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BOW FLOWS WITH SMOOTH SEPARATION IN WATER OF FINITE DEPTH

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Abstract

The bow flow generated by a wide flat-bottomed ship moving in water of finite depth is examined. Solutions obtained using an integral equation technique are presented for a range of different depths and for a range of angles of the front of the bow. The solution for the limiting case of infinite Froude number is obtained as an integral, and numerical solutions are found for the nonlinear problem in which the Froude number is finite. Solutions with smooth separation are shown to exist for all values of Froude number greater than unity, for any bow slope.

1. Introduction

The flow of water under the bow or stern of a ship has been the subject of considerable research over the years. A knowledge of the flow past the body can assist in finding the most efficient design, minimising drag. In a fluid of finite depth, the flow can be characterised by the ratio D/H of the draft D (submergence depth of the body) to the depth H of the free stream, and the Froude number F, given by

$$F = \frac{U}{\sqrt{gH}}$$

where U is the velocity of the free stream and g is gravity.

Several studies have been performed on the bow flow in water of infinite depth [12, 14, 15]. Evidence was provided that there are no waveless solutions

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which approach a uniform stream for the case of a semi-infinite body with a flat bottom and an oblique sloping front. Madurasinghe [6] and Tuck and Vanden-Broeck [10], have computed flows past bodies of arbitrary shape which have either smooth separation from the body, or a stagnation point on the front of the body, and no waves in the free stream.

In a fluid of finite depth Vanden-Broeck [11] has shown the existence of flows with a stagnation point on the front of a body with a vertical face for a small range of Froude number between 1.22 and $\sqrt{2}$.

At infinite Froude number, solutions to the related problem in which the gap beneath the ship reduces to a line sink, i.e. withdrawal from a fluid of finite depth, exhibits behaviour in which the free surface rises up to some maximum height before turning downward and attaching to the wall above the sink [2, 5]. At finite values of Froude number, solutions to this problem have been computed including both smooth separation from the wall [3, 13] and solutions with a stagnation point at the point of separation [1, 4, 7].

The present investigation was conducted to determine if similar flows could be obtained for the case in which the flow is into a finite channel, as beneath a moving ship.

In this paper, the flow past a flat-bottomed object with a bow of arbitrary but constant slope $\pi\gamma$, in water of finite depth (see Figure 1c), is considered. The problem is formulated as an integral equation for the angle which the free surface makes to the horizontal. After removing a part of the solution which can be computed exactly, the linearised equations which result from taking the limit as $F \to \infty$ can be transformed to the well known airfoil equation [8, 9] and the solution can be calculated as a singular integral. If the Froude number is finite, the resulting equations become nonlinear and must be solved numerically.

Solutions are found to exist for all bow slopes between 0 and $\pi/2$ and even for larger values, and all Froude numbers in the range $1 \leq F < \infty$. No solutions are found for any geometry for values of Froude number less than unity. This is consistent with the results for the solution to the problem of the flow of a source into a fluid of finite depth beneath a free surface [3, 13]. However, it is in contrast to the results obtained for the infinite-depth case [12, 4, 15], in which no such waveless solutions were obtained. No solutions with a bow-wave-like formation similar to those found by Hocking [3] are obtained.

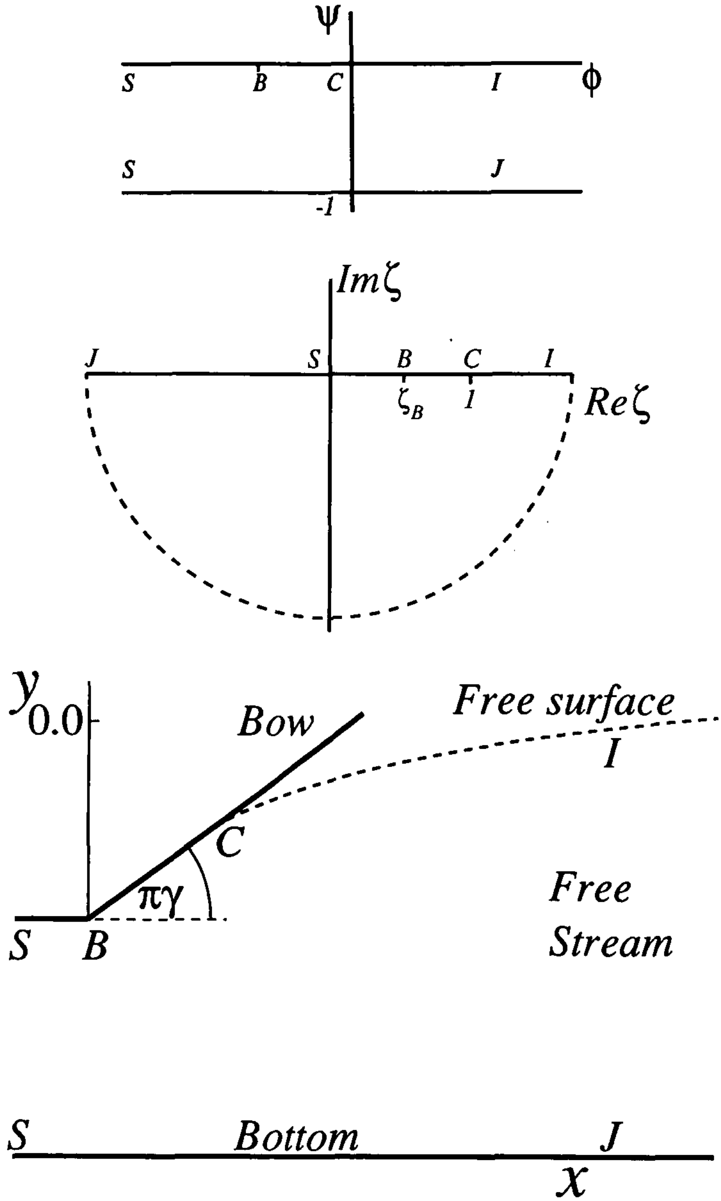

FIGURE 1. Mapped planes used in the problem formulation;
(a) the complex velocity potential w-plane,
(b) the lower half ζ-plane,
(c) the physical z-plane.

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2. Problem formulation

The steady irrotational motion of an inviscid, incompressible fluid in the absence of gravity in two dimensions is to be examined. The fluid is of finite depth and has a free surface on the right-hand side of the y-axis, and a narrow channel to the left (see Figure 1c).

Let $z = x + iy$ be the physical plane, with the origin directly above the opening of the narrow channel, and at the level of the free surface far away from the sink. The mathematical problem is to find a complex potential $w = \phi(x, y) + i\psi(x, y)$, which satisfies Laplace's equation ($\nabla^2\phi = 0$) within the flow domain, conditions of no flow across the solid boundaries and the free surface, and the condition of constant pressure on the free surface provided by Bernoulli's equation, which, if we nondimensionalise with respect to the free stream velocity U, and the free stream depth H, takes the form

$$2F^{-2}y + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = 1 \tag{2.1}$$

on $y = \eta(x)$, where $\eta(x)$ is the equation of the free surface shape. Since the equations are independent of the direction of the flow, the solutions are equally valid for a bow or stern flow. We shall henceforth work in nondimensional variables.

To derive an integral equation for this problem, we follow a similar procedure to that used in Hocking [2, 3]. The transformation

$$e^{\pi w} = \zeta \tag{2.2}$$

maps the infinite strip between $\psi = 0$ and $\psi = -1.0$ in the w-plane to the lower half of the ζ-plane (see Figure 1). Without loss of generality we may choose to let $w = 0$ correspond to the point of separation, so that the free surface $\psi = 0$, $\phi > 0$, lies along the real ζ-axis where $\zeta \geq 1$. The bottom corner of the bow is situated at ζ_B where $0 \leq \zeta_B < 1$, the point far downstream of the opening beneath the ship corresponds to the origin, and the bottom of the channel to the negative real axis (see Figure 1). The case in which $\zeta_B = 0$ corresponds to the case of a line sink in the bottom corner of a channel.

We seek w by solving for $\Omega(\zeta) = \delta(\zeta) + i\tau(\zeta)$, defined in relation to the complex conjugate of the velocity field by

$$w'(z(\zeta)) = e^{-i\Omega(\zeta)}. \tag{2.3}$$

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The magnitude of the velocity at any point is then given by $|w'(z)| = e^{\tau(\zeta)}$, and the angle any streamline makes with the horizontal is $\delta(\zeta)$. The total flux is one, and since the free stream velocity is also one, the depth of the free stream is one. Thus, for solutions which attach smoothly to the bow, we have $\delta = \pi\gamma$ at $\zeta = 1$, and $\tau, \delta \to 0$ as $\zeta \to \infty$.

On the region of the real ζ-axis given by $-\infty < \zeta < 1$, which corresponds to the solid boundaries of the flow domain, the streamlines must be parallel to the walls, so that the condition that there be no flow normal to the solid boundaries is satisfied if we choose $\delta(\zeta)$ to be the angle of the wall to the horizontal, i.e.

$$\delta(\zeta) = \begin{cases} 0 & \text{if } -\infty < \zeta < 0; \\ 0 & \text{if } 0 < \zeta < \zeta_B; \\ \pi\gamma & \text{if } \zeta_B < \zeta \leq 1. \end{cases} \tag{2.4}$$

The only singularities of the function $\Omega(\zeta)$ in the ζ-plane are those at the origin and at $\zeta = \zeta_B$, corresponding to the point at downstream infinity, and the bottom corner of the bow of the ship, respectively. Both of these singularities can be shown to be weaker than a simple pole, so that Cauchy's theorem can be applied to $\Omega(\zeta)$ on a path consisting of the real ζ-axis, a semi-circle at $|z| = \infty$ in the lower half plane, and a circle of vanishing radius about the point ζ. Hence, for $\mathrm{Im}\{\zeta\} < 0$ we have

$$\Omega(\zeta) = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\Omega(\zeta_0)}{\zeta_0 - \zeta} d\zeta_0, \tag{2.5}$$

since $\Omega \to 0$ as $|\zeta| \to \infty$. If we let $\mathrm{Im}\{\zeta\} \to 0^-$, we obtain

$$\tau(\zeta) = \frac{1}{\pi} \fint_{-\infty}^{+\infty} \frac{\delta(\zeta_0)}{\zeta_0 - \zeta} d\zeta_0 \tag{2.6a}$$

and

$$\delta(\zeta) = -\frac{1}{\pi} \fint_{-\infty}^{+\infty} \frac{\tau(\zeta_0)}{\zeta_0 - \zeta} d\zeta_0, \tag{2.6b}$$

where the integrals are of Cauchy principal-value form.

Substituting the known values of $\delta(\zeta)$ given by (2.4) into the equation for $\tau(\zeta)$ gives

$$\tau(\zeta) = \gamma \ln\left[\frac{1-\zeta}{\zeta - \zeta_B}\right] + \frac{1}{\pi} \fint_{1}^{\infty} \frac{\delta(\zeta_0)}{\zeta_0 - \zeta} d\zeta_0. \tag{2.7}$$

The singularity at $\zeta = 1$ in (2.7) can be removed by noting that on $1 < \zeta < \infty$,

$$-\frac{1}{\pi} \fint_{1}^{\infty} \frac{\arcsin \zeta_0^{-1/2}}{\zeta_0 - \zeta} d\zeta_0 = \frac{1}{2} \ln\left(\frac{1-\zeta}{\zeta}\right). \tag{2.8}$$

Writing $\delta(\zeta) = 2\gamma \arcsin \zeta^{-1/2} + \delta_b(\zeta)$, (2.7) becomes

$$\tau(\zeta) = \gamma \ln(\frac{\zeta}{\zeta - \zeta_B}) + \frac{1}{\pi} \fint_1^\infty \frac{\delta_b(\zeta_0)}{\zeta_0 - \zeta} d\zeta_0, \tag{2.9}$$

on $1 < \zeta < \infty$.

On the remainder of the boundary, $-\infty < \zeta < 1$, τ can be shown to take the form

$$\tau(\zeta) = 2\gamma \ln(1 + \sqrt{1-\zeta}) - \gamma \ln(\zeta - \zeta_B) + \frac{1}{\pi} \int_1^\infty \frac{\delta_b(\zeta_0)}{\zeta_0 - \zeta} d\zeta_0. \tag{2.10}$$

To obtain the value of τ, we must satisfy the Bernoulli condition of constant pressure. This can be achieved by combining (2.1), (2.2) and (2.3), to give after some manipulation

$$\tau(\zeta) = \tfrac{1}{3} \ln\left(1 + \frac{3}{\pi F^2} \int_\zeta^\infty \frac{\sin\delta(\zeta_0)}{\zeta_0} d\zeta_0\right). \tag{2.11}$$

on $1 \le \zeta < \infty$. Combining this with (2.9) gives a nonlinear integral equation for δ on the free surface.

Once this is solved, δ is known everywhere on the boundary, and hence we can obtain τ from (2.9) or (2.10). Using δ and τ, it is possible to integrate (2.3) to obtain the location of points on the free surface. These may be written as

$$x(\zeta) = x(\zeta^*) + \frac{1}{\pi} \int_{\zeta^*}^{\zeta} \frac{e^{-\tau(\zeta_0)} \cos\delta(\zeta_0)}{\zeta_0} d\zeta_0, \tag{2.12}$$

and

$$y(\zeta) = y(\zeta^*) + \frac{1}{\pi} \int_{\zeta^*}^{\zeta} \frac{e^{-\tau(\zeta_0)} \sin\delta(\zeta_0)}{\zeta_0} d\zeta_0. \tag{2.13}$$

Since $y \to 0$ as $\zeta \to \infty$, the depth of the separation point is

$$h_C = \frac{1}{\pi} \int_1^\infty \frac{e^{-\tau(\zeta_0)} \sin\delta(\zeta_0)}{\zeta_0} d\zeta_0, \tag{2.14}$$

and the draft D is given by

$$D = h_C + \frac{\sin\pi\gamma}{\pi} \int_{\zeta_B}^{1} \frac{e^{-\tau(\zeta_0)}}{\zeta_0} d\zeta_0. \tag{2.15}$$

The downstream velocity in the gap can be obtained from the value of $\tau(\zeta)$ at $\zeta = 0$ since $e^{\tau(\zeta)}$ is the velocity at any point, and we know the flux is one, so that the height of the gap is $G = e^{-\tau(0)}$ and must equal $1 - D$.

Thus, if we can find $\delta_b(\zeta)$ on $1 \le \zeta < \infty$ by solving the integral equation given by (2.9) and (2.10), we can compute all aspects of the flow.

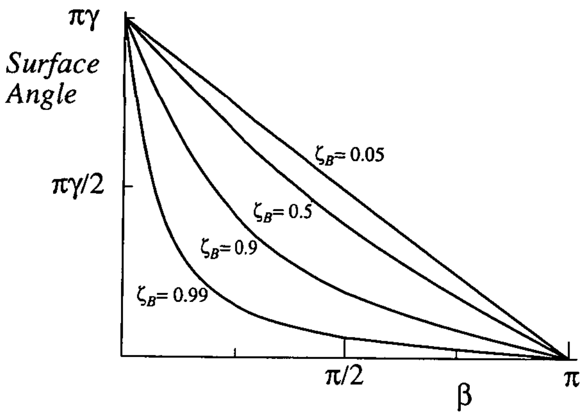

FIGURE 2. Diagram showing the angle of the free surface for several values of ζ_B. The solutions are the same if scaled by the bow slope, γ.

3. Solution in the infinite Froude number limit

In the limit as the Froude number approaches infinity, the velocity on the free surface approaches a constant value given by (2.2). Thus the solution in this limit can be obtained by setting $\tau = 0$ on $1 < \zeta < \infty$, in equation (2.9). There exists a solution to this integral equation which is written as a principal-value integral. A simple transformation takes (2.9) into the form of the airfoil equation

$$\frac{1}{\pi} \fint_{-1/2}^{1/2} \frac{\Gamma(\alpha)}{\beta - \alpha} d\alpha = \frac{\gamma}{\beta + 1/2} \ln \left[1 - \zeta_B(\beta + 1/2) \right] \tag{3.1}$$

where

$$\Gamma(\alpha) = \frac{\delta_b(\alpha)}{\alpha + 1/2}$$

and $\zeta = (\beta + 1/2)^{-1}$, for which a particular solution is given by ([8, 9]),

$$\Gamma_{PS}(\beta) = -\frac{\gamma}{\pi} \sqrt{\frac{1/2 + \beta}{1/2 - \beta}} \fint_{-1/2}^{1/2} \sqrt{\frac{1/2 - \alpha}{1/2 + \alpha}} \ln(1 - \zeta_B(\alpha + 1/2)) \frac{d\alpha}{\beta - \alpha}. \tag{3.2}$$

The solution to the homogeneous airfoil equation, in which the right-hand side of (3.1) is zero, is given by ([8, 9])

$$\Gamma_0(\beta) = \frac{\chi}{\pi \sqrt{1/4 - \beta^2}} \tag{3.3}$$

where χ is a constant which must be chosen to satisfy the boundary conditions. The only solution which is bounded at both ends of the interval is that for which $\delta_b(\beta) = 0$ at $\beta = 1/2, -1/2$. These conditions give a smooth separation from the body and a horizontal free stream surface as $x \to \infty$. Combining (3.2) and (3.3), and choosing χ so that both boundary conditions are satisfied, the solution for δ_b is given by

$$\delta_b(\beta) = -\frac{\gamma}{\pi}\sqrt{(1/2+\beta)(1/2-\beta)} \int_{-1/2}^{1/2} \frac{\ln(1-\zeta_B(\alpha+1/2))}{\sqrt{(1/2+\alpha)(1/2-\alpha)}} \frac{d\alpha}{\beta-\alpha} \tag{3.4}$$

or, in the original coordinates

$$\delta_b(\zeta) = -\frac{\gamma}{\pi}(\zeta-1)^{1/2} \int_{1}^{\infty} (\zeta_0-1)^{-1/2} \ln(1-\frac{\zeta_B}{\zeta_0}) \frac{d\zeta_0}{\zeta_0-\zeta} \tag{3.5}$$

on $1 < \zeta < \infty$. This integral can be solved using standard techniques (see, e.g., Tuck [9]), and remembering that the angle of the free streamline is given by $\delta(\zeta) = 2\gamma \arcsin \zeta^{-1/2} + \delta_b(\zeta)$, the angle of the free streamline and hence the free surface shape, draft and separation depths can be computed very accurately.

The shape of the free surface was computed for a range of bow angles and gap sizes. In Figure 2, the angle of the free surface, δ, is plotted against a mapped variable $\beta = \arcsin \zeta^{-1/2}$ for a range of values of ζ_B. As ζ_B approaches zero, the solution approaches the line $\delta = 2\gamma\beta$, as it should. It is an interesting fact that the solution, $\delta(\zeta)$, for a given value of ζ_B is a solution for any value of γ if multiplied by the appropriate factor, as can be seen from equation (2.9). Thus the solution for a line source in the corner of a wedge with internal angle $\pi\gamma$ is given by $\delta(\zeta) = 2\gamma \arcsin \zeta^{-1/2}$. This is not true of the free streamline shape in the physical plane, however.

4. The nonlinear problem

The full nonlinear integral equation for δ given by equations (2.9) and (2.11) on $1 < \zeta < \infty$ can not be solved analytically, and we must resort to numerical methods. This can be done by truncating the integrals to some large value of $\zeta = \zeta_T$, and generating a set of nonlinear algebraic equations by evaluating the integral at N evenly spaced points on the interval $1 < \zeta < \zeta_T$, using a guess for the value of δ_b at the same N points. This guess can be successively updated using a Newton iteration scheme until the residual error on the surface is less than some small value (e.g. 10^{-9}) at every point on the interval. The value of

ζ_T was increased until it had minimal effect on the calculations; usually a value of around $\zeta = 30$ was found to be sufficient. The principal-value integral can be treated by letting

$$\frac{1}{\pi}\fint_1^{\zeta_T} \frac{\delta_b(\zeta_0)}{\zeta_0 - \zeta} d\zeta_0 = \frac{1}{\pi}\int_1^{\zeta_T} \frac{\delta_b(\zeta_0) - \delta_b(\zeta)}{\zeta_0 - \zeta} d\zeta_0 + \frac{\delta_b(\zeta)}{\pi} \ln\left(\frac{\zeta_T - \zeta}{\zeta - 1}\right). \quad (4.1)$$

The singular point is thus removed, and all of the integration can be performed using standard techniques. In this paper, cubic splines were used for all of the calculations.

A value of $N = 240$ was found to give anwers accurate to four decimal places, and was used for most calculations. For each bow slope and value of ζ_B, a solution was computed for some value of $F > 1$, and the Froude number was gradually decreased, using the previous solution as a starting guess, until the method failed to converge. The iteration scheme converged rapidly, usually within four steps, for all values of bow slope, gap size and Froude number greater than unity. However for Froude numbers less than one, the method failed to converge for any combination of parameters despite numerous attempts using different starting values, thus providing evidence that no solutions of this type exist for subcritical values of F. Several attempts were made to compute solutions at a unique Froude number, as found by Vanden-Broeck and Keller [13] for the line source problem, by allowing F to be a variable rather than a fixed quantity, but without success.

5. Results and Conclusions

All of the computed solutions exhibit a smooth separation of the free surface from the bow (see Figures 3, 4, 5, 6) and asymptote to the free stream level from below. There is no sign of the beginning of a bow wave for any geometrical configuration of the flow boundaries.

Figure 3 shows typical free streamlines for the case of a bow sloping forward at an angle of 45^o with various values of the submergence depth. The solid line represents the bow, and the dashed line the free surface.

Figure 4 shows flows with a fixed draft for $F = \infty$, but varying angles of the bow face. It can be seen that the point of separation occurs relatively closer to the surface for a bow with a smaller value of γ. As x increases, the free surface converges to much the same shape, which is as it should be since the effect of the bow slope is only important locally for a body of fixed draft.

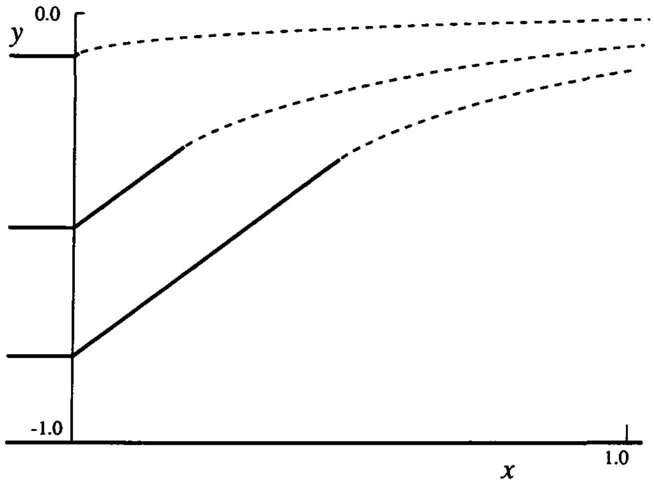

FIGURE 3. Some free surface shapes for a fixed bow slope with different submergence depths for the infinite Froude number case. All asymptote to the level $y = 0$, and the bottom of the channel is at $y = -1$. Solid lines depict the bow, dashed lines the free surface.

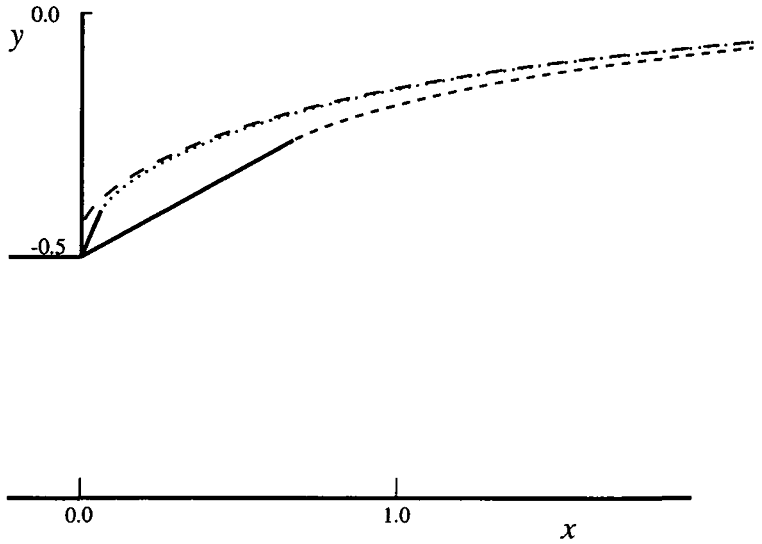

FIGURE 4. Free surface shapes computed for a fixed submergence depth at infinite Froude number, but with different bow slopes. The effect of the bow is local, with all free surfaces asymptoting to the same shape as x increases.

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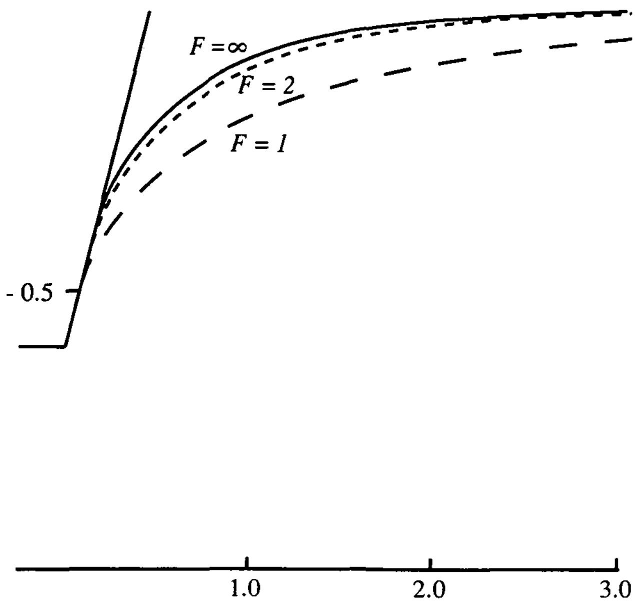

FIGURE 5. Comparison of free surface shapes for different values of the Froude number, $F = \infty$, $F = 2$, $F = 1$, where the slope is 54^o and the draft $D/H = 0.6$.

Figure 5 shows the effect of varying the Froude number on the shape of the free surface. A fixed geometry of $\gamma = 54^o$ and $D/H = 0.6$ is shown for several different values of the Froude number. The free surface shapes obtained for finite values of Froude number differ little from those obtained in the limit as $F \to \infty$, except as $F \to 1$.

The computed solutions are also valid for values of $\gamma > 1/2$, and it is interesting to note that for such flows the separation point occurs very close to the corner compared to the cases where $\gamma < 1/2$ (see Figure 6), i.e as γ increases, the separation point occurs closer to the corner.

The related problem in which the gap is replaced by a line sink on the bottom of a channel produced solutions with smooth separation for finite values of F ([3, 13]) above some lower bound (usually $F = 1$), and this qualitative behaviour is repeated in this problem. At very low values of the Froude number for this related problem, solutions with a stagnation point above the sink were found [1, 4], and there is no reason to believe that solutions with a stagnation point on the bow could not exist at low Froude number. Vanden-Broeck [11] has already computed stagnation-point solutions over a small range of Froude numbers,

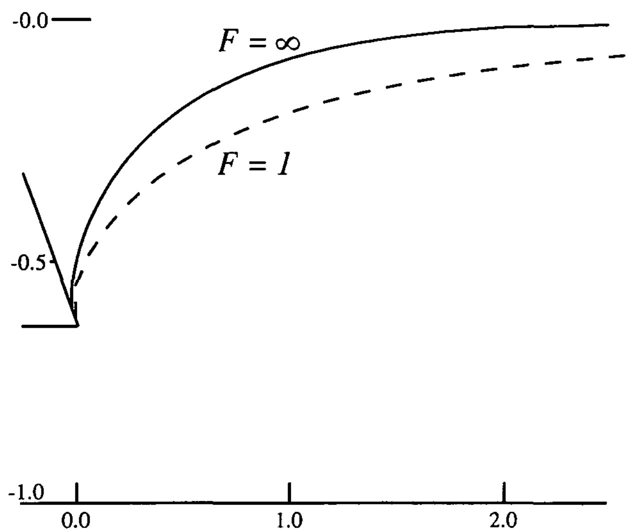

FIGURE 6. Free surface shapes for a bow which slopes backward for $F = 1$ and $F = \infty$. Note that the separation is still smooth, but very close to the corner. The slope is 126^0 and $D/H = 0.63$.

$1.22 < F < \sqrt{2}$. In addition, Vanden-Broeck [12] computed solutions in which the flow separates smoothly from the bottom corner of the body rather than the upslope for a fluid of infinite depth. Work is continuing to attempt to compute both of these types of solution, and the results will be reported at a later time.

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