, \Phi)$ for J where $q : (K; \varphi) \to (L; \psi)$ is an extension of topological dynamical systems. We obtain that $\Phi(\mathrm{C}(L)) = A = \mathrm{L}^\infty(\mathrm{Y})$. Since the system $(\mathrm{X}; T)$ is ergodic, $\mathrm{fix}(T_\varphi)$ is one-dimensional and therefore $(K; \varphi)$ is topologically ergodic. Also, since we have chosen A to be the whole space $\mathrm{L}^\infty(\mathrm{Y})$, the induced extension q is open by [**Ell87**, Corollary 1.9]. Finally, since $B = \Phi(\mathrm{C}(K))$ we obtain that $\mathrm{C}(K)$ is generated as a unital C*-algebra by all closed, invariant, finitely generated, projective $\mathrm{C}(L)$-submodules. Thus, q is pseudoisometric by Theorem 2.8. □

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Note that the construction of the topological model in the proof of Theorem 4.5 is completely canonical. However, it is still unsatisfactory in some ways. For example, by relying on the 'Stone model' (that is, the topological model for the algebra $A = \mathrm{L}^\infty(\mathrm{Y})$) in the proof of Theorem 4.5, we cannot find metrizable models for extensions between separable probability spaces since $\mathrm{L}^\infty(\mathrm{Y})$ is only separable if it is finite-dimensional. A yet more serious problem is that, given two extensions $J_1 : (\mathrm{Z}; R) \to (\mathrm{Y}; S)$ and $J_2 : (\mathrm{Y}; S) \to (\mathrm{X}; T)$ with relative discrete spectrum, Theorem 4.5 does not allow us to construct pseudoisometric models q_1 and q_2 which 'fit together' since in Theorem 4.5 the base of the constructed extension is the Stone model. This will be essential in the construction of distal models by means of successive extensions. The following result fixes these problems, at least in certain situations, by allowing us to impose that the topological model at the bottom of an extension be any specific given topological model instead of the Stone model.

Recall that a measure $\nu \in \mathrm{P}_\psi(L)$ is called *ergodic* if the induced measure-preserving system $(L, \nu; T_\psi)$ is ergodic.

THEOREM 4.6. *Let $(L; \psi)$ be a minimal topological dynamical system, $\nu \in \mathrm{P}_\psi(L)$ a fully supported ergodic measure and $J : (L, \nu; T_\psi) \to (\mathrm{X}; T)$ an extension of ergodic systems with relative discrete spectrum. Then there are*

(i) *an open pseudoisometric extension $q : (K; \varphi) \to (L; \psi)$,*

(ii) *a fully supported ergodic measure $\mu \in \mathrm{P}_\varphi(K)$ with $q_*\mu = \nu$, and*

(iii) *an isomorphism $\Phi : (K, \mu; T_\varphi) \to (\mathrm{X}; T)$,*

such that $(q, \mu; \mathrm{Id}, \Phi)$ is a topological model for J. Moreover, if X is separable and L is metrizable, then K can be (non-canonically) chosen to be metrizable such that q is an isometric extension.

The challenge in proving Theorem 4.6 is, given an abundance of finitely generated invariant $\mathrm{L}^\infty(L, \nu)$-submodules, to find an abundance of finitely generated invariant $\mathrm{C}(L)$-submodules. It is non-trivial that this can be done, and it is this and only this point that forces us to restrict to $\mathbb{Z}$-actions in this paper. The following lemma shows that, at least for $\mathbb{Z}$-actions, finitely generated invariant $\mathrm{L}^\infty(L, \nu)$-submodules can indeed be approximated by finitely generated invariant $\mathrm{C}(L)$-submodules.

LEMMA 4.7. *Let $(L; \psi)$ be a topological dynamical system, $\nu \in \mathrm{P}_\psi(L)$ fully supported and ergodic, and $J : (L, \nu; T_\psi) \to (\mathrm{X}; T)$ an extension. Let $M \subseteq \mathrm{L}^\infty(\mathrm{X})$ be an invariant $\mathrm{L}^\infty(L, \nu)$-submodule with orthonormal basis $\{e_1, \ldots, e_n\}$. For every $\varepsilon > 0$ there is an (L, ν)-orthonormal set $\{d_1, \ldots, d_n\} \subseteq M$ such that:*

(i) *the $\mathrm{C}(L)$-submodule generated by $d_1, \ldots, d_n$ is invariant (as well as closed in $\mathrm{L}^\infty(\mathrm{X})$ and projective); and*

(ii) $\|d_i - e_i\|_{\mathrm{L}^1(\mathrm{X})} \leqslant \varepsilon$ *for all* $i \in \{1, \ldots, n\}$.

The proof rests on the following approximation result which is, in essence, due to Derrien (see [**Der00**]). It shows that, given a measurable map with values in the compact group $\mathrm{U}(n)$ of unitary $n \times n$ matrices, one can find an arbitrarily close continuous map that is cohomologous (cf. [**Lin99**, Theorem 3.1]).

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LEMMA 4.8. *Let $(L;\psi)$ be a metrizable topological dynamical system and $\mu\in \mathrm{P}_{\psi}(L)$ fully supported and ergodic. Assume that $F: L\to \mathrm{U}(n)$ is a Borel measurable map. For every $\varepsilon>0$ there are a Borel measurable map $G: L\to \mathrm{U}(n)$ and a continuous map $H: L\to \mathrm{U}(n)$ such that:*

(i) $\nu(\{l\in L\mid G(l)\neq \mathrm{Id}\})\leqslant\varepsilon$; *and*
(ii) $(G\circ\psi)\cdot F\cdot G^{-1}=H$ *almost everywhere.*

Proof. If ν has no atoms, then [Der00, Theorem 1.1] shows the existence of a Borel measurable map $G: L\to \mathrm{U}(n)$ satisfying (ii). However, an inspection of the proof (see the remarks after [Der00, Theorem 1.2]) reveals that for a given $\varepsilon>0$ one can also ensure (i).

Now assume that ν has an atom. Then there is a periodic finite orbit of measure one and therefore, since ν is fully supported, L is a discrete finite space. In particular, every (Borel measurable) map $G: L\to \mathrm{U}(n)$ is continuous and there is nothing to prove. □

Proof of Lemma 4.7. As a first step, we reduce the problem to the case of a metrizable space L. So let $M\subseteq \mathrm{L}^\infty(\mathrm{X})$ be a finitely generated invariant $\mathrm{L}^\infty(L,\nu)$-submodule with orthonormal basis $e_1,\ldots,e_n$. Then

$$Te_i=\sum_{j=1}^n f_{ij}e_j \quad \text{for every } i\in\{1,\ldots n\},$$

where $f_{ij}:=\mathbb{E}_{(L,\nu)}(Te_i\cdot\overline{e_j})\in \mathrm{L}^\infty(L,\nu)$ for $i,j\in\{1,\ldots,n\}$ (see Remark 3.7). Let $B\subseteq \mathrm{L}^\infty(L,\nu)$ be the unital invariant C*-subalgebra generated by the coefficients f_{ij} for $i,j\in\{1,\ldots n\}$. Then B is separable. Since $\mathrm{C}(L)$ is dense in $\mathrm{L}^1(L,\nu)$, we find a separable invariant C*-subalgebra A of $\mathrm{C}(L)$, the L^1-closure of which contains B. The canonical inclusion map $A\hookrightarrow \mathrm{C}(L)$ induces an extension $q:(L;\psi)\to(M;\vartheta)$ (see Remark 2.2), that is, $T_q(\mathrm{C}(M))=A$. Since A is separable, the space M is metrizable (see [EFHN15, Theorem 4.7]). We equip M with the pushforward measure $q_*\nu$. Then T_q defines an extension $T_q:((M,q_*\nu);T_\vartheta)\to((L,\nu);T_\psi)$. In particular, we obtain a new extension $JT_q:((M,q_*\nu);T_\vartheta)\to(\mathrm{X};T)$ and $\mathrm{L}^1(\mathrm{X})$ is thus a $\mathrm{L}^\infty(M,q_*\nu)$-module (cf. Remark 3.3). By choice of A, the $\mathrm{L}^\infty(M,q_*\nu)$-submodule generated by $e_1,\ldots,e_n$ is still invariant. Replacing $(L;\psi)$ by $(M;\vartheta)$ and ν by $q_*\nu$, we may therefore assume that L is metrizable.

Next, we show that we can pick representatives for the coefficients $f_{ij}\in \mathrm{L}^\infty(L,\nu)$ which define a $\mathrm{U}(n)$-valued function. To that end, note that

$$\sum_{k=1}^n f_{ik}\overline{f_{jk}}=\mathbb{E}_{(L,\nu)}(Te_i\cdot T\overline{e_j})=T_\psi\mathbb{E}_{(L,\nu)}(e_i\overline{e_j})=T_\psi\delta_{ij}\mathbb{1}=\delta_{ij}\mathbb{1}.$$

Picking suitable representatives for $f_{ij}\in \mathrm{L}^\infty(\mathrm{Y})$ for $i,j\in\{1,\ldots,n\}$, which we denote by the same symbol, we therefore obtain a Borel measurable map

$$F: L\mapsto \mathrm{U}(n),\quad l\mapsto (f_{ij}(l))_{i,j}$$

from L to $\mathrm{U}(n)$. For $\varepsilon>0$ set

$$\delta:=\varepsilon\cdot[2n\cdot\max\{\|e_i\|_{\mathrm{L}^\infty(\mathrm{X})}\mid i\in\{1,\ldots,n\}\}]^{-1}.$$

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Apply Lemma 4.8 to find a Borel measurable map

$$G : L \to \mathrm{U}(n), \quad l \mapsto (g_{ij}(l))_{i,j}$$

and a continuous map

$$H : L \to \mathrm{U}(n), \quad l \mapsto (h_{ij}(l))_{i,j}$$

such that:

(i) $\nu(\{l \in L \mid G(l) \neq \mathrm{Id}\}) \leqslant \delta$; and

(ii) $(G \circ \psi) \cdot F = H \cdot G$ almost everywhere.

Now consider the elements $d_i := \sum_{k=1}^n g_{ik} e_k \in M \subseteq \mathrm{L}^\infty(\mathrm{X})$ for $i \in \{1, \ldots, n\}$. Since

$$\begin{aligned} T d_i &= \sum_{k=1}^n (T_\psi g_{ik}) \cdot T e_k = \sum_{k=1}^n \sum_{j=1}^n (T_\psi g_{ik}) \cdot f_{kj} \cdot e_j \\ &= \sum_{j=1}^n \bigg(\sum_{k=1}^n (T_\psi g_{ik}) \cdot f_{kj} \bigg) e_j = \sum_{j=1}^n \bigg(\sum_{k=1}^n h_{ik} g_{kj} \bigg) e_j = \sum_{k=1}^n h_{ik} d_k \end{aligned}$$

for every $i \in \{1, \ldots n\}$, the $\mathrm{C}(L)$-submodule of $\mathrm{L}^\infty(\mathrm{X})$ generated by $d_1, \ldots, d_n$ is invariant. Moreover, a quick computation confirms that $\mathbb{E}_{(L,\nu)}(d_i \overline{d_j}) = \delta_{i,j} \mathbb{1}$ for $i, j \in \{1, \ldots, n\}$, which shows that $\{d_1, \ldots, d_n\}$ is an (L, ν)-orthonormal set. In particular, the $\mathrm{C}(L)$-submodule generated by $d_1, \ldots, d_n$ is free and hence closed in $\mathrm{L}^\infty(\mathrm{X})$ and projective. Finally, since $\nu(\{l \in L \mid G(l) \neq \mathrm{Id}\}) \leqslant \delta$ we obtain, for $i \in \{1, \ldots, n\}$,

$$\begin{aligned} \|d_i - e_i\|_{\mathrm{L}^1(\mathrm{X})} = \|\mathbb{E}_{(L,\nu)}(d_i - e_i)\|_{\mathrm{L}^1(L,\nu)} &= \bigg\| \sum_{k=1}^n g_{ik} \mathbb{E}_{(L,\nu)} e_k - \mathbb{E}_{(L,\nu)} e_i \bigg\|_{\mathrm{L}^1(L,\nu)} \\ &\leqslant \delta \cdot \bigg\| \sum_{k=1}^n g_{ik} \mathbb{E}_{(L,\nu)} e_k - \mathbb{E}_{(L,\nu)} e_i \bigg\|_{\mathrm{L}^\infty(L,\nu)} \\ &\leqslant \delta \cdot \bigg\| \sum_{k=1}^n g_{ik} e_k - e_i \bigg\|_{\mathrm{L}^\infty(\mathrm{X})} \leqslant \varepsilon. \end{aligned}$$

□

We can now prove Theorem 4.6.

Proof of Theorem 4.6. Let B be the unital C^*-subalgebra generated by all closed, invariant, finitely generated, projective $\mathrm{C}(L)$-submodules of $\mathrm{L}^\infty(\mathrm{X})$. We show that B is dense in $\mathrm{L}^1(\mathrm{X})$. Since J has relative discrete spectrum, it suffices to approximate elements f contained in a finitely generated $\mathrm{L}^\infty(L, \nu)$-submodule of $\mathrm{L}^\infty(\mathrm{X})$ with an orthonormal basis (see Proposition 3.8). Take an orthonormal basis $\{e_1, \ldots, e_n\}$ of such a module M. Let $f = \sum_{i=1}^n f_i e_i \in M$ for $f_1, \ldots, f_n \in \mathrm{L}^\infty(L, \nu)$ and $\varepsilon > 0$. Set $c_1 := \sum_{i=1}^n \|f_i\|_{\mathrm{L}^\infty(L,\nu)} + 1 > 0$. Using Lemma 4.7, we find an (L, ν)-orthonormal set $\{d_1, \ldots, d_n\} \subseteq M$ such that its $\mathrm{C}(L)$-linear hull N is invariant and

$$\|d_i - e_i\|_{\mathrm{L}^1(\mathrm{X})} \leqslant \frac{\varepsilon}{2c_1}$$

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for all $i \in \{1, \ldots, n\}$. Set $c_2 := \sum_{i=1}^n \|d_i\|_{\mathrm{L}^\infty(\mathrm{X})} + 1 > 0$. Since $\mathrm{C}(L)$ is dense in $\mathrm{L}^1(L, \nu)$, we now also find $g_1, \ldots, g_n \in \mathrm{C}(L)$ such that

$$\|f_i - g_i\|_{\mathrm{L}^1(L,\nu)} \leqslant \frac{\varepsilon}{2c_2}.$$

For $g := \sum_{i=1}^n g_i d_i \in N$ we then obtain

$$\begin{aligned}\|f - g\|_{\mathrm{L}^1(\mathrm{X})} &\leqslant \sum_{i=1}^n \|f_i\|_{\mathrm{L}^\infty(L,\nu)} \|e_i - d_i\|_{\mathrm{L}^1(\mathrm{X})} + \sum_{i=1}^n \|f_i - g_i\|_{\mathrm{L}^1(L,\nu)} \|d_i\|_{\mathrm{L}^\infty(\mathrm{X})} \\ &\leqslant \varepsilon.\end{aligned}$$

Since N is a closed, invariant, finitely generated, projective $\mathrm{C}(L)$-submodule of $\mathrm{L}^\infty(\mathrm{X})$ (use Remark 3.7), this shows that B is dense in $\mathrm{L}^1(\mathrm{X})$. By Gelfand theory we now find an extension $q : (K; \varphi) \to (L; \psi)$, an ergodic measure $\mu \in \mathrm{P}_\varphi(K)$ with $q_*\mu = \nu$ and an isomorphism $\Phi : (K, \mu; T_\varphi) \to (\mathrm{X}; T)$ with $\Phi(\mathrm{C}(L)) = B$ such that $(q, \mu; \mathrm{Id}, \Phi)$ is a topological model for J (see Remark 4.4). By definition of B and Theorem 2.8, the extension q is pseudoisometric.

Finally, assume that X is separable and L is metrizable. Then we find a sequence $(M_n)_{n\in\mathbb{N}}$ of finitely generated $\mathrm{L}^\infty(L, \nu)$-submodules of $\mathrm{L}^\infty(\mathrm{X})$ with an orthonormal basis such that their union is dense in $\mathrm{L}^1(\mathrm{X})$. For every $n \in \mathbb{N}$ we find a sequence $(N_{n,k})_{k\in\mathbb{N}}$ of closed, invariant, finitely generated, projective $\mathrm{C}(L)$-submodules contained in M_n the union of which is dense in M_n with respect to the L^1-norm. Let B the unital C^*-subalgebra of $\mathrm{L}^\infty(\mathrm{X})$ generated by $\{N_{n,k} \mid n, k \in \mathbb{N}\}$. Since $\mathrm{C}(L)$ is separable (see [**EFHN15**, Theorem 4.7]), $N_{n,k}$ is separable for all $n, k \in \mathbb{N}$. Therefore B is separable. Proceeding as above yields a pseudoisometric extension $q : (K; \varphi) \to (L; \psi)$, an ergodic measure $\mu \in \mathrm{P}_\varphi(K)$ with $q_*\mu = \nu$ and an isomorphism $\Phi : (K, \mu; T_\varphi) \to (\mathrm{X}; T)$ with $\Phi(\mathrm{C}(L)) = B$ such that $(q, \mu; \mathrm{Id}, \Phi)$ is a topological model for J. Again using [**EFHN15**, Theorem 4.7], we conclude that K is metrizable and therefore q is isometric (see Theorem 2.8). □

5. *Topological models for distal systems*

With the help of Theorem 4.6, we now prove the existence of minimal distal topological models for ergodic distal measure-preserving systems. Recall that a topological dynamical system $(K; \varphi)$ is *distal* if the extension $q : (K; \varphi) \to (\{\mathrm{pt}\}; \mathrm{id})$ over a one-point system is distal in the sense of Definition 2.10, that is, if the following condition is satisfied: whenever $(x, y) \in K \times K$ and $(\varphi^{n_\alpha})_{\alpha\in A}$ is a net with $\lim_\alpha \varphi^{n_\alpha}(x) = \lim_\alpha \varphi^{n_\alpha}(y)$, then $x = y$.

A typical example of a distal system is the skew-torus discussed in Example 2.5 which is given by an isometric extension of an isometric system. Put differently, it can be built from a trivial system by performing two isometric extensions. Furstenberg's structure theorem extends this observation, stating that in fact any minimal distal system can be built up from a trivial system via successive (pseudo)isometric extensions and *projective limits*. We recall the latter concept (see also [**dV93**, §E.12]).

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Definition 5.1. Let I be a directed set. For every $i \in I$ let $(K_i; \varphi_i)$ be a topological dynamical system, and for $i \leqslant j$ let $q_i^j : (K_j; \varphi_j) \to (K_i; \varphi)$ be an extension. Assume that:

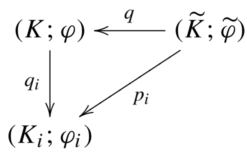
(i) $q_i^j \circ q_j^k = q_i^k$ for all $i \leqslant j \leqslant k$; and

(ii) $q_i^i = \mathrm{id}_{K_i}$ for every $i \in I$.

Then the pair $(((K_i; \varphi_i))_{i \in I}, (q_i^j)_{i \leqslant j})$ is a *projective system.*

A topological dynamical system $(K; \varphi)$, together with extensions $q_i : (K; \varphi) \to (K_i; \varphi_i)$ for every $i \in I$ such that $q_i = q_i^j \circ q_j$ for all $i \leqslant j$, is a *projective limit of* $(((K_i; \varphi_i))_{i \in I}, (q_i^j)_{i \leqslant j})$ if it satisfies the following universal property.

- Whenever $(\widetilde{K}; \widetilde{\varphi})$ is a topological dynamical system and $p_i : (\widetilde{K}; \widetilde{\varphi}) \to (K_i; \varphi_i)$ are extensions for every $i \in I$ such that $p_i = q_i^j \circ p_j$ for all $i \leqslant j$, then there is a unique extension $q : (\widetilde{K}; \widetilde{\varphi}) \to (K; \varphi)$ such that the diagram



commutes for every $i \in I$.

In this case, we write

$$(K; \varphi) = \varprojlim_i (K_i; \varphi_i).$$

Remark 5.2. Every projective system $(((K_i; \varphi_i))_{i \in I}, (q_i^j)_{i \leqslant j})$ has a projective limit, and it is unique up to isomorphy. In fact, we obtain a concrete construction of a projective limit by considering the dynamics on the compact space

$$\left\{ (x_i)_{i \in I} \in \prod_{i \in I} K_i \;\middle|\; \pi_i^j(x_j) = x_i \text{ for all } i \leqslant j \right\}$$

induced by the product action on $\prod_{i \in I} K_i$ (see [**EFHN15**, Exercise 2.18]). Moreover, if $(K_i; \varphi_i)$ is minimal for every $i \in I$, then every projective limit of $(((K_i; \varphi_i))_{i \in I}, (q_i^j)_{i \leqslant j})$ is also minimal (see [**EFHN15**, Exercise 3.19]).

Remark 5.3. The following is an operator-theoretic view of projective limits. Suppose that $(((K_i; \varphi_i))_{i \in I}, (q_i^j)_{i \leqslant j})$ is a projective system and $(K; \varphi)$, together with extensions $q_i : (K; \varphi) \to (K_i; \varphi_i)$ for $i \in I$, is a projective limit. Then the corresponding invariant unital C*-subalgebras $A_i := T_{q_i}(\mathrm{C}(K_i))$ for $i \in I$ (see Remark 2.2) satisfy:

(i) $A_i \subseteq A_j$ for $i \leqslant j$; and

(ii) the union

$$\bigcup_{i \in I} A_i$$

is dense in $\mathrm{C}(K)$

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(see [**EFHN15**, Exercise 4.16]). Conversely, assume that $(A_i)_{i\in I}$ is a net of invariant unital C*-subalgebras $A_i \subseteq \mathrm{C}(K)$ for $i \in I$ satisfying (i) and (ii). For every $i \in I$ we then find an extension $q_i : (K;\varphi) \to (K_i;\varphi_i)$ such that $A_i = T_{q_i}(\mathrm{C}(K_i))$ (see Remark 2.2) and the canonical inclusion maps $A_i \hookrightarrow A_j$ for $i \leqslant j$ induce extensions $q_i^j : (K_j;\varphi_j) \to (K_i;\varphi_i)$ between the associated systems. A moment's thought reveals that $(K;\varphi)$ is a projective limit of the projective system $(((K_i;\varphi_i))_{i\in I}, (q_i^j)_{i\leqslant j})$.

The observations discussed in Remark 5.3 are helpful for showing that a system is a projective limit of certain factors with specific properties. We demonstrate this by proving the following lemma which will soon be important.

LEMMA 5.4. *Let $(K;\varphi)$ be a topological dynamical system. Then there are*

(i) *an inductive system $(((K_i;\varphi_i))_{i\in I}, (q_i^j)_{i\leqslant j})$ of metrizable systems, and*
(ii) *extensions $q_i : (K;\varphi) \to (K_i;\varphi_i)$ for every $i \in I$,*

such that $(K;\varphi)$ together with the extensions q_i for $i \in I$ is a projective limit of $(((K_i;\varphi_i))_{i\in I}, (q_i^j)_{i\leqslant j})$.

Proof. Let I be the family of finite subsets of $\mathrm{C}(K)$ ordered by set inclusion. For every $i \in I$ let A_i be the invariant unital C*-subalgebra generated by i. Then A_i is separable for every $i \in I$ and the net $(A_i)_{i\in I}$ satisfies properties (i) and (ii) of Remark 5.3. By Remark 5.3 we therefore find a projective system $(((K_i;\varphi_i))_{i\in I}, (q_i^j)_{i\leqslant j})$ and extensions $q_i : (K;\varphi) \to (K_i;\varphi_i)$ for $i \in I$ such that $(K;\varphi)$ is a projective limit of $(((K_i;\varphi_i))_{i\in I}, (q_i^j)_{i\leqslant j})$ and $T_{q_i}(\mathrm{C}(K_i)) = A_i$ for every $i \in I$. Since A_i is separable, K_i is metrizable for every $i \in I$ (see [**EFHN15**, Theorem 4.7]), which proves the claim. □

Let us now recall the famous Furstenberg structure theorem for minimal distal systems (see [**Aus88**, Ch. 7] and [**dV93**, §V.3]).

THEOREM 5.5. *For a minimal system $(K;\varphi)$ the following assertions are equivalent.*

(a) *The system $(K;\varphi)$ is distal.*
(b) *There are an ordinal η_0 and a projective system $(((K_\eta;\varphi_\eta))_{\eta\leqslant\eta_0}, (q_\eta^\sigma)_{\eta\leqslant\sigma})$ such that:*
    (i) *$(K_1;\varphi_1)$ is a trivial system $(\{\mathrm{pt}\};\mathrm{id})$;*
    (ii) *$q_\eta^{\eta+1}$ is pseudoisometric for every $\eta < \eta_0$;*
    (iii) *$(K_\eta;\varphi_\eta) = \lim_{\gamma<\eta}(K_\gamma;\varphi_\gamma)$ for every limit ordinal $\eta \leqslant \eta_0$.*

One can take part (b) of Theorem 5.5 as an inspiration for the concept of measurably distal systems. To formulate this concept, we briefly recall the notion of inductive limits for measure-preserving systems (see [**EFHN15**, §13.5]).

Definition 5.6. Let I be a directed set. For every $i \in I$, let $(\mathrm{X}_i;T_i)$ be a measure-preserving system, and for $i \leqslant j$ let $J_i^j : (\mathrm{X}_i;T_i) \to (\mathrm{X}_j;T_j)$ be an extension. Suppose that:

(i) $J_j^k J_i^j = J_i^k$ for $i \leqslant j \leqslant k$; and
(ii) $J_i^i = \mathrm{Id}$ for every $i \in I$.

Then the pair $(((\mathrm{X}_i;T_i))_{i\in I}, (J_i^j)_{i\leqslant j})$ is an *inductive system*.

A measure-preserving system $(\mathrm{X}; T)$, together with extensions $J_i : (\mathrm{X}_i; T_i) \to (\mathrm{X}; T)$ such that $J_i = J_j J_i^j$ for $i \leqslant j$, is an *inductive limit of* $(((\mathrm{X}_i; T_i))_{i \in I}, (J_i^j)_{i \leqslant j})$ if it satisfies the following universal property.

- Whenever $(\mathrm{Y}; S)$ is a measure-preserving system and $I_i : (\mathrm{X}_i; T_i) \to (\mathrm{Y}; S)$ are extensions with $I_i = I_j J_i^j$ for $i \leqslant j$, then there is a unique extension $J : (\mathrm{X}; T) \to (\mathrm{Y}; S)$ such that the diagram

$$\begin{array}{ccc} & & (\mathrm{Y}; S) \\ & \overset{I_i}{\nearrow} & \uparrow J \\ (\mathrm{X}_i; T_i) & \xrightarrow[J_i]{} & (\mathrm{X}; T) \end{array}$$

  commutes for every i.

We then write

$$(\mathrm{X}; T) = \varinjlim_{i} (\mathrm{X}_i; T_i).$$

Every inductive system has an inductive limit (see [**EFHN15**, Theorem 13.38]) and it is unique up to isomorphy. We now recall the definition of distal systems used by Furstenberg [**Fur77**, Definition 8.3].

Definition 5.7. A measure-preserving system $(\mathrm{X}; T)$ is *distal* if there are an ordinal η_0 and an inductive system $(((X_\eta; T_\eta))_{\eta \leqslant \eta_0}, (J_\eta^\sigma)_{\eta \leqslant \sigma})$ such that:

(i) $(\mathrm{X}_1; T_1)$ is a trivial system $(\{\mathrm{pt}\}; \mathrm{Id})$;
(ii) $J_\eta^{\eta+1}$ has relatively discrete spectrum for every $\eta < \eta_0$;
(iii) $(X_\eta; T_\eta) = \lim_{\mu < \eta} (X_\mu; T_\mu)$ for every limit ordinal $\mu \leqslant \eta_0$.

Remark 5.8. If X is a standard probability space, then there is an equivalent definition in terms of so-called *separating sieves* (see [**Par68**] and [**Zim76**, Theorem 8.7]).

The measure-preserving system given by the skew-torus (see Example 3.5) is a standard example for a distal measure-preserving system. By definition, it is obtained by equipping a topologically distal system with an invariant probability measure. Our main result, generalizing [**Lin99**, Theorem 4.4], shows that, up to an isomorphism, every ergodic distal system can be obtained in this way. Moreover, the proof reveals a canonical choice for such a minimal distal model of a given distal ergodic measure-preserving system.

THEOREM 5.9. *Let* $(\mathrm{X}; T)$ *be an ergodic distal measure-preserving system. Then there are a minimal distal topological dynamical system* $(K; \varphi)$ *and a fully supported ergodic measure* $\mu \in \mathrm{P}_\varphi(K)$ *such that* $(\mathrm{X}; T)$ *is isomorphic to* $(K, \mu; T_\varphi)$*. If* X *is separable, then* K *can be (non-canonically) chosen to be metrizable.*

The following lemma (cf. the proof of [**Lin99**, Theorem 4.4]) is the last missing ingredient for the proof of Theorem 5.9.

LEMMA 5.10. *If* $(K; \varphi)$ *is a distal topological dynamical system and there is a fully supported ergodic measure* $\mu \in \mathrm{P}_\varphi(K)$*, then* $(K; \varphi)$ *is minimal.*

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Proof. Assume first that K is metrizable. Then the existence of a fully supported ergodic measure guarantees the existence of a point $x \in K$ with dense orbit $\{\varphi^n(x) \mid n \in \mathbb{Z}\}$, use Poincaré recurrence or Birkhoff's ergodic theorem (see [**KH95**, Proposition 4.1.13]). But then $(K;\varphi)$ is already minimal since a distal system decomposes into a disjoint union of minimal systems (see [**Aus88**, Corollary 7]).

If K is not metrizable, we use Lemma 5.4 to write $(K;\varphi)$ as a projective limit of metrizable factors $(K_i;\varphi_i)$ for $i \in I$. Since $(K_i;\varphi_i)$ is distal and admits a fully supported ergodic measure (recall that the pushforward of an ergodic measure is again ergodic), we obtain that $(K_i;\varphi)$ is minimal for every $i \in I$. Using that a projective limit of minimal systems is minimal (see Remark 5.2), we obtain that $(K;\varphi)$ is itself minimal. □

Proof of Theorem 5.9. For an ergodic distal measure-preserving system $(\mathrm{X};T)$ take an ordinal η_0 and an inductive system

$$(((X_\eta;T_\eta))_{\eta\leqslant\eta_0},(J_\eta^\sigma)_{\eta\leqslant\sigma})$$

as in Definition 5.7. Moreover, we write $J_\eta : (\mathrm{X}_\eta;T_\eta) \to (\mathrm{X};T)$ for the corresponding extensions for every $\eta \leqslant \eta_0$. We now recursively construct

- a projective system $(((K_\eta;\varphi_\eta))_{\eta\leqslant\eta_0},(q_\eta^\sigma)_{\eta\leqslant\sigma})$,
- ergodic measures $\mu_\eta \in \mathrm{P}_{\varphi_\eta}(K_\eta)$ for every $\eta \leqslant \eta_0$, and
- isomorphisms $\Phi_\eta : (K_\eta,\mu_\eta;T_{\varphi_\eta}) \to (\mathrm{X}_\eta;T_\eta)$ for every $\eta \leqslant \eta_0$,

such that $(q_\eta^\sigma,\mu_\sigma;\Phi_\eta,\Phi_\sigma)$ is a topological model for J_η^σ for all $\eta \leqslant \sigma \leqslant \eta_0$ and such that $(((K_\eta;\varphi_\eta))_{\eta\leqslant\eta_0},(q_\eta^\sigma)_{\eta\leqslant\sigma})$ is a projective system of minimal distal systems satisfying all the properties of Theorem 5.5(b). From this the claim follows.

Let $(K_1;\varphi_1)$ be a trivial system $(\{\mathrm{pt}\};\mathrm{id})$, μ_1 the unique probability measure on K_1 and $\Phi_1 : (K_1,\mu_1;T_{\varphi_1}) \to (\mathrm{X}_1;T_1)$ the identity operator. Now assume that $\eta \leqslant \eta_0$ is an ordinal and suppose we have already constructed $(K_\gamma;\varphi_\gamma)$ for every $\gamma < \eta$; q_γ^σ for $\gamma \leqslant \sigma < \eta$; μ_γ for $\gamma < \eta$; and Φ_γ for $\gamma < \eta$. We have to consider two cases.

(i) Assume that η is a successor ordinal, that is, $\eta = \gamma + 1$ for an ordinal γ. Since $(K_\gamma,\mu_\gamma;T_{\varphi_\gamma})$ is isomorphic to $(X_\gamma;T_\gamma)$ via Φ_γ, we can apply Theorem 4.6 to find
   - a pseudoisometric extension $q_\gamma^\eta : (K_\eta;\varphi_\gamma) \to (K_\gamma;\varphi_\gamma)$,
   - a fully supported ergodic measure $\mu_\eta \in \mathrm{P}_{\varphi_\eta}(K_\eta)$ with $(q_\gamma^\eta)_*\mu_\eta = \mu_\gamma$, and
   - an isomorphism $\Phi_\eta : (K_\eta,\mu_\eta;T_{\varphi_\eta}) \to (X_\eta;T_\eta)$,

   such that $(q_\gamma^\eta,\mu_\eta;\Phi_\gamma,\Phi_\eta)$ is a topological model for J_γ^η. Since $(K_\gamma;\varphi_\gamma)$ is distal and q_γ^η is pseudoisometric, the system $(K_\eta;\varphi_\eta)$ is also distal. Moreover, $(K_\eta;\varphi_\eta)$ is minimal by Lemma 5.10. We set $q_\sigma^\eta := q_\sigma^\gamma \circ q_\gamma^\eta$ for every $\sigma < \gamma$.

(ii) If $\eta \leqslant \eta_0$ is a limit ordinal, we let $(K_\eta;\varphi_\eta)$, together with maps $q_\gamma^\eta : (K_\eta;\varphi_\eta) \to (K_\gamma;\varphi_\gamma)$ for $\gamma < \eta$, be a projective limit of the projective system $(((K_\gamma;\varphi_\gamma))_{\gamma<\eta},(q_\gamma^\sigma)_{\gamma\leqslant\sigma})$. Moreover, let μ_η be the ergodic measure on $(K_\eta;\varphi_\eta)$ induced by the net $(\mu_\gamma)_{\gamma<\eta}$ (cf. [**EFHN15**, Exercise 10.13]), that is, $\mu_\eta \in \mathrm{C}(K_\eta)'$ is uniquely determined by the identity $(q_\gamma^\eta)_*\mu_\eta = \mu_\gamma$ for every $\gamma < \eta$. By setting

$$\Phi_\eta(T_{q_\gamma^\eta}f) := J_\gamma^\eta\Phi_\gamma f$$

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for every $f \in \mathrm{C}(K_\gamma)$ and $\gamma < \eta$ we obtain a (well-defined) map

$$\Phi_\eta : \bigcup_{\gamma<\eta} T_{q_\gamma^\eta}(\mathrm{C}(K_\gamma)) \subseteq \mathrm{C}(K_\eta) \to \mathrm{L}^1(\mathrm{X}_\eta)$$

which extends to an isometric isomorphism $\Phi_\eta : \mathrm{L}^1(K_\eta, \mu_\eta) \to \mathrm{L}^1(\mathrm{X}_\eta)$ intertwining the dynamics.

It is clear from the construction that $(q_\eta^\sigma, \mu_\sigma; \Phi_\eta, \Phi_\sigma)$ is a topological model for J_η^σ for all $\eta \leqslant \sigma \leqslant \eta_0$.

Finally, if X is separable, then we can choose metric models in (i) (see Theorem 4.6). Moreover, in (ii) we can find a subsequence $(((X_{\gamma_n}; T_{\gamma_n}))_{n\in\mathbb{N}}, (J_{\gamma_n}^{\gamma_k})_{n\leqslant k})$ of the projective system $(((X_\gamma; T_\gamma))_{\gamma\leqslant\eta}, (J_\gamma^\sigma)_{\gamma\leqslant\sigma})$ such that $(\mathrm{X}_\eta; T_\eta)$ is still the inductive limit of that subsequence (this is an easy consequence of the characterization (iii) of inductive limits in [**EFHN15**, Theorem 13.35]). By considering the now metrizable projective limit of $(((K_{\gamma_n}; \varphi_{\gamma_n}))_n, (q_{\gamma_n}^{\gamma_k})_{n\leqslant k})$ in (ii) and then proceeding as before, we also obtain metrizable models in (ii). □

Remark 5.11. Our approach to the theorem of Lindenstrauss unveils a connection between topological and measure-preserving distal systems at a categorical level. Inspecting the definition of the canonical minimal distal model $\mathrm{Mod}(\mathrm{X}; T) := (K; \varphi)$ of an ergodic distal measure-preserving system $(\mathrm{X}; T)$ in the proof of Theorem 5.9 shows that the assignment $(\mathrm{X}; T) \mapsto \mathrm{Mod}(\mathrm{X}; T)$ is actually functorial: every extension $J : (\mathrm{Y}; S) \to (\mathrm{X}; T)$ of ergodic distal measure-preserving systems induces an extension $\mathrm{Mod}(J) : \mathrm{Mod}(\mathrm{X}; T) \to \mathrm{Mod}(\mathrm{Y}; S)$ between the corresponding canonical topological models. In this way, we obtain a (contravariant) functor Mod from the category of ergodic distal measure-preserving systems to the category of minimal distal topological dynamical systems. It is noteworthy, however, that, even though we can also construct distal ergodic measure-preserving systems from distal minimal systems (by simply choosing an ergodic invariant probability measure), the functor Mod does not define an equivalence between the two categories. In fact, if $(\mathrm{X}; T)$ is an ergodic distal measure-preserving system and $(K; \varphi)$ its canonical model, then every eigenfunction of T corresponds to a continuous eigenfunction of T_φ. With this observation one can readily show that a minimal distal system $(K; \varphi)$ possessing

(i) a unique invariant Borel probability measure μ, and
(ii) an eigenfunction $f \in \mathrm{L}^\infty(K, \mu) \setminus \mathrm{C}(K)$ with respect to T_φ

cannot be isomorphic to any canonical model $\mathrm{Mod}(\mathrm{X}; T)$ of an ergodic distal measure-preserving system $(\mathrm{X}; T)$. An example due to Parry (see [**Par74**, §3]) demonstrates that such systems indeed exist and hence Mod does not define a categorical equivalence.

Remark 5.12. In his paper [**Lin99**], Lindenstrauss also discusses under what conditions an ergodic distal measure-preserving system has a distal model which is strictly ergodic (that is, minimal with a unique invariant Borel probability measure; see also [**GL19**]). It is therefore an interesting problem to determine the cases in which the canonical model constructed in this paper is strictly ergodic.

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