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COMPACT 16-DIMENSIONAL PROJECTIVE PLANES WITH LARGE COLLINEATION GROUPS. IV

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Let $\mathscr{P}$ be a topological projective plane with compact point set P of finite (covering) dimension. In the compact-open topology (of uniform convergence), the group Σ of continuous collineations of $\mathscr{P}$ is a locally compact transformation group of P.

THEOREM. *If* $\dim \Sigma > 40$, *then* $\mathscr{P}$ *is isomorphic to the Moufang plane* $\mathscr{O}$ *over the real octonions* (*and* $\dim \Sigma = 78$).

By [**3**] the translation planes with $\dim \Sigma = 40$ form a one-parameter family and have Lenz type V. Presumably, there are no other planes with $\dim \Sigma = 40$, cp. [**17**].

If $\dim \Sigma > 35$, then $\dim P > 8$, each line is homotopy equivalent to the sphere $\mathbf{S}^8$, and $\dim P = 16$, see [**11**, (4.0)] and [**5**]. Moreover, any connected closed subgroup $\Delta \leqq \Sigma$ is a Lie group [**6**], and Δ is semisimple or fixes a point or a line [**16**, (2.1)]. In each of the following cases, $\mathscr{P} \cong \mathscr{O}$ has already been shown:

(I) $\dim \Delta \geqq 37$, and Δ is semisimple [**15**],

(II) $\dim \Delta \geqq 39$, and Δ fixes exactly one element (point or line) [**17**, (C)] or a non-incident point-line pair [**15**, (2.2)],

(III) $\dim \Delta \geqq 40$, and Δ fixes two points or two lines [**16**, Section 5].

If Δ has more fixed elements, then $\dim \Delta \leqq 38$ by [**12**]. In the only remaining case, the fixed elements of Δ form a flag (v, W), and Δ has a minimal normal subgroup $\Theta \cong \mathbf{R}^t$ consisting [**16**, (2.2)] of translations with axis W and center v. The theorem will be proved in the following main steps: For $a \notin W$ the connected component Γ of the stabilizer Δ_a cannot be semisimple, and there is a normal subgroup $\Xi \cong \mathbf{R}^s$ which consists of elations with axis av. Dually, there is a group $\Pi \cong \mathbf{R}^r$ of translations with center $u \in W \setminus v$. Up to duality, $s \leqq r$. The stabilizer ∇ of the triangle (a, u, v) induces irreducible representations on subgroups of Θ, Ξ, and Π. The representation on the product of two of these groups is faithful (∇ is reductive). By a combination of group theoretic and geometric arguments, $r < 8$ turns out to be impossible. Hence $\mathscr{P}^W$ is a translation plane, and the result follows from Hähl's classification [**3**, p. 264] of all translation planes with $\dim \Sigma \geqq 38$.

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By [**5**], P is of dimension $d = 2^{m+1}$ $(0 \leqq m \leqq 3)$. The theorem may then be combined with analogous results [**9**; **10**; **13**] for planes of dimension $d = 2^{m+1}$ $(0 \leqq m < 3)$ to obtain the following corollary. Let $g = g(m)$ denote the dimension of the full automorphism group of the "classical" plane over the real or complex numbers, the quaternions or octonions respectively.

COROLLARY. *If $\mathscr{P}$ is a compact d-dimensional projective plane, and if*

$$\dim \Sigma > \left[\frac{g}{2}\right] + \left[\frac{m}{2}\right],$$

then $\mathscr{P}$ is classical. The given bound is sharp.

Since $\dim \Delta \leqq 40$ for proper translation planes, it will be assumed throughout that the group T of translations in Δ with axis W is not transitive and also that $\mathscr{P}$ is not a dual translation plane. The group of translations with center $z \in W$ will be denoted by T_z. The next theorem is also due to Hähl [**4**, Corollary 1.3], and will play a key role:

(H) *If Ω is a connected subgroup of Δ and if $a^{\Omega} \neq a \notin W = W^{\Omega}$, then either Ω_a acts effectively on W or $a^{\Omega} = a^{\mathrm{T} \cap \Omega}$.*

Another useful fact is a topological analogue [**18**] of a well-known theorem of Gleason:

(T) *If $\mathrm{T}_z \cong \mathbf{R}^k$ for all $z \in W$ and some fixed $k > 0$, then T is transitive.*

As in the case of 8-dimensional planes [**11**, (1.2)], and with an analogous proof one has

(R) *There are at most* 3 *pairwise commuting reflections.*

Many steps in the proof of the theorem require information about the connected component Λ of the stabilizer of a quadrangle (the automorphism group of a corresponding ternary field).

(A) *Let $\mathscr{F}$ be the subplane of the fixed elements of Λ.*

(i) $\Lambda \cong G_2$, *the compact* 14-*dimensional automorphism group of the octonions*, *or* $\dim \Lambda < 14$.

(ii) *If Λ contains a pair of commuting involutions, then Λ is compact.*

(iii) *If Λ is compact, then* $\Lambda \cong G_2$, SU_3, *or* SO_4, *or* $\dim \Lambda < 5$.

(iv) *If* $\dim \mathscr{F} > 2$, *then* $\Lambda \cong SU_3$ *or* $\dim \Lambda < 8$.

(v) *If* $\dim \mathscr{F} = 8$ (*i.e.*, *if $\mathscr{F}$ is a Baer subplane*), *then* $\Lambda \cong SU_2$ *or* SO_2.

For a proof see [**12**] and [**16**, Corollary], and note that Λ is a Lie group. Assertion (v) follows from [**12**, (1.7) and (2.3)].

More can be said exploiting the existence of an invariant group $\Theta \cong \mathbf{R}^t$ of translations [**17**; **16**; **13**]:

(B) *Assume that* Λ *fixes* $a \notin W$ *and* $c \in a^{\Theta}$ *and* 3 *points* $u, v, w \in W$ *where* $a^{\Theta} \subseteq av$.

(i) *If* $t = 1$, *then* $\Lambda \cong G_2$ *or* $\dim \Lambda \leqq 10$.

(ii) *If* $t = 2$, *then* $\Lambda \cong SU_3$ *or* $\dim \Lambda < 8$.

(iii) *If* $2 < t < 8$, *then* $\dim \Lambda \leqq 6$, *or* $\Lambda \cong SU_3$ *and* $t = 7$.

(iv) *If* $t = 8$, *then* Λ *is compact and* (A, iii) *applies.*

The last part of (iii) is a consequence of the fact that the action of Λ on a^{Θ} is naturally equivalent to the action on Θ and that SU_3 has no representation in dimension <6.

(C) *If the assumption* $c \in a^{\Theta}$ *in* (B) *is replaced by* $c \in av \backslash a^{\Theta} \backslash v$, *then* Λ *is compact or* $\dim \Lambda \leqq 6$ *or* $t = 1$ *and* $\dim \Lambda < 8$, *see Section* 1 *below.*

It will be proved in (2.3) that $\dim \Delta > 40$ and $\dim \mathsf{T} < 16$ imply $\Lambda \ncong G_2$. Hence

(B′) $\dim \Lambda \leqq 8$ *or* $t = 1$ *and* $\dim \Lambda \leqq 10$.

Together with (2.2) and its dual follows immediately

(D) *If* ∇ *fixes a triangle* (a, u, v), *then*

$$17 \leqq \dim \nabla \leqq 22.$$

Another useful application of (B′) is

(E) *If* $\dim \Delta > 40$, *and if the translation group satisfies* $\mathsf{T}_v < \mathsf{T}$, *then* $\dim \mathsf{T} = n > 8$, *and* $\dim \mathsf{T}_z > 0$ *for each* $z \in W$. *Moreover,* T *is the centralizer of its connected component* T^1.

Proof. Let $b \in a^{\mathsf{T} \backslash \mathsf{T}_v}$ and $c \in a^{\Theta} \backslash a$, and denote the connected component of $\Delta_{a,b,c}$ by Λ. Then

$$24 - n < \dim \Delta_{a,b} \leqq t + \dim \Lambda \leqq 16, \text{ and } n > 8.$$

By the definition of translations, $\tau \mapsto a^{\tau} z$ induces an injective map of T/T_z into the pencil $\mathcal{L}_z$, and the dimension $\mathsf{T}:\mathsf{T}_z$ is at most 8. Hence each $z \in W$ is the center of some connected subgroup of T and is fixed by the centralizer of T^1. Note in particular

(F) $\dim \mathsf{T} \leqq \dim \mathsf{T}_z + 8$.

The following fact [**7**, **19** or **22**] will be needed repeatedly

(G) *If* G *is a connected transitive subgroup of* $GL_6\mathbf{R}$, *then a maximal compact subgroup of* G *is isomorphic to* SU_3, U_3, *or* SO_6, *and* $\dim G \leqq 10$ *or* $\dim G \geqq 16$. *Moreover,* G' *is compact or* $\dim G' \geqq 16$.

Notation is mostly standard, and is in accordance with that in parts I-III ([**15**, **16**, **17**]). The meaning is often indicated in the text. We note that

$$\Gamma:\Delta = \dim \Gamma - \dim \Delta$$

is the dimension of the coset space Γ/Δ, so that

$$\Gamma:\Gamma_x = \dim x^\Gamma;$$

and distinguish between the commutator group $\Delta' = \Delta \circ \Delta$ and the connected component Δ^1 of the identity.

1. The stabilizer of a quadrangle. For the proof of (C), introduce coordinates from a ternary field K as in [**16**, Section 1]. The translations in Θ are given by $(x, y) \mapsto (x, y + s)$, where $s \in S = S^\Lambda \cong \mathbf{R}^t$ and $1 \notin S$ by the hypothesis of (C). Let $0 \neq d \in S$ and denote the subternary of the fixed elements of Λ_d by D. Then D properly contains the one-parameter group spanned by d in S. Hence D is connected [**14**, (1.8)], and $\dim D = 2^k > 1$. If there is a closed subternary H with $D < H < K$, then Λ is compact by [**12**, Zusatz]. For $t = 1$ the assertion is but a restatement of [**16**, Corollary]. If $t > 1$, then Λ is compact or $\Lambda_{d,d'} = 1$, so that (C) is true for $t \leqq 3$. Now choose S minimal and assume $t \geqq 4$. Then Λ acts faithfully and irreducibly on S. Hence Λ' is semisimple and $\Lambda:\Lambda' \leqq 2$, see [**2**, (19.17)]. If Λ contains a pair of commuting involutions, then Λ is compact by (A, ii). Otherwise Λ' is quasisimple and $\dim \Lambda' \leqq 3$ or a maximal compact subgroup of Λ' is isomorphic to Spin_3. In the latter case, Λ contains a central involution α. The fixed elements of α coordinatize an invariant Baer subplane. Now [**11**, (2.13)] and (A, v) imply again $\dim \Lambda' \leqq 3$.

2. The stabilizer of an affine point. In the sequel, $\mathscr{P}$ will always denote a compact 16-dimensional projective plane such that neither $\mathscr{P}$ nor its dual is a translation plane; Δ is a connected Lie group of automorphisms of $\mathscr{P}$ with $\dim \Delta > 40$ fixing a line W, a point $v \in W$ and no other elements. These general assumptions will usually not be repeated. By [**16**, Section 2], the group $\mathsf{T}_v = \Delta_{[v,W]}$ of translations in Δ with center v has an invariant subgroup $\Theta \cong \mathbf{R}^t$. In this section, the connected component Γ of the stabilizer Δ_a of a point $a \notin W$ will be investigated. Note that $25 \leqq \dim \Gamma \leqq 38$ by (A, i).

(1) *If* $u \in W \backslash v$, *then* $\dim u^\Gamma > 4$.

Proof. (a) First assume $u^\Gamma = u$. Let $K = av$ and consider the connected component Ψ of Δ_K. With (A, i) follows $\dim \Gamma \leqq 30 < \dim \Psi$. If $u^\Psi = u \neq u^\delta$ then $K^\delta = L \neq K$ and $\Psi^\delta = \Delta_L^1$. Therefore $\Gamma_L^1 \leqq \Psi \cap \Psi^\delta$ fixes a quadrangle, but $\dim \Gamma_L \geqq 17$. Hence $\Psi:\Psi_u > 0$ and there is some $\delta \in \Psi$ with $u^\delta \neq u$ and $a^\delta = c \neq a$. Now $\Gamma_c^1 \leqq \Gamma \cap \Gamma^\delta$ fixes also u and u^δ, which again contradicts (A, i). Consequently, $\dim u^\Gamma = k > 0$.

(b) If Λ is the connected component of the stabilizer of a, $c \in a^\Theta$ and two points in u^Γ, then $\Gamma:\Lambda \leqq 2k + t$, and $k > 1$. Moreover, $k > 4$ or $\Lambda \cong \mathrm{G}_2$ acts in the standard way on W by (B) and [**15**, (1.2)], and u^Γ contains an orbit $z^\Lambda \approx \mathbf{S}^6$.

(2) Γ_u *is not transitive on* $W \setminus \{u, v\}$ *for any* $u \in W \setminus v$.

Proof. Assume that the effective action

$$\Omega = \Gamma^W = \Gamma/\Gamma_{[W]}$$

is doubly transitive on $W \setminus v \approx \mathbf{R}^8$. Then Ω is an extension of $\mathbf{R}^8$ by a transitive linear group, and the latter contains a subgroup $\Phi \cong \mathrm{Spin}_k$ with $5 \leqq k \leqq 7$, see [**19**, IV C; **7** or **22**]. There is an isomorphic copy of Φ in a maximal semisimple subgroup of Γ, and Φ fixes a triangle. Since Φ does not act on a proper subplane by [**11**, (**)], the central involution $\sigma \in \Phi$ is a reflection with center $u \in W \setminus v$ or axis au. Transitivity of Γ implies that the elation group $\Delta_{[v,av]}$ is transitive. The axis av not being fixed by the general assumption, $\mathscr{P}$ is then even a dual translation plane, a contradiction.

(3) *The group* G_2 *is not contained in* Δ.

Proof. The fixed elements of a group $\Lambda \cong G_2$ form a flat (= 2-dimensional) subplane $\mathscr{E}$ by [**15**, (1.2)]. Choosing a in $\mathscr{E}$, one has $\Lambda < \Gamma$. Let $\Omega = \sqrt{\Gamma}$ denote the radical (maximal connected solvable normal subgroup) of Γ. Either $\Gamma = \Lambda\Omega$ or Λ is properly contained in a semi-simple subgroup Ψ of Γ. In the latter case, Λ is normal in Ψ or there is even a quasisimple group Ψ. Inspection of the list of simple Lie groups shows that Ψ is then the complexification $G_2^{\mathbf{C}}$ or contains a compact group Φ isomorphic to SO_7 or Spin_7. These 5 possibilities will be treated separately.

(a) $\Lambda \lhd \Psi$. Then Ψ/Λ induces on $\mathscr{E}$ a quasisimple group fixing 2 points and two lines. This contradicts [**9**, (5.2)].

(b) $\Psi \cong G_2^{\mathbf{C}}$. Then Λ is a maximal subgroup of Ψ, and $\Psi{:}\Lambda = 14$. Hence Ψ fixes each point of $\mathscr{E} \cap W$ and dually. This contradicts (A, i).

(c) $\Phi \cong SO_7$. Then the diagonal involution $\alpha = (-1)^6 \times (1)$ and each of its conjugates has a centralizer SO_6. Hence α is not planar by the first part of [**11**, (*)], and α cannot be a reflection by (R).

(d) $\Phi \cong \mathrm{Spin}_7$. Then the central involution $\sigma \in \Phi$ is a reflection. If σ has axis W and center a, the translation group T is connected by (H), and $\tau^\sigma = \tau^{-1}$ for each $\tau \in \mathsf{T}$. Hence Φ acts faithfully on each invariant component $\Xi \leqq \mathsf{T}$ and dim Ξ is even and >6. Since $\mathsf{T} \neq \mathsf{T}_v$ and the action of Φ is completely reducible (see e.g. [**2**, (35.4)]), $\mathsf{T} = \Theta \times \Pi$ is a product of two irreducible components, and dim $\mathsf{T} = 16$ contrary to the assumption. By analogous arguments, σ cannot have a center on W.

(e) $\Gamma = \Lambda\Omega$. Choose $u \in W \setminus v$ in $\mathscr{E}$, and consider the stabilizer $\nabla = \Gamma_u^1 = \Lambda\mathsf{P}$ where

$$\mathsf{P} = \sqrt{\nabla} = \Omega_u^1$$

is the radical of ∇. From dim $\Delta > 40$ follows dim $\mathsf{P} > 2$. On the other hand,

$\dim(\nabla \cap \mathrm{Cs}\,\Lambda) \leqq 2$

by [**8**, Section 3]. Consequently $\Lambda \circ \mathsf{P} \neq 1$, and Λ is faithfully represented on the Lie algebra $\ell\mathsf{P}$. This implies $\dim \mathsf{P} \geqq 7$. Being solvable, Ω has a normal subgroup N such that $1 \leqq \dim u^{\mathsf{N}} \leqq 2$. If $u \neq w \in u^{\mathsf{N}}$, then $w^{\mathsf{P}} \subseteq u^{\mathsf{N}}$ and $\mathsf{P}{:}\mathsf{P}_w \leqq 2$. Also, there is $c \in a^{\Theta}\backslash a$ with $\mathsf{P}{:}\mathsf{P}_c \leqq 2$. Now $\dim \mathsf{P}_{c,w} \geqq 3$, and $\mathsf{P}_{c,w} \lhd \nabla_{c,w}$. Not being simple, $\nabla_{c,w} \not\cong \mathrm{G}_2$. From $\nabla = \Lambda\mathsf{P}$ follows $\dim \nabla \geqq 21$, $\dim \nabla_w \geqq 13$, and (B) implies $t \geqq 7$.

If $t = 7$, then ∇ induces an irreducible group on Θ, and $\mathsf{K} = \mathsf{P} \cap \mathrm{Cs}\,\Theta$ acts freely on $W\backslash\{u, v\}$ since a^{Θ} is not contained in any proper subplane. The radical P/K consists of scalar multiplications of $\Theta \cong \mathbf{R}^7$ and $\mathsf{P}{:}\mathsf{K} \leqq 1$. But this would imply $6 \leqq \dim w^{\mathsf{K}} \leqq 2$. For $t = 8$, finally, $\mathsf{P}^1_{c,w}$ is solvable and compact by (B, iv) and hence contains a torus $\mathbf{T}^3$ in contradiction to [**12**, (1.9)].

If a group acts transitively on $\mathbf{R}^7$, then a maximal compact semisimple subgroup is transitive on $\mathbf{S}^6$ and contains G_2, see [**7** or **22**]. Therefore, (3) has the following corollary:

(4) *No subgroup of* Δ *has a transitive representation on* $\mathbf{R}^7$.

(5) *If* Γ *is semisimple*, *then* Γ *is even simple.*

Proof. (a) Assume $\mathsf{T}_v < \mathsf{T}$. Then Γ acts faithfully and completely reducibly on $\mathsf{T}^1 \cong \mathbf{R}^n$ by (E) and semisimplicity. Hence there are $b \in a^{\mathsf{T}}$ and $c \in a^{\Theta}$ such that $\Lambda = \Gamma^1_{b,c}$ fixes a quadrangle and $\Gamma{:}\Lambda \leqq n < 16$. Now $\dim \Lambda \geqq 25 - n \geqq 10$ in contradiction to (B′) and (F). This shows $\mathsf{T} = \mathsf{T}_v$.

(b) $\Gamma_{[W]} = 1$ by (H) and (a) and the fact that av is not fixed.

(c) The centre Z of Γ is trivial: If $u^{\mathsf{Z}} \neq u \in W$, then $\Lambda = \Gamma^1_{c,u}$ fixes a quadrangle, and $\dim \Lambda \geqq 17 - t$ in contradiction to (B′).

(d) Each involution in Γ is planar: a reflection would have center v or axis av by (b). Because of (1), the elation group $\Gamma_{[v,av]}$ would be a commutative normal subgroup of positive dimension.

(e) Consider an involution $\alpha \in \Gamma$, the subplane $\mathscr{F}$ of its fixed elements, a connected subgroup Ψ in the centralizer of α in Γ, and the effective action $\Psi^{\mathscr{F}} = \Psi/\Phi$ on $\mathscr{F}$. The kernel satisfies $\dim \Phi \leqq 3$ by (A, v), and $\Psi{:}\Phi \leqq 4 + 11$ by [**11**, (*)]. Moreover, if Ψ is quasisimple, then $\dim \Psi < 14$, because Ψ cannot act doubly transitively on the points of $W\backslash v$ in $\mathscr{F}$ by [**19**], cp. [**15**, (1.1)], and Ψ is not of type G_2 by [**11**, (**) and (†)].

(f) Note that $\mathsf{Z} = 1$ by (c). If $\Gamma = \mathsf{A} \times \mathsf{B}$, where A is a proper simple factor, apply (e) to an involution $\alpha \in \mathsf{A}$. Then successively $\dim \mathsf{B} \leqq 18$, $\dim \mathsf{A} \geqq 8$, $\Phi = 1$, $\dim \mathsf{B} < 14$ and B is simple, $\dim \mathsf{B} \leqq 10$, $\dim \mathsf{A} \leqq 10$, but $\dim \Gamma \geqq 25$.

(6) Γ *is not semisimple.*

Proof. If Γ is simple, then $25 \leqq \dim \Gamma \leqq 30$ by (D), and $\Gamma \cong \mathrm{PSL}_4\mathbf{C}$ or Γ is an orthogonal group $\mathrm{PSO}_8(r)$. With the notation of (5e), there is a group $\Psi \cong \mathrm{SL}_3\mathbf{C}$ or $\mathrm{SO}_6(r)$ respectively in the centralizer of some involution α. This contradicts the last part of (5e).

The next aim is to show that the elation group $\mathsf{E} = \Gamma_{[v,av]}$ has dimension >1. The proof is rather involved. It will follow from (H) if Γ contains any homology, from (6) otherwise.

(7) *If* $1 \neq \Xi \lhd \Gamma$, *then* $\dim \Xi \geqq 2$.

Proof. The orbit $u^{\Xi} \subseteq W \backslash v$ is invariant under Γ_u, and $17 \leqq \dim \Gamma_u \leqq \dim \Xi + 7 + 8$ by (B′). (In the case $\Theta \cong \mathbf{R}^8$ use the dual of (2).)

(8) *If* Γ *contains a (non-trivial) homology with axis av or with center v, then* $\dim \mathsf{E} > 4$ *by* (H) *and* (1).

(9) *If* Γ *does not contain homologies with axis av or center v, then* Γ *acts effectively on W.*

Proof. Assume $\Gamma_{[W]} \neq 1$. Then (H) implies $a^{\Delta} = a^{\mathsf{T}}$. Consequently, T is connected and $\Delta{:}\Gamma = \dim \mathsf{T} < 16$. Choose $u \in W \backslash v$ and put again $\nabla = \Gamma_u^1$. Then $\dim \nabla \geqq 18$. By (E) and because $\mathscr{P}$ is not a translation plane, $0 < r = \dim \mathsf{T}_u < 8$.

(a) $\mathsf{T}_u \cong \mathbf{R}^7$ and $\mathsf{T}_v \cong \mathbf{R}^8$: From (H), (F) and (2) follows

$$(*) \quad 25 - r \leqq \dim \nabla \leqq 7 + r + \dim \Lambda,$$

where Λ fixes a quadrangle. Applying (B) to T_u instead of Θ, this gives $r = 7$ or $\dim \Lambda = 6 = r$. But the latter is impossible by $(*)$ and (G). Hence $\mathsf{T}_u \cong \mathbf{R}^7$ for any $u \neq v$. Similarly, $\dim \mathsf{T}_v > 6$, and T_v is transitive by (T).

(b) ∇ does not contain any reflection, and each involution has 4-dimensional eigenspaces in T_v: If σ is a reflection, then σ has center a, and $\tau^{\sigma} = \tau^{-1}$ for each $\tau \in \mathsf{T} \cong \mathbf{R}^{15}$, but the negative eigenspace of σ has even dimension because ∇ is connected.

(c) ∇ acts faithfully and irreducibly on T_v: By (4) there is some $b \in a^{\mathsf{T}_u} \backslash a$ with $\dim \nabla_b \geqq 12$. Let $\Psi = \nabla_b^1$ and consider a minimal Ψ-invariant subgroup Θ_1 of T_v. From (B) follows $\dim \Theta_1 \geqq 6$ so that Ψ is faithful and irreducible on Θ_1. The radical $\sqrt{\Psi}$ induces real or complex scalar multiplications on Θ_1 ([**2**, (19.17)], cp. [**17**, p. 186]). Now (b) implies $\sqrt{\Psi} \not\cong \mathbf{C}^{\times}$ and $\dim \Psi' \geqq 11$. Being semisimple, Ψ acts completely reducibly on T_v, and (B) shows that T_v cannot split into proper invariant subgroups.

(d) ∇' is semisimple and $17 \leqq \dim \nabla' \leqq 21$ by (c), (B′), (2) and (4).

(e) ∇ induces also an irreducible action on T_u: From (d), (B) and (2) follows easily that T_u is not a sum of two invariant subgroups.

Noting that each involution in ∇' is planar and hence has proper

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eigenspaces in T_u and T_v, a study of the possible representations will reveal a contradiction. The details will be given in Section 3 where a few similar situations will be treated together.

For steps (10)-(14), assume in view of (8) and (9) that Γ does not contain any homology so that, in particular, Γ acts effectively on W. Changing the previous notation, $\Theta \cong \mathbf{R}^t$ shall denote a minimal Γ-invariant subgroup of T_v, it need no longer be normal in Δ.

(10) Γ *has a minimal normal subgroup* $\Xi \cong \mathbf{R}^s$.

Proof. Because of (6) there is either a normal vector group or a central torus, but the latter is impossible by (5e).

(11) $\Xi \circ \Theta = 1$ *and* Ξ *acts freely on* $W \backslash v$.

Proof. From (B′) follows as in (7) that $s + t \geqq 9$ or $t = 1$ and $s \geqq 6$. If $t < s$, then obviously $\Xi \cong \mathbf{R}^s$ cannot act faithfully on Θ. If $s \leqq t$, then $s \geqq 2$ by (7), and $t \geqq 5$. Because Θ is minimal, Γ acts irreducibly on Θ, and Ξ induces a group of real or complex scalar multiplications, so that again

$$1 \neq \Xi \cap \mathrm{Cs}\,\Theta \lhd \Gamma.$$

Now $\Xi \leqq \mathrm{Cs}\,\Theta$ by (7) and the minimality of Ξ. Consequently, Ξ fixes each point of a^{Θ}, and (1) implies $u^{\Xi} \neq u$ for each $u \in W \backslash v$. Because Ξ is commutative, Ξ_u induces the identity on the subplane $\mathscr{F}$ generated by a^{Θ} and u^{Ξ}. From (B′) follows $\dim u^{\Xi} > 4$ or $t > 4$ and hence $\mathscr{F} = \mathscr{P}$ and $\Xi_u = 1$.

(12) $s < 8$ *or* $s = t$.

Proof. If $s = 8$, then $u^{\Xi} = W \backslash v$. By assumption, $\nabla = \Gamma_u$ does not contain any homology. Hence (10) implies that ∇ acts faithfully and irreducibly on Ξ. Now ∇' is semisimple, $\sqrt{\nabla} \not\cong \mathbf{C}^{\times}$, and $16 \leqq \dim \nabla' \leqq 22$. For $t < 8$ this possibility will be excluded in Section 3, case (β).

(13) $t > 1$.

Proof. If $\Theta \cong \mathbf{R}$, then $\Psi = \nabla \cap \mathrm{Cs}\,\Theta$ acts faithfully on Ξ, $6 \leqq s \leqq 7$ and $\dim \Psi = 16$ by (B′), (4) and (12). Moreover, Ψ is transitive on a 6-dimensional invariant subgroup $\Xi_1 \leqq \Xi$ or irreducible on $\Xi \cong \mathbf{R}^7$.

(a) In the first case, (G) implies easily $\Psi \cong \mathrm{SL}_3\mathbf{C}$ and hence $s = 6$. For $u \neq w \in u^{\Xi}$ the stabilizer Ψ_w fixes a 2-dimensional subset of u^{Ξ} pointwise, and $\dim \Psi_w \geqq 10$. This contradicts (A, iv).

(b) In the second case, Ψ' is semisimple and $\dim \Psi' \geqq 15$. Therefore, Ψ contains a 2-torus Φ which fixes some $w \in u^{\Xi} \backslash u$. Now Ψ_w is compact by (A, ii), and $\dim \Psi_w \geqq 9$. But this is impossible by (A, iii) and (3).

(14) Ξ *fixes each line through* v *and hence consists of elations in* $\mathsf{E} = \Gamma_{[v,av]}$.

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Proof. By (12) and (13) either $s \leqq t$ or $1 < t < s < 8$. In the latter case, (B, iii) and (4) imply $t \geqq 5$. If $s = 6$ and $c \in a^{\Theta}\backslash a$, then $\dim \nabla_c = 12$ and ∇_c is transitive on Ξ. This contradicts (G) and shows $s = 7$. Let $\mathbf{R} \cong \mathsf{P} < \Xi$, $\Psi = \Gamma\Theta \cap \mathrm{Cs}\,\mathsf{P}$ and $x \notin W \cup av$. Then Ψ_x fixes each point of $x^{\mathsf{P}} \neq x$ and

$$\dim \Psi_x \geqq 25 + t - s - 16 > 6.$$

If x^{P} is not contained in a line, then $\Psi_x \cong \mathrm{SU}_3$ by (B) and (C). This is only possible if $t = 6$ and $s = 7$. In that case, Γ is not transitive on Ξ by (4), and there is some P such that $\dim \Psi_x > 8$ for all x. Hence x^{P} is contained in a line $L = L^{\mathsf{P}}$, and $L \cap W = v$ by (11). Now $\mathsf{P} \leqq \Xi_{[v]} \leqq \mathsf{E}$, and $\Xi = \Xi_{[v]}$ because Ξ is a minimal normal subgroup of Γ.

The result of (7-10) and (14) is

(15) $\dim \mathsf{E} > 1$. *Dually*, $\dim \mathsf{T}_u > 1$ *for each* $u \in W\backslash v$.

As before, put $\nabla = \Gamma_u^1$ and consider minimal ∇-invariant subgroups $\Pi \leqq \mathsf{T}_u$, $\Xi \leqq \mathsf{E}$, and $\Theta \leqq \mathsf{T}_v$ of dimensions r, s, and t respectively. Remember that $\mathscr{P}$ is not a translation plane. Hence up to duality

(16) $s \leqq r \leqq 7$, *and* $\dim \nabla \leqq 20$ *by* (B′) *and* (4).

On the other hand, $\dim \nabla \geqq 17$. Applying the dual of (B) to Ξ and Θ, we obtain

(17) $r, s, t \geqq 5$. *Moreover*, $r + s \geqq 12$ *by* (G).

(18) *Each involution in* ∇ *is planar.*

Proof. If the connected group ∇ contains a reflection with center v, then $\dim \mathsf{E} = 6$ and $\dim \nabla > 18$ by the dual of (H). But (4) and the dual of (B, iii) imply $\dim \nabla \leqq 3 \cdot 6$. If there is a reflection with axis av or with center a, then $\dim \mathsf{T}_u = 6$, and an analogous argument leads to a contradiction.

Consider an involution $\alpha \in \nabla$, the subplane $\mathscr{F}$ of its fixed elements, the connected component Ψ of $\nabla \cap \mathrm{Cs}\,\alpha$ and its effective action $\Psi^{\mathscr{F}} = \Psi/\Phi$ on $\mathscr{F}$. Then, [**11**, $(*)$ and $({}^{*}_{**})$] and (A, v) imply

(19) $\Psi{:}\Phi < 11$ *or* Ψ/Φ *is isomorphic to the stabilizer of a triangle in the quaternion plane, and* Φ^1 *is a subgroup of* Spin_3. *In particular,* $\dim \Psi \leqq 14$.

Because of (17),

$$\nabla \cap \mathrm{Cs}\,\Pi \cap \mathrm{Cs}\,\Xi = 1.$$

Hence ∇ acts faithfully on the external direct product $\Pi \times \Xi$ (which is not a subgroup of Δ), and irreducibly on each factor: ∇ is reductive, in particular, ∇' is semisimple and the radical $\sqrt{\nabla}$ is in the centre of ∇, see

[**1**, I, Section 6, no. 4 or **21**, Theorem 3.16.3] for the corresponding Lie algebras. $\sqrt{\nabla}$ induces real or complex scalar multiplications on Π and Ξ and does not contain any involution by (19). Now $\sqrt{\nabla} \cap \mathrm{Cs}\ \Xi$ is a closed proper subgroup of $\mathbf{C}^{\times}$, and $\dim \sqrt{\nabla} < 3$. Hence

(20) ∇' *is semisimple*, $\dim \sqrt{\nabla} \leqq 2$ *and* $\dim \nabla' \geqq 15$.

In Section 3, case (γ), the representations of ∇ on Π and Ξ and statement (19) will be used to show that no group with the above properties can exist; this will then complete the proof of the theorem.

3. The stabilizer of a triangle. With the previous notation and conventions, the situations encountered in Section 2, (9), (12), and (20) have the following in common: ∇ is a reductive Lie group without reflections acting irreducibly on two of the vector groups Ξ, Π, and Θ and faithfully on their product. ∇' is semisimple and the radical $\sqrt{\nabla}$ is a vector group of dimension at most 2. Moreover, $17 \leqq \dim \nabla \leqq 22$ by (D). The respective additional information obtained in the three cases is

(α) $\Pi \cong \mathbf{R}^7$, $\Theta \cong \mathbf{R}^8$, $\nabla \leqq \mathrm{Aut}\ \Theta$, and $17 \leqq \dim \nabla' \leqq 21$.

(β) $\Theta \ncong \mathbf{R}^8 \cong \Xi$, $\nabla \leqq \mathrm{Aut}\ \Xi$, and $16 \leqq \dim \nabla'$.

(γ) $\Pi \cong \Xi \cong \mathbf{R}^6$ or $\Pi \cong \mathbf{R}^7$, and $5 \leqq \dim \Xi \leqq 7$.

Moreover, Ξ consists of elations and $\dim \nabla \leqq 20$.

It will turn out that ∇' is then necessarily quasisimple. In the few remaining cases, the representations of ∇' will reveal non-planar involutions, a contradiction. For a list of simple (real) Lie groups and their representations see [**20**].

(1) ∇' *is quasisimple. Hence* ∇' *is a complex group* A_2 *or* B_2 *of* (*real*) *dimension* 16 *or* 20 *or a real form of type* A_3 *and dimension* 15 *or of type* B_3 *or* C_3 *and dimension* 21.

Proof. Let $\nabla = \mathsf{AB}$ where $\mathsf{A} \neq \nabla'$ is a quasisimple factor of minimal dimension and $\mathsf{A} \circ \mathsf{B} = 1$. Since ∇ has a faithful linear representation, there is an involution $\alpha \in \mathsf{A}$ to which Section 2 (19) can be applied. Choose α so that $\Omega = \mathsf{A} \cap \Psi$ has maximal dimension. Then $\Psi = \Omega\mathsf{B}$, $\dim \mathsf{B} < 14$, $\dim \mathsf{A} \geqq 6$, and $\dim(\mathsf{B}' \cap \Phi) = 0$ by minimality of A. Hence $\dim \mathsf{B}' < 11$ and B' is quasisimple. If $\dim \mathsf{A} = 6$, then $\dim \Omega = 2$, $\dim \mathsf{B}' = 10$, and $\mathrm{Spin}_3 \cong \Phi \leqq \Omega$, a contradiction. Now $\dim \mathsf{A} \geqq 8$, $\mathsf{A}{:}\Omega = 4$, $\dim \mathsf{B}' = 8 = \dim \mathsf{A}$, and again $\mathrm{Spin}_3 \cong \Phi \leqq \Omega$ for each admissible choice of α. Therefore, A is compact and so is B'. But the fixed points of α on W form a 4-sphere, and SU_3 cannot act on $\mathbf{S}^4$, cp. [**11**, (†)].

(2) $\dim \nabla' < 21$. *Consequently*, ∇' *has no irreducible representation in dimension* 7.

Proof. This is true in case (γ). In the other two cases, ∇' has an irreducible representation in dimension 8. But each linear group of type B_3 or C_3 contains a torus T^3 which cannot act on $\mathbf{R}^8$ in such a way that each involution has 4-dimensional eigenspaces.

The second part of (2) excludes case (α) and reduces (γ) to $\Pi \cong \Xi \cong \mathbf{R}^6$.

(3) $\dim \nabla = 17$.

Proof. The group $\mathrm{Sp}_4\mathbf{C}$ of type B_2 can only act on $\mathbf{R}^8$, and $\dim \nabla' \leqq 16$. Moreover, $\nabla{:}\nabla' \leqq 1$ in case (β), and (G) implies $\dim \nabla < 18$ in case (γ).

(4) ∇' *is locally isomorphic to* $\mathrm{SL}_3\mathbf{C}$.

Proof. The only other possibility is $\dim \nabla' = 15$ in case (γ). Then ∇ is transitive on Π or on Ξ by (B, iii), and ∇' induces a group SO_6 by (G). Hence ∇ would contain a central involution.

(5) *Case* (β) *is impossible.*

Proof. Denote again by $\mathscr{F}$ the subplane of the fixed elements of an involution $\alpha \in \nabla'$. Then

$$\Psi = \nabla' \cap \mathrm{Cs}\,\alpha \cong \mathrm{GL}_2\mathbf{C}.$$

Because of (B) either $\Theta \cong \mathbf{R}$ or $\Theta \cong \mathbf{R}^6$. In the first case $\Theta \circ \Psi = 1$ and $\dim \Psi^{\mathscr{F}} = 7$ by (A, v), but this contradicts [**11**, (**)]. In the second case, ∇' acts on Θ in the standard way, and Ψ' fixes the positive eigenspace $\Theta_\alpha^+ \cong \mathbf{R}^2$ element-wise. Now [**11**, (2.5′) or (*)] would imply $\dim \Psi' < 6$.

Now $\nabla' \cong \mathrm{SL}_3\mathbf{C}$ acts equivalently on Π and Ξ. For $1 \neq \xi \in \Xi$ let

$$\Lambda = \nabla' \cap \mathrm{Cs}\,\xi.$$

Then $\dim \Lambda = 10$, and the fixed elements of Λ form a 4-dimensional subplane. This final contradiction proves that $\mathscr{P}$ or its dual is a translation plane.

Remark. Presumably, the same is still true if $\dim \Delta = 40$, but several steps of the proof depend essentially on the stronger assumption. With the techniques of this paper, the following can be shown, however:

THEOREM. *A compact* 8-*dimensional plane with* $\dim \Sigma = 18$ *is a translation plane* (*and hence belongs to one of the* 3 *families of planes of Lenz type V determined by Hähl*).

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