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CONVOLUTIONS OF GENERIC ORBITAL MEASURES IN COMPACT SYMMETRIC SPACES

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Abstract

We prove that in any compact symmetric space, G/K, there is a dense set of $a_1, a_2 \in G$ such that if $\mu_j = m_K * \delta_{a_j} * m_k$ is the K-bi-invariant measure supported on Ka_jK, then $\mu_1 * \mu_2$ is absolutely continuous with respect to Haar measure on G. Moreover, the product of double cosets, Ka_1Ka_2K, has nonempty interior in G.

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1. Introduction

In a now classical paper [4], Dunkl proved that the convolution of the surface measure of a sphere in $\mathbb{R}^n$ with itself is absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^n$. Motivated by this result, Ragozin [9] considered the analogous problem in the setting of a compact, symmetric space G/K and showed that if μ_j are K-bi-invariant, continuous measures, then $\mu_1 * \cdots * \mu_{\dim G/K}$ is absolutely continuous with respect to the Haar measure on G. In particular, this is true when μ_j are the K-orbital surface measures supported on the double cosets Ka_jK, with a_j not in the normalizer of K in G. These singular measures are given by

$$\mu_j = \mu_{a_j} = m_K * \delta_{a_j} * m_K$$

where m_K denotes the Haar measure on K. Equivalently, the $\dim G/K$-fold product of the double cosets Ka_jK has nonempty interior for all such a_j.

Recently, the authors [5] proved that for the special case of the symmetric space $SU(n)/SO(n)$ the number of convolution powers (or double cosets in the product) could be reduced from the dimension of the symmetric space to the rank $+1$, and that this is sharp for particular $a_j \in SU(n)$.

In this paper, we prove that for any compact symmetric space there is a dense subset $D \subseteq G$ such that if $a_1, a_2 \in D$, then $\mu_{a_1} * \mu_{a_2}$ is absolutely continuous with respect to

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the Haar measure on G and the product of double cosets, Ka_1Ka_2K, has nonempty interior. General results of Ricci and Stein [10] then imply that the convolution product actually belongs to $L^p(G)$ for some $p > 1$.

One example of this is when H is a compact, simple, connected Lie group, $G = H \times H$ and $K = \{(h, h) \mid h \in H\}$. Then G/K is homeomorphic to H, double cosets correspond to conjugacy classes, and the K-bi-inviariant measures on G/K can be identified with the central measures on H. Using the representation theory of compact Lie groups, a stronger result has been proved in this case; namely, $\mu_{a_1} * \mu_{a_2} \in L^2(H)$ for a dense set of elements of H [6]. It would be interesting to know whether this stronger result holds for general compact symmetric spaces as well. This may require further development of the L^2 theory for symmetric spaces (see [2, 8]). Smoothness properties of these orbital measures were also investigated in [3, 11].

2. Notation and basic facts

2.1. Restricted roots and root vectors Let G be a compact, connected, semi-simple Lie group and suppose θ is a Cartan involution that fixes the closed Lie subgroup K. The quotient space G/K is known as a compact symmetric space. We denote by $N_G(K)$ the normalizer of K in G.

We shall write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G.[1] Thus $\mathfrak{p}$ is the -1 eigenspace of θ. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace and extend this to a maximal abelian subalgebra, t, of the Lie algebra $\mathfrak{g}$. We write $\mathfrak{m}$ for the subspace of $\mathfrak{k}$ that commutes with $\mathfrak{a}$. For a classification of compact symmetric spaces we refer the reader to [1, p. 219] or [7, p. 518].

Let τ be the conjugation of $\mathfrak{g}$ which gives the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and extend θ by linearity to $\mathfrak{g}^{\mathbb{C}}$. More generally $\mathfrak{a}^{\mathbb{C}}$, $t^{\mathbb{C}}$, and so on, will denote the complexification of the corresponding subspace.

Suppose Φ is the set of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $t^{\mathbb{C}}$. We consider the roots which do not vanish identically on $\mathfrak{a}^{\mathbb{C}}$ and let Σ (Σ^+) denote the corresponding set of (positive) restricted roots. We denote by g_α^R the restricted root space for the restricted root $\alpha \in \Sigma$:

$$g_\alpha^R = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [H, X] = i\alpha(H)X \text{ for all } H \in \mathfrak{a}\}.$$

The restricted root vectors are the nonzero vectors in g_α^R. Similarly, g_α will denote the root space of the root $\alpha \in \Phi$.

In contrast to the situation for root spaces, restricted root spaces need not be one-dimensional. Indeed,

$$g_\alpha^R = \sum g_\beta$$

where the sum is over all root vectors β such that $\beta|_{\mathfrak{a}} = \alpha$. The complexified Lie algebra can be decomposed as

$$\mathfrak{g}^{\mathbb{C}} = t^{\mathbb{C}} \oplus \sum_{\alpha \in \Phi} g_\alpha = \mathfrak{a}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}} \oplus \sum_{\alpha \in \Sigma} g_\alpha^R.$$

[1] Following Ragozin, we define our Lie algebras as right-invariant vector fields.

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If $H \in \mathfrak{a}$, then $\theta(H) = -H$ and $\tau(H) = H$. Thus if $X_\alpha \in g_\alpha^R$, then

$$[H, \theta(X_\alpha)] = \theta[\theta(H), X_\alpha] = -\theta[H, X_\alpha] = -i\alpha(H)\theta(X_\alpha)$$

and

$$[H, \tau(X_\alpha)] = \tau[\tau(H), X_\alpha] = \tau[H, X_\alpha] = -i\alpha(H)\tau(X_\alpha),$$

with the final inequality coming because τ is conjugate linear. Hence both θ and τ map g_α^R to $g_{-\alpha}^R$.

2.2. Regular elements Given a restricted root $\alpha \in \Sigma$ and $a \in \exp \mathfrak{a}$, say $a = \exp(A)$ for $A \in \mathfrak{a}$, we set $\alpha(a) = \alpha(A)$. We call the element $a \in \exp \mathfrak{a}$ *regular* if $\alpha(a) \neq 0 \mod \pi$ for any $\alpha \in \Sigma$.

It follows from the Cartan decomposition that the double cosets, KaK, can be indexed by the elements in $\exp \mathfrak{a} \subseteq G$. The regular elements in $\exp \mathfrak{a}$ are dense in $\exp \mathfrak{a}$, and the elements $g \in G$ with $KgK = KaK$ for some regular $a \in \exp \mathfrak{a}$ are dense in G. We shall show in Corollary 2.3 that if a is regular, then $a \notin N_G(K)$.

2.3. Preliminary results For $E_\alpha \in g_\alpha^R$ set

$$\begin{aligned} F_\alpha &= E_\alpha + \tau E_\alpha + \theta(E_\alpha + \tau E_\alpha), \\ F'_\alpha &= i(E_\alpha - \tau E_\alpha + \theta(E_\alpha - \tau E_\alpha)), \\ G_\alpha &= E_\alpha + \tau E_\alpha - \theta(E_\alpha + \tau E_\alpha), \\ G'_\alpha &= i(E_\alpha - \tau E_\alpha - \theta(E_\alpha - \tau E_\alpha)). \end{aligned}$$

Of course, $F_\alpha, F'_\alpha, G_\alpha, G'_\alpha \in g_\alpha^R \oplus g_{-\alpha}^R$. All four vectors are fixed by τ and hence belong to $\mathfrak{g}$. Moreover, F_α, F'_α are fixed by θ and thus belong to $\mathfrak{k}$, while G_α, G'_α are negated by θ and hence are in $\mathfrak{p}$. If $E_\alpha^{(1)}, \ldots, E_\alpha^{(m_\alpha)}$ is a basis for g_α^R and $F_\alpha^{(j)}, F_\alpha^{(j)\prime}, G_\alpha^{(j)}, G_\alpha^{(j)\prime}$, $j = 1, \ldots, m_\alpha$ are the corresponding vectors, then

$$\mathfrak{k} = \mathrm{span}\{F_\alpha^{(j)}, F_\alpha^{(j)\prime} \mid j = 1, \ldots, m_\alpha; \alpha \in \Sigma\} \oplus \mathfrak{m}$$

and

$$\mathfrak{p} = \mathrm{span}\{G_\alpha^{(j)}, G_\alpha^{(j)\prime} \mid j = 1, \ldots, m_\alpha; \alpha \in \Sigma\} \oplus \mathfrak{a}.$$

We shall follow the usual practice of writing $\mathrm{Ad}(a)$ for the action of the group on the Lie algebra. For $E_\alpha \in g_\alpha^R$, we have $\mathrm{Ad}(a)E_\alpha = e^{i\alpha(a)}E_\alpha$, thus

$$\begin{aligned} \mathrm{Ad}(a)\theta E_\alpha &= e^{-i\alpha(a)}\theta E_\alpha, \\ \mathrm{Ad}(a)\tau E_\alpha &= e^{-i\alpha(a)}\tau E_\alpha. \end{aligned}$$

Simple calculation shows that this implies the following result.

LEMMA 2.1.

(i) $\mathrm{Ad}(a)F_\alpha = \cos\alpha(a)F_\alpha + \sin\alpha(a)G'_\alpha$.
(ii) $\mathrm{Ad}(a)F'_\alpha = \cos\alpha(a)F'_\alpha - \sin\alpha(a)G_\alpha$.

COROLLARY 2.2. *If a is regular, then* $\mathrm{Ad}(a)\mathfrak{k} + \mathfrak{k} = \mathfrak{g} \ominus \mathfrak{a}$.

PROOF. Since a is regular, $\sin \alpha(a) \neq 0$ for any $\alpha \in \Sigma$ and thus

$$\mathrm{span}\{F_\alpha, F'_\alpha, \mathrm{Ad}(a)F_\alpha, \mathrm{Ad}(a)F'_\alpha\} = \mathrm{span}\{F_\alpha, F'_\alpha, G_\alpha, G'_\alpha\}. \qquad \square$$

Since $a \in N_G(K)$ if and only if $\mathrm{Ad}(a)\mathfrak{k} \subseteq \mathfrak{k}$, similar reasoning shows the following result.

COROLLARY 2.3. *An element a belongs to the $N_G(K)$ if and only if $\alpha(a) = 0 \bmod \pi$ for all $\alpha \in \Sigma$.*

There is a particular restricted root vector that we shall be interested in.

LEMMA 2.4. *For each restricted root α, there is a restricted root vector $E_\alpha \in g^R_\alpha$ such that $[E_\alpha, \theta(E_\alpha)] \in i\mathfrak{a}$.*

PROOF. Let $\widetilde{\mathfrak{g}} = \mathfrak{k} + i\mathfrak{p}$. By [1, Proposition 32.5] there is a choice $E_\alpha \in \widetilde{\mathfrak{g}} \bigcap g^R_\alpha$ with $\theta(E_\alpha) \in \widetilde{\mathfrak{g}} \bigcap g^R_{-\alpha}$. Hence $[E_\alpha, \theta(E_\alpha)] \in \widetilde{\mathfrak{g}}$.

Note that $\theta[X, \theta(X)] = -[X, \theta(X)]$, so $[X, \theta(X)] \in \mathfrak{p}^{\mathbb{C}}$. An application of the Jacobi identity proves that for any $H \in \mathfrak{a}^{\mathbb{C}}$ and $X \in g^R_\alpha$,

$$\begin{aligned}[H, [X, \theta(X)]] &= -[X, [\theta(X), H]] - [\theta(X), [H, X]] \\ &= -[X, i\alpha(H)\theta(X)] - [\theta(X), i\alpha(H)X] = 0.\end{aligned}$$

Hence $[X, \theta(X)]$ commutes with all $H \in \mathfrak{a}^{\mathbb{C}}$. Since $\mathfrak{a}^{\mathbb{C}}$ is a maximal abelian subspace of $\mathfrak{p}^{\mathbb{C}}$, it follows that $[X, \theta(X)] \in \mathfrak{a}^{\mathbb{C}}$.

Consequently, $[E_\alpha, \theta(E_\alpha)] \in \widetilde{\mathfrak{g}} \bigcap \mathfrak{a}^{\mathbb{C}} = i\mathfrak{a}$. $\square$

Let $\mathcal{P} : \mathfrak{g} \to \mathfrak{a}$ denote the projection map. Here are some other elementary facts that will be of use to us later.

LEMMA 2.5.

(i) $[F_\alpha, G'_\alpha] - [F'_\alpha, G_\alpha] = -4(I - \theta)i[E_\alpha, \tau(E_\alpha)] = -8\mathcal{P}(i[E_\alpha, \tau E_\alpha])$.
(ii) *If E_α is chosen with $[E_\alpha, \theta(E_\alpha)] \in i\mathfrak{a}$, then $[F_\alpha, G_\alpha] = [F'_\alpha, G'_\alpha]$.*

PROOF. The first equality in (i) is a straightforward computation. Because $\tau(E_\alpha) \in g^R_{-\alpha}$, then $i[E_\alpha, \tau E_\alpha] \in \mathfrak{m}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}$. But also $\tau(i[E_\alpha, \tau E_\alpha]) = -i[\tau E_\alpha, E_\alpha] = i[E_\alpha, \tau E_\alpha]$ and therefore $i[E_\alpha, \tau E_\alpha] \in \mathfrak{g}$. Since $(I - \theta)/2$ projects from $\mathfrak{g}$ onto $\mathfrak{p}$, we obtain the second equality.

For (ii) one can first check that, for any root α,

$$\begin{aligned}&[F_\alpha, G_\alpha] - [F'_\alpha, G'_\alpha] \\ &\quad = 2[\theta(E_\alpha + \tau(E_\alpha)), E_\alpha + \tau(E_\alpha)] + 2[\theta(E_\alpha - \tau(E_\alpha)), E_\alpha - \tau(E_\alpha)] \\ &\quad = 4([\theta(E_\alpha), E_\alpha] + \tau[\theta(E_\alpha), E_\alpha]),\end{aligned}$$

with the latter equality due to the fact that $\theta\tau = \tau\theta$. But $[\theta(E_\alpha), E_\alpha] \in i\mathfrak{a}$, so $\tau[\theta(E_\alpha), E_\alpha] = -[\theta(E_\alpha), E_\alpha]$. $\square$

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We shall also make use of the following technical result which we could not find in the literature. We recall that $\mathfrak{g}^{\mathbb{C}}$ admits a Weyl basis $\{X_\beta \mid \beta \in \Phi^+\}$ where $X_\beta \in g_\beta$ [7, p. 421]. Such a basis has the property that $\tau(X_\beta) = -X_{-\beta}$ and $[X_\beta, X_{-\beta}] = H_\beta$ where H_β is the linear functional on $\mathfrak{t}^{\mathbb{C}}$ given by $H_\beta(t) = \beta(t)$.

LEMMA 2.6. *For any nonzero* $E_\alpha \in g_\alpha^R$, $\mathcal{P}(i[E_\alpha, \tau E_\alpha]) = c_\alpha H_\alpha|_{\mathfrak{a}}$ *where* c_α *is a nonzero constant (depending on* E_α*).*

PROOF. Since $g_\alpha^R = \sum_{\beta|_{\mathfrak{a}}=\alpha} g_\beta$, we can write

$$E_\alpha = \sum_{\beta|_{\mathfrak{a}}=\alpha} b_\beta X_\beta,$$

where $\{X_\beta \mid \beta \in \Phi\}$ is a Weyl basis of $\mathfrak{g}^{\mathbb{C}}$. Thus

$$\begin{aligned}
[E_\alpha, \tau E_\alpha] &= \Bigg[\sum_{\beta|_{\mathfrak{a}}=\alpha} b_\beta X_\beta, \tau\Bigg(\sum_{\beta|_{\mathfrak{a}}=\alpha} b_\beta X_\beta\Bigg)\Bigg] \\
&= \Bigg[\sum_{\beta|_{\mathfrak{a}}=\alpha} b_\beta X_\beta, -\sum_{\beta|_{\mathfrak{a}}=\alpha} \overline{b_\beta} X_{-\beta}\Bigg] \\
&= -\sum_{\beta,\gamma|_{\mathfrak{a}}=\alpha} b_\beta \overline{b_\gamma}[X_\beta, X_{-\gamma}].
\end{aligned}$$

Consequently,

$$\mathcal{P}(i[E_\alpha, \tau E_\alpha]) = -\mathcal{P}\Bigg(\sum_{\beta|_{\mathfrak{a}}=\alpha} i|b_\beta|^2[X_\beta, X_{-\beta}]\Bigg) - \mathcal{P}\Bigg(\sum_{\beta\neq\gamma} i b_\beta \overline{b_\gamma}[X_\beta, X_{-\gamma}]\Bigg).$$

When $\beta \neq \gamma$, then $[X_\beta, X_{-\gamma}]$ either belongs to the root space $g_{\beta-\gamma}$ (if $\beta - \gamma$ is a root) or is zero. In either case, the projection onto $\mathfrak{a}$ is zero. Hence

$$\mathcal{P}(i[E_\alpha, \tau E_\alpha]) = -\sum_{\beta|_{\mathfrak{a}}=\alpha} i|b_\beta|^2 H_\beta|_{\mathfrak{a}}.$$

Since $\beta|_{\mathfrak{a}} = \alpha$, $H_\beta|_{\mathfrak{a}} = H_\alpha|_{\mathfrak{a}}$. Thus $\mathcal{P}(i[E_\alpha, \tau E_\alpha]) = c_\alpha H_\alpha|_{\mathfrak{a}}$ where

$$c_\alpha = -i \sum_{\beta|_{\mathfrak{a}}=\alpha} |b_\beta|^2 \neq 0$$

as $E_\alpha \neq 0$. □

3. Main theorem

By a measure we mean a finite regular Borel measure on G. The measure μ is K-bi-invariant if $\mu(k_1 S k_2) = \mu(S)$ for all $k_1, k_2 \in K$ and Borel sets $S \subseteq G$. An example of a K-bi-invariant measure is the K-orbital measure

$$\mu_a = m_K * \delta_a * m_K$$

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where m_K denotes the normalized Haar measure on K and δ_a denotes the point mass measure at a. The K-orbital measure, μ_a, is a singular probability measure which is supported on KaK, and is continuous (meaning nonatomic) if $a \notin N_G(K)$ when viewed as a measure on the symmetric space G/K. These measures are the extreme points of the unit ball of the space of K-bi-invariant, continuous measures (see [9]). Of course, if $KgK = KaK$, then $\mu_g = \mu_a$.

Ragozin proved that if $d \geq \dim G/K$, then $\mu_{a_1} * \mu_{a_2} * \cdots * \mu_{a_d}$ is absolutely continuous with respect to Haar measure on G and the d-fold product of double cosets $Ka_1Ka_2 \cdots Ka_dK$ has nonempty interior if $a_j \notin N_G(K)$. For special orbital measures the number of convolution powers can be reduced to two. Here is our main result.

THEOREM 3.1. *Suppose $a_1, a_2 \in \exp \mathfrak{a}$ are regular elements and μ_{a_1}, μ_{a_2} are the associated K-orbital measures. Then $\mu_{a_1} * \mu_{a_2}$ is absolutely continuous with respect to Haar measure on G and Ka_1Ka_2K has nonempty interior in G.*

PROOF. For any two elements $a_1, a_2 \in \exp \mathfrak{a}$, let $f_{a_1,a_2} : K^3 \to G$ be given by

$$f(k_0, k_1, k_2) = k_0a_1k_1a_2k_2.$$

The proof of [9, Theorem 2.5] (an application of the implicit function theorem) shows that if the rank of f_{a_1,a_2} is full, except possibly on a set of Haar measure zero, for each a_1, a_2 in the support of the K-bi-invariant measures μ_1, μ_2, then $\mu_1 * \mu_2$ is absolutely continuous and Ka_1Ka_2K has nonempty interior. However, an analyticity argument proves that if the rank of f_{a_1,a_2} is full at one point, then it is full on a set whose complement has measure zero.

Thus to prove our theorem it will be enough to show that whenever a_1, a_2 are two regular elements in $\exp \mathfrak{a}$, then the rank f_{a_1,a_2} is full at one point, and this is what we shall prove. For notational convenience we shall write f for f_{a_1,a_2}.

The differential of f at the point (k_0, k_1, k_2), $df|_{(k_0,k_1,k_2)}$, is the map from $\mathfrak{k}^3$ to $\mathfrak{g}$ given by

$$df|_{(k_0,k_1,k_2)}(X_0, X_1, X_2) = -(X_0 + \mathrm{Ad}(k_0a_1)X_1 + \mathrm{Ad}(k_0a_1k_1a_2)X_2)k_0a_1k_1a_2k_2$$

for $X_i \in k$. (This is true because of our convention of using right invariant vector fields.) Thus rank f at (k_0, k_1, k_2) is the dimension of

$$\mathrm{span}\{X_0 + \mathrm{Ad}(k_0a_1)X_1 + \mathrm{Ad}(k_0a_1k_1a_2)X_2 \mid X_0, X_1, X_2 \in \mathfrak{k}\},$$

which is equal to the dimension of

$$\mathrm{span}\{X_0 + \mathrm{Ad}(a_1)X_1 + \mathrm{Ad}(a_1k_1a_2)X_2 \mid X_0, X_1, X_2 \in \mathfrak{k}\}.$$

Hence it is enough to show that there exists a point $k_1 \in K$ such that

$$\mathfrak{k} + \mathrm{Ad}(a_1)\mathfrak{k} + \mathrm{Ad}(a_1k_1a_2)\mathfrak{k} = \mathfrak{g},$$

or, equivalently,

$$\mathfrak{k} + \mathrm{Ad}(a_1^{-1})\mathfrak{k} + \mathrm{Ad}(k_1 a_2)\mathfrak{k} = \mathfrak{g}.$$

Note that Corollary 2.2 implies that $\mathfrak{k} + \mathrm{Ad}(a_1^{-1})\mathfrak{k} = \mathfrak{g} \ominus \mathfrak{a}$.

As in the previous section, let $\mathcal{P}$ be the projection operator defined from $\mathfrak{g}$ onto $\mathfrak{a}$. Using this notation, it follows that to prove the theorem it suffices to show that for each a regular, there exists some $k \in K$ for which $\dim(\mathcal{P}(\mathrm{Ad}(ka)\mathfrak{k})) = \dim(\mathfrak{a})$.

According to Lemma 2.4, for each positive, restricted root α it is possible to choose a restricted root vector $E_\alpha \in g_\alpha^R$ satisfying $[E_\alpha, \theta(E_\alpha)] \in i\mathfrak{a}$. Define $F_\alpha, F'_\alpha, G_\alpha, G'_\alpha$ as described in the previous section with this choice of E_α. Set

$$Z = \sum_{\beta \in \Sigma^+} F_\beta + F'_\beta$$

and for any real number s put $k_s = \exp(sZ)$. Since $Z \in \mathfrak{k}$, k_s belongs to the subgroup K.

Fix $a \in \exp \mathfrak{a}$, a regular. For $\alpha \in \Sigma^+$,

$$\begin{aligned}\mathrm{Ad}(k_s a)(F_\alpha + F'_\alpha) &= \mathrm{Ad}(k_s)(\cos\alpha(a)(F_\alpha + F'_\alpha) - \sin\alpha(a)(G_\alpha - G'_\alpha)) \\ &= \exp(ad(sZ))(\cos\alpha(a)(F_\alpha + F'_\alpha) - \sin\alpha(a)(G_\alpha - G'_\alpha)) \\ &= \cos\alpha(a)(F_\alpha + F'_\alpha) - \sin\alpha(a)(G_\alpha - G'_\alpha) + R + S,\end{aligned}$$

where

$$R = s[Z, \cos\alpha(a)(F_\alpha + F'_\alpha) - \sin\alpha(a)(G_\alpha - G'_\alpha)]$$

and

$$S = \sum_{l=2}^{\infty} \frac{s^l}{l!}(ad\ Z)^l(\cos\alpha(a)(F_\alpha + F'_\alpha) - \sin\alpha(a)(G_\alpha - G'_\alpha)).$$

Since $Z, F_\alpha, F'_\alpha \in \mathfrak{k}$, we have $\mathcal{P}[Z, F_\alpha + F'_\alpha] = 0$ for all $\alpha \in \Sigma$. Also, $F_\alpha, F'_\alpha, G_\alpha, G'_\alpha \in \sum_{\alpha \in \Sigma} g_\alpha^R$, hence $\mathcal{P}(F_\alpha + F'_\alpha) = 0 = P(G_\alpha - G'_\alpha)$. Therefore

$$\mathcal{P}(\mathrm{Ad}(k_s a)(F_\alpha + F'_\alpha)) = -s\sin\alpha(a)\mathcal{P}([Z, G_\alpha - G'_\alpha] + sY_{\alpha,s})$$

where

$$Y_{\alpha,s} = \sum_{l=2}^{\infty} \frac{s^{l-2}}{l!}(ad\ Z)^l(G_\alpha - G'_\alpha).$$

First, consider

$$\begin{aligned}[Z, G_\alpha - G'_\alpha] &= \left[\sum_{\beta \in \Sigma^+} F_\beta + F'_\beta, G_\alpha - G'_\alpha\right] \\ &= [F_\alpha + F'_\alpha, G_\alpha - G'_\alpha] + \sum_{\beta \neq \alpha}[F_\beta + F'_\beta, G_\alpha - G'_\alpha].\end{aligned}$$

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If $\beta \neq \alpha$, then also $\beta \neq -\alpha$ since β and α are positive, restricted roots. Hence either $[F_\beta + F'_\beta, G_\alpha - G'_\alpha] \in \sum_{\gamma=\pm\alpha\pm\beta} g^R_\gamma$ or none of $\pm\alpha \pm \beta$ are roots, in which case $[F_\beta + F'_\beta, G_\alpha - G'_\alpha] = 0$ [1, Proposition 32.5]. In either case, $\mathcal{P}[F_\beta + F'_\beta, G_\alpha - G'_\alpha] = 0$.

Combined with Lemmas 2.5 and 2.6, this observation implies that

$$\begin{aligned}\mathcal{P}([Z, (G_\alpha - G'_\alpha)]) &= \mathcal{P}([F_\alpha + F'_\alpha, G_\alpha - G'_\alpha]) \\ &= \mathcal{P}([F'_\alpha, G_\alpha] - [F_\alpha, G'_\alpha]) \\ &= -8\mathcal{P}i[E_\alpha, \tau E_\alpha] = c_\alpha H_\alpha|_\mathfrak{a}\end{aligned}$$

for some nonzero constant c_α. Hence

$$\mathcal{P}(\mathrm{Ad}(k_s a)(F_\alpha + F'_\alpha)) = -s \sin\alpha(a)(c_\alpha H_\alpha|_\mathfrak{a} + s\mathcal{P}(Y_{\alpha,s})).$$

Thus to show that $\dim(\mathcal{P}(\mathrm{Ad}(ka)\mathfrak{k})) = \dim(\mathfrak{a})$ it is enough to prove that for suitably small s, the set

$$\{\mathcal{P}(\mathrm{Ad}(k_s a)(F_\alpha + F'_\alpha)) \mid \alpha \in \Sigma\}$$

or, equivalently,

$$\{c_\alpha H_\alpha|_\mathfrak{a} + s\mathcal{P}(Y_{\alpha,s}) \mid \alpha \in \Sigma\}$$

contains a linearly independent set of size $\dim \mathfrak{a} \equiv r$. To see that this is true, choose positive, restricted roots, $\alpha_1, \ldots, \alpha_r$, such that $\{H_{\alpha_j}|_\mathfrak{a} \mid j = 1, \ldots, r\}$ is a basis for $\mathfrak{a}$. We claim that the set of vectors

$$\{c_{\alpha_j} H_{\alpha_j}|_\mathfrak{a} + s\mathcal{P}(Y_{\alpha_j,s}) \mid j = 1, \ldots, r\}$$

is linearly independent for sufficiently small s.

Assume otherwise, say,

$$\sum_{j=1}^{r} d_j(c_{\alpha_j} H_{\alpha_j}|_\mathfrak{a} + s\mathcal{P}(Y_{\alpha_j,s})) = 0 \tag{3.1}$$

with not all $d_j = 0$. Since all norms are comparable on a finite-dimensional space, there exists a positive constant C_0 such that

$$\left\| \sum_{j=1}^{r} d_j c_{\alpha_j} H_{\alpha_j}|_\mathfrak{a} \right\| \geq C_0 \sum_{j=1}^{r} |d_j c_{\alpha_j}| \geq C_0 \min |c_{\alpha_j}| \sum_{j=1}^{r} |d_j|.$$

For any $0 < s < 1$,

$$\|\mathcal{P}Y_{\alpha,s}\| \leq \|Y_{\alpha,s}\| \leq \sum_{l=2}^{\infty} \frac{\|ad\ Z\|^l \max_{\alpha\in\Sigma} \|(G_\alpha - G'_\alpha\|}{l!} \equiv C_Z$$

where C_Z is independent of α and s. Hence

$$\left\| \sum_{j=1}^{r} d_j s \mathcal{P}(Y_{\alpha_j,s}) \right\| \leq s \sum_{j=1}^{r} |d_j| C_Z.$$

If we take $s < C_0 \min |c_{\alpha_j}|/C_Z$ we clearly cannot satisfy (3.1) and therefore $\dim(\mathcal{P}(\mathrm{Ad}(ka)\mathfrak{k})) = \dim(\mathfrak{a})$. This completes the proof that f has full rank at one point. □

COROLLARY 3.2. *Suppose μ_1, μ_2 are K-bi-invariant measures, compactly supported on $\bigcup_{a \in D} KaK$ where D is the dense set of regular elements. Then $\mu_1 * \mu_2$ is absolutely continuous.*

PROOF. This can also be deduced from the same proof, as per the remarks in the first paragraph. □

COROLLARY 3.3. *Suppose G/K is a compact symmetric space which admits only one positive restricted root. Then for any $a_1, a_2 \notin N_G(K)$, $\mu_{a_1} * \mu_{a_2}$ is absolutely continuous.*

PROOF. When there is only one positive restricted root any element of $\exp \mathfrak{a}$ is either in the normalizer or regular. □

REMARK. Many of the rank-one symmetric spaces, including $SU(2)/SO(2)$ and $SO(p+1)/O(p)$, have only one positive restricted root. It would be interesting to know if the conclusion of the corollary holds for all rank-one spaces.

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