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SOME EXPLICIT GENERATORS FOR $SL(3, 3^n)$, $SU(3, 3^n)$, $Sp(4, 3^n)$ AND $SL(4, 3^n)$

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1. Introduction.

This is a generalization of the results in [**3**, § 5]. Some of the proofs presented here are actually the original proofs presented in [**3**]. Although we can find alternate proofs in the case $p = 3$, since [**3**] will not be published for a while yet, we feel that it is worthwhile to present the proof in [**3**] whenever it carries over in the case $p = 3$. The results in this paper will be used in the investigation of the quadratic pairs for the prime 3.

We will show here some explicit generators for $SL(3, 3^n)$ and $SU(3, 3^n)$. Together with the results in [**3**], we complete the situation for the case of odd characteristic.

Some authors use the notation $SU(3, q^2)$; here we adopt the notation $SU(3, q)$ for the special unitary group.

2.

LEMMA 2.1. *Suppose $q = 3^n$, $n \geqq 1$, and K is a field with $|K| = q$. Then $SL(3, q)$ is generated by*

$$X_{12}(t) = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, X_{21}(t) = \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{and}$$

$$\xi = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \text{where } t \text{ ranges over } K.$$

Proof. Let $S = \langle X_{12}(1), X_{21}(-1), \xi \rangle$ and let $S_1 = \langle X_{12}(1), X_{21}(-1) \rangle$. Then S_1 is isomorphic to $SL(2, 3)$. Set

$$\omega = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\omega \in S_1$, we get $\xi\xi^\omega \in S$. As $\operatorname{tr}(\xi\xi^\omega) = -1 \neq 0$, $\langle \xi, \xi^\omega \rangle$ is not a 3-group by Sylow's theorem. Thus $\xi \notin O_3(S)$. We claim S contains a four group. Since $S \subset SL(3, 3)$ if $13 \mid |S|$, then $|SL(3, 3) : S| \leqq 6$. But this implies $S = SL(3, 3)$. So we may assume $13 \nmid |S|$. Hence S is a $\{2, 3\}$ group. Thus $O(S) = O_3(S)$. Since $\xi \notin O_3(S)$, we get $\xi \notin O(S)$. Let i be the unique involu-

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tion of S_1. If $\langle i, O(S)\rangle \lhd S$, then $\xi^2 = i\xi^{-1}i\xi = [i, \xi] \in \langle i, O(S)\rangle$ which in turn forces $\xi^2 \in O(S)$. Since this is false, we cannot have $\langle i, O(S)\rangle \lhd S$. By a Brauer-Suzuki theorem [**2**, p. 527], Sylow 2-subgroups of S are not quaternion. This establishes our claim. Let j be any involution of S which centralizes i. Then $j = \begin{bmatrix} A & \\ & a \end{bmatrix}$, where $A \in GL(2, 3)$ and $A^2 = I_2$, $a^2 = 1$. Since $j \notin S_1$, $a = -1$. Hence $\det A = -1$, and the stability subgroup of the negative space of i in S induces $GL(2, 3)$ on the negative space of i. Therefore we may assume $j_0 = \mathrm{diag}[-1, 1, -1]$. Now

$$\xi\xi^{j_0} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and S contains

$$Z = X_{12}(1)(\xi\xi^{j_0}) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $h(t) = \mathrm{diag}[t^{-1}, t, 1]$ where $t \in K^\times$, and let N be the group generated by the matrices in the statement of our lemma. Then $h(t) \in N$, and N contains

$$Z^{h(t)} = \begin{bmatrix} 1 & 0 & -t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, t \in K^\times.$$

Since

$$\xi X_{12}(1)Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \in N, N \text{ contains} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{bmatrix}, t \in K.$$

From here it is straightforward to verify $N = SL(3, q)$.

LEMMA 2.2. *Assume the notation for $X_{12}(t)$, $X_{21}(t)$, K as in Lemma 2.1. Let $y \in K^\times$. Then $SL(3, q)$ is generated by*

$$X_{12}(t), X_{21}(t) \text{ and } \xi = \begin{bmatrix} 1 & y & 1 \\ 0 & 1 & 0 \\ 0 & 2y & 1 \end{bmatrix}.$$

Proof. Let N be the subgroup of $SL(3, q)$ which is generated by the matrices in the statement of Lemma 2.2. We claim N is irreducible on the 3-dimensional underlying space V. Suppose not.

Case (1). There is a one-dimensional irreducible N-submodule M_0 of V: Since N is generated by 3-elements, $N|M_0 = \mathrm{id}$. Since V is a complete reducible $S_1 = \langle X_{12}(t), X_{21}(t)\rangle$ module, $V = M_1 \oplus M_0$ where M_1 is a 2-dimensional standard module for S_1. We can choose a basis of M_1 together with a vector in M_0 to form a basis B of V such that with respect to this basis $X_{12}(t)$

is represented by

$$\begin{bmatrix} 1 & \sigma(t) & \\ & 1 & \\ & & 1 \end{bmatrix}$$

where $\sigma(t)$ ranges over K as t ranges over K. Since M_0 is an N-module, ξ is represented by

$$\begin{bmatrix} & & \alpha \\ & A & \beta \\ 0 & 0 & 1 \end{bmatrix}$$

with respect to B. Here A is a 2×2 matrix over K. Since $[\xi, X_{12}(t)] = 1$, ξ must be represented by a matrix of the form

$$\begin{bmatrix} 1 & \delta & \epsilon \\ 0 & 1 & 0 \\ 0 & \eta & 1 \end{bmatrix}.$$

But then $\eta = 0$, and $(\xi - 1)^2 = 0$ which is false. Therefore Case (1) cannot occur.

Case (2). V has a 2-dimensional irreducible N-submodule M_1: Since V is a complete reducible $S_1 = \langle X_{12}(t), X_{21}(t) \rangle$ module, $V = M_1 \oplus M_0$ where M_0 is a 1-dimensional S_1 module. In particular $S_1|M_0 = \text{id}$, and N induces identity on V/M_1. We can choose a basis for M_1 together with a vector from M_0 to form a basis B for V such that with respect to B, $X_{12}(t)$ is represented by

$$\begin{bmatrix} 1 & \sigma(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\sigma(t)$ ranges over K when t ranges over K. Since $[\xi, X_{12}(t)] = 1$, we get with respect to B, ξ is represented by

$$\begin{bmatrix} 1 & \gamma & \alpha \\ 0 & 1 & 0 \\ 0 & \beta & 1 \end{bmatrix}.$$

Since M_1 is an N-submodule, $\alpha = 0$. But then $(\xi - 1)^2 = 0$, a contradiction to $y \in K^\times$. Therefore Case (2) cannot arise either. Hence we conclude that N is irreducible on V. In particular, we get $O_3(N) = 1$. Since $|K| = 3^n$, we have $PSL_3(q) = SL(3, q)$. By Lemma 1 we may assume $y \neq -1$. A similar argument in the proof of Lemma 1 will enable us to assume $y \neq 1$. Hence we may assume $n > 1$. In [**1**, Theorem 7.1] the cases (4) and (5) are out as $O_3(N) = 1$. The case (3) is also out by $O_3(N) = 1$ and the Sylow 3-subgroups of N have order $>q$. Since $n > 1$, $S_1 = \langle X_{12}(t), X_{21}(t) \rangle$ is perfect. If we were in Case (2) of [**1**, Theorem 7.1], then S_1 must lie inside the diagonal normal subgroup stated there. But then S_1 would be solvable which is false. So this case cannot arise. If N contains a cyclic normal subgroup, then S_1 induces the identity on it. Let this cyclic group be generated by Z. Then $[S_1, Z] = 1$ and

$$Z = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}.$$

Since $\det Z = 1$, $b = a^{-2}$. But

$$\xi^{-1}Z\xi = \begin{bmatrix} 1 & y & -1 \\ 0 & 1 & 0 \\ 0 & -2y & 1 \end{bmatrix} Z \begin{bmatrix} 1 & y & 1 \\ & 1 & 0 \\ & 2y & 1 \end{bmatrix} = \begin{bmatrix} a & * & a-b \\ 0 & a & 0 \\ 0 & * & b \end{bmatrix}.$$

Since $a - b \neq 0$, ξ does not normalize Z. Hence Case (1) of [**1**, Theorem 7.1] can not occur. Since $n > 1$ and $q = 3^n$ and Sylow 3-subgroups have order $> q$, the cases (l), $3 \leqq l \leqq 9$ of [**1**, Theorem 1.1] cannot occur. Therefore N always contains a four-group. A similar argument as in the proof of Lemma 2.1 completes the proof of Lemma 2.2.

LEMMA 2.3. *Suppose A is a quadratic extension field of K with odd characteristic p and H is a subgroup of $SL(2, A)$ such that $SL(2, K) \subseteq H \subseteq SL(2, A)$. Then either $SL(2, K) \lhd K$ or $H = SL(2, A)$ unless $|K| = 3$. The normalizer of $SL(2, K)$ in $GL(2, A)$ is $SL(2, K) \cdot D$ where*

$$D = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b \in A^{\times}, a^{-1}b \in K^{\times} \right\}.$$

Proof. Let V be the 2-dimensional underlying space over A. Since $SL(2, K)$ is irreducible on V, $O_p(H) = 1$. Let $S \in \mathrm{Syl}_p(SL(2, K))$. Then $[V, S, S] = 0$. Since $S \cdot O(H)$ is p-solvable, by Theorem (B) of Hall-Higman, we get $[S, O(H)] = 1$. Hence $S \subset C(O(H))$. Since this is true for all

$$S \in \mathrm{Syl}_p(SL(2, K)),$$

we get $O(H) = 1$. The Sylow 2-subgroups of H are generalized quaternion; the theorem of Gorenstein-Walter [**2**, p. 462] or Hauptsatz 8.27 of Huppert's *Endliche Gruppen* completes the proof of the first part.

Since $SL(2, K)$ is 2-transitive on its Sylow p-subgroups, the normalizer of $SL(2, K)$ in $GL(2, A)$ is $SL(2, K) \cdot D$, where D is a group of diagonal matrices. Let $d = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in D$, so that

$$d^{-1}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} d = \begin{bmatrix} 1 & a^{-1}b \\ & 1 \end{bmatrix} \in SL(2, K),$$

and so $a^{-1}b \in K$. The proof is complete.

Remark. In the case when $|K| = 3$, we do have a counter-example of Lemma 2.3, namely, $SL(2, 3) \subset SL(2, 5) \subset SL(2, 9)$.

LEMMA 2.4. *Suppose A is a quadratic extension field of K with odd characteristic p, j is an involution of $GL(2, A)$ of determinant -1, j normalizes $SL(2, K)$, and $H = \langle SL(2, K), j\rangle$. Then either* $\mathrm{diag}(-1, 1) \in H$, *or* $\mathrm{diag}(a, -a^{-1}) \in H$ *where $a \in A\backslash K$ and $a^2 \in K$.*

Proof. By Lemma 2.3 $j = j_0 d$, where $j_0 \in SL(2, K)$ and $d = \mathrm{diag}(a, b)$ with $a^{-1}b \in K^\times$. Since $\det j = -1$, we have $ab = -1$. If $a \in K^\times$, then $d = \mathrm{diag}(a, -a^{-1}) \in H$ as well as $\mathrm{diag}(-a^{-1}, -a) \in H$, so we get $\mathrm{diag}(-1, 1) \in H$. If $a \notin K^\times$, then the second possibility occurs.

THEOREM 2.1. *Suppose A is a quadratic extension field of K, $|K| = 3^n = q$, $y \in A$. Then $N = \langle X_{12}(t), X_{21}(t), \xi\rangle$ is isomorphic to $SL(3, A)$ or $SU_3(K)$, where $t \in K$, and*

$$\xi = \begin{bmatrix} 1 & y & 1 \\ 0 & 1 & 0 \\ 0 & 2y & 1 \end{bmatrix}.$$

N is isomorphic to $SU_3(K)$ provided $y^q = -y$.

Proof. As in the proof of Lemma 2.2 we get N irreducible on the 3-dimensional space V over A, and $O_3(N) = 1$. Thus, Cases (4), (5) of [**1**, Theorem 7.1] cannot arise. Case (2) of that theorem can easily be seen not to arise. The argument in the proof of Lemma 2.2 also rules out the Case (1) of [**1**, Theorem 7.1]. If we were in the case (3) of that theorem, then the involution of $\langle X_{12}(1), X_{21}(-1)\rangle$ would be central. But then ξ has to act quadratically on V which is false. Hence this case cannot arise either.

If $q = 3$, then $q \not\equiv 1$ or 19 (30). If $q > 3$, then the Sylow 3-subgroups of N have order >9. Therefore Case (9) of [**1**, Theorem 1.1] cannot arise. Similar arguments rule out the Cases (5), (6), (7) of that theorem. Certainly Cases (3), (4), (8) of that theorem cannot arise. Hence $N/N \cap Z(SL(3, A))$ is isomorphic to either $PSL(3, 3^\beta)$ with $\beta|2n$ or $PSU(3, 3^\beta)$ with $2\beta|2n$. Hence N always contains a four group. Suppose $d = \mathrm{diag}(-1, 1, -1) \in N$. Then

$$\eta = [\xi, \xi^d] = \begin{bmatrix} 1 & -4y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in N.$$

Since $\mathrm{diag}(t^{-1}, t, 1) \in H$ for all $t \in K^\times$, we get $X_{12}(-4yt^2) \in N$ for all $t \in K^\times$. Since $A = K + Ky$, we get $X_{12}(t) \in N$ for all $t \in A$. Since

$$\omega = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in N,$$

we get $X_{21}(t) \in N$ for all $t \in A$. By Lemma 2.2 we have $N = SL(3, A)$.

So we may assume that $\operatorname{diag}(-1, 1, -1) \notin N$. Let j be an involution of N which centralizes i. Then $j = \begin{bmatrix} J & 0 \\ 0 & a \end{bmatrix}$ where $j \in GL(2, A)$, and $a \in A^{\times}$. Since $j \notin \langle X_{12}(t), X_{21}(t) | t \in K^{\times} \rangle = S_1, a \neq 1$. Thus $a = -1$, and $\det J = -1$.

Case (1). $|K| = 3$: Then $|A| = 3^2$. Since $\operatorname{diag}(-1, 1, -1) \notin N$, N is not isomorphic to $SL(3, 3^{\beta})$. Hence $N/N \cap Z(SL(3, A))$ is isomorphic to $PSU(3, 3^{\beta})$ with $2\beta | 2$. This implies N is isomorphic to $SU(3, 3)$.

Case (2). $|K| > 3$: Set

$$H = N \cap \left\{ \begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix} \middle| L \in SL(2, A) \right\}.$$

If we let H_0 be the set of such L, then $SL(2, K) \subset H_0 \subset SL(2, A)$. If $H_0 = SL(2, A)$, Lemma 2.1 implies that $N = SL(3, A)$ which is not the case since $\operatorname{diag}(-1, 1, -1) \notin N$; whence by Lemma 2.3 we get $SL(2, K) \lhd H_0$. By the same lemma we have $SL(2, K) \operatorname{char} H_0$. Since $|\langle S_1, j \rangle : H| = 2$, then j normalizes S_1. By Lemma 2.4, we get $\delta = \operatorname{diag}(a, -a^{-1}, -1) \in N$, where $a \in A \backslash K$ and $a^2 \in K$. Since $q = 3^n$, $q - 1$ is even. Thus $a^{q-1} = c \in K^{\times}$. Since $a^{q^2-1} = 1$, $c^{q+1} = 1$. But $c^q = c$ as $c \in K$. Therefore $c^2 = 1$. If $c = 1$, then $a^q = a \cdot a^{q-1} = a$ forces $a \in K$. This is false, so $c = -1$, $a^q = -a$. Now we get

$$\delta \xi \delta^{-1} = \begin{bmatrix} 1 & -a^2 y & a \\ 0 & 1 & 0 \\ 0 & -2ay & -1 \end{bmatrix},$$

and so

$$[\delta \xi \delta^{-1}, \xi] = \begin{bmatrix} 1 & ay & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $ay \notin K$, then N contains $X_{12}(t)$, $t \in A$, and so contains $X_{21}(t)$ with $t \in A$. By Lemma 2.2 we get $N = SL(3, A)$ which is not the case. Hence $ay = k \in K$. Thus $(ay)^q = ay = a^q y^q = -ay^q$, and so $-y = y^q$. Let $k_1 = -a^{-1} y = -a^2 \cdot ay$. Then $k_1 \in K$, and so N contains $\delta \operatorname{diag}(k_1, k_1^{-1}, 1) = \operatorname{diag}(-y, y^{-1}, -1)$. Let $\xi_t = h(t)^{-1} \xi h(t)$ with $h(t) = \operatorname{diag}(t^{-1}, t, 1)$ $(t \in K^{\times})$, and let

$$\xi_t' = \pi^{-1} \xi_t \pi = \begin{bmatrix} 1 & -t^2 y & y^{-1} t \\ 0 & 1 & 0 \\ 0 & -2t & 1 \end{bmatrix}.$$

Set $U = \langle \xi_t, \xi_t' | t \in K^{\times} \rangle$. Then $|U| = q^3$. Since $N/N \cap Z(SL(3, A))$ is isomorphic to $PSU(3, 3^{\beta})$ with $\beta | n$, we get $\beta = n$. Hence $N = SU(3, q)$.

With the aid of the above results we get Lemma 5.6, Lemmas 6.1, 6.2, 6.3 of [**3**] in the case $p = 3$. The proof is essentially the proof presented in [**3**].

LEMMA 2.5. *Suppose* $A = K[y]$ *is a commutative ring*, $|K| = 3^s$, $y \notin K$,

$y^2 = m^2$, $m \in K^{\times}$ and S is the subgroup of $SL(3, A)$ generated by $X_{12}(t)$, $X_{21}(t)$, where t ranges over K and

$$\xi = \begin{bmatrix} 1 & y & 1 \\ 0 & 1 & 0 \\ 0 & 2y & 1 \end{bmatrix}.$$

Let $e_1 = (1 + m^{-1}y)/2$, $e_2 = 1 - e_1$. Then $A = Ke_1 \oplus Ke_2$ and there is an automorphism α of A of order 2 such that $e_1{}^{\alpha} = e_2$ and such that α fixes K element-wise. Let α be the automorphism of $SL(3, A)$ induced by α, let Ω be the automorphism given by

$$x^{\Omega} = \omega^{-1}x\omega, \text{ with } \omega = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1/2y \end{bmatrix},$$

and let Φ be the automorphism $x^{\Phi} = {}^t x^{-1}$. Finally let $\pi = \alpha\Phi\Omega$. Then S is the set of fixed points of π on $SL(3, A)$ and $S \cong SL(3, K)$.

Proof. Let L be the set of the fixed points of π on $SL(3, A)$. We check that $S \subset L$. Since $y = me_1 - me_2$, $y^{\alpha} = -y$. We have $e_1 + e_2 = 1$, $e_1e_2 = 0$, and $e_i{}^2 = e_i$, $i = 1, 2$.

Let $L_i = \{x \in SL(3, A) | x \equiv 1(Ae_i)\}$. Since $0 \to Ae_j \to A \to Ae_i \to 0$, $i \neq j$ is exact, $1 \to L_j \to SL(3, A) \to SL(3, Ae_i) \to 1$ is also exact. Thus

$$L_j \lhd SL(3, A).$$

Since $Ke_i \cong K$, we get $L_i \cong SL(3, K)$ and $SL(3, A) = L_1 \times L_2$. Since $L_1{}^{\pi} = L_2$, $L = \{x \cdot x^{\pi} | x \in L_1\} \cong SL(3, K)$. Let $\tilde{L}_1 = \{x \in L_1 | x^{-1}z \in L_2$ for some $z \in S\}$. Since $S \subset L$, we get $z = a \cdot a^{\pi}$ with $a \in L_1$, $a^{\pi} \in L_2$. Hence $(x^{-1}a)(a^{\pi}) \in L_2$, which implies $x = a$, and $z = x \cdot x^{\pi}$. Thus

$$\tilde{L}_1 = \{x \in L_1 | x \cdot x^{\pi} \in S\}.$$

For $t \in K$, we have

$$\begin{bmatrix} 1 & te_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\pi} = \begin{bmatrix} 1 & te_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\Phi\Omega} =$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -te_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1/2y \end{bmatrix} = \begin{bmatrix} 1 & te_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

$$\begin{bmatrix} 1 & te_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & te_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\pi} = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in S.$$

Hence

$$\begin{bmatrix} 1 & te_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \tilde{L}_1.$$

A similar argument shows

$$\begin{bmatrix} 1 & 0 & 0 \\ te_1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \tilde{L}_1.$$

Let

$$\xi_1 = \begin{bmatrix} 1 & me_1 & e_1 \\ 0 & 1 & 0 \\ 0 & 2me_1 & 1 \end{bmatrix}.$$

Then

$$\xi_1^\pi = \begin{bmatrix} 1 & -me_2 & -me_2/y \\ 0 & 1 & 0 \\ 0 & 2ye_2 & 1 \end{bmatrix}, \qquad \text{so } \xi_1\xi_1^\pi = \begin{bmatrix} 1 & y & 1 \\ 0 & 1 & 0 \\ 0 & 2y & 1 \end{bmatrix} \in S.$$

Therefore $\tilde{L}_1$ contains ξ_1. The mapping which sends $X = e_2 I_3 + X_1 \in \tilde{L}_1$ with $X_1 \in M_{Ke_1}(3, 3)$ to X_1 is an isomorphism of $\tilde{L}_1$ to

$$\left\{ \begin{bmatrix} e_1 & te_1 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_1 \end{bmatrix}, \begin{bmatrix} e_1 & 0 & 0 \\ te_1 & e_1 & 0 \\ 0 & 0 & e_1 \end{bmatrix}, \begin{bmatrix} e_1 & me_1 & e_1 \\ 0 & e_1 & 0 \\ 0 & 2me_1 & e_1 \end{bmatrix} \middle| \, t \in K \right\}.$$

The latter is isomorphic to $SL(3, Ke_1) \cong SL(3, K)$ by Lemma 2.2 as $m \in K$. Since $\tilde{L}_1 \subset L_1 \cong SL(3, K)$ we get $\tilde{L}_1 = L_1$. But $|S| \geqq |\tilde{L}_1|$, and $S \subset L \cong SL(3, K)$ forces $S = L$. This completes the proof of Lemma 2.5.

3. SL_4. Suppose K is a field with $q = 3^n$, and $A = K[y]$ is a commutative algebra over K generated by an element y with $y^2 \in K$. We allow the possibility that $y \in K$. Let e_{ij} be the matrix units, $i \neq j$, $1 \leqq i, j \leqq 4$ and $X_{ij}(t) = 1 + te_{ij}$, $X_{ij}(R) = \{X_{ij}(t) | t \in R\}$ for some subring R of A. Let

$$\omega = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$S = \langle X_{12}(K), X_{21}(K), X_{13}(t)X_{42}(t), X_{14}(-t)X_{32}(t), X_{43}(K) | t \in K \rangle.$$

Then $\omega \in S$.

LEMMA 3.1. $S \cong Sp(4, K)$.

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Proof. Let $V = Kv_1 \oplus \ldots \oplus Kv_4$ be a 4-space over K. Define a skew-symmetric form f on V by

$$\begin{aligned} f(v_1v_2) = 1, \quad & f(V_1, V_3) = 0, \quad f(V_1, V_4) = 0, \\ & f(V_2, V_3) = 0, \quad f(V_2, V_4) = 0 \\ & \qquad\qquad\qquad f(V_3, V_4) = -1. \end{aligned}$$

Thus, V is the direct sum of two hyperplanes, $P_1 = KV_1 \oplus KV_2$ and $P_2 = Kv_3 \oplus Kv_4$. Using (v_1, v_2, v_3, v_4) as a basis for V, we see that $S \subset Sp(4, K)$ as f is non-singular.

Let

$$P = \langle X_{12}(t), X_{13}(t)X_{42}(t), X_{43}(t), X_{14}(-t)X_{32}(t) | t \in K\rangle,$$

so that $|P| = q^4$; P is a Sylow p-subgroup of $Sp(4, K)$. Let $X_b(t) = (X_{13}(t)X_{42}(t))^\omega = X_{23}(t)X_{42}(-t)$ so that $X_b(t) \in S$ for all $t \in K$. Let $X_{-b}(t) = X_{14}(-t)X_{32}(t)$ so that $X_{-b}(t) \in S$. Let

$$\omega_b = X_b(1)X_{-b}(-1)X_b(1) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

$h(t) = \text{diag}(t^{-1}, t, 1, 1) \in S$. So $S \supset H = \langle h(t), h(t)^{\omega_b} | t \in K^\times\rangle$ and $H \subset N(P)$. (Note $X_{12}(K)^{\omega_b} = X_{43}(K)$ and $X_{43}(K)^{\omega_b} = X_{12}(K)$.) Since

$$\omega^2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \omega_b{}^2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \in H$$

and

$$\omega\omega_b = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \text{ has order 4 mod } H,$$

so that $W = \langle \omega, \omega_b \rangle$ normalizes H and $W/W \cap H$ is dihedral of order 8. The lemma follows from the well-known properties of the Bruhat decomposition.

Let

$$\xi = \begin{bmatrix} 1 & 0 & y & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -y & 0 & 1 \end{bmatrix}, \text{ and } L = \langle S, \xi \rangle.$$

LEMMA 3.2. *If $y \in K^\times$ then $L = SL(4, K)$.*

Proof. Since $\xi X_{13}(y)X_{42}(y) \in L$, L contains $X_{13}(2y)$. Hence L contains $X_{13}(K)$, $X_{23}(K) = X_{13}(K)^\omega$. Since $\xi X_{13}(-y) = X_{42}(-y)$, L contains $X_{42}(K)$,

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$X_{42}(K)^{\omega} = X_{41}(K)$. Since $X_{12}(K)^{\omega b} = X_{43}(K)$, L contains $X_{13}(K)$. Since $X_{21}(K)^{\omega b} = X_{34}(K)$, L contains $X_{34}(K)$. Let

$$\omega_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Then $\omega_1 \in \langle X_{34}(K), X_{43}(K) \rangle \subset L$. Hence L contains $X_{13}{}^{\omega_1}(K) = X_{14}(K)$, $X_{41}(K)^{\omega_1} = X_{31}(K)$, $X_{42}(K)^{\omega_1} = X_{32}(K)$, $X_{23}(K)^{\omega_1} = X_{24}(K)$. Therefore $L = SL(4, K)$.

Suppose $y^2 = m^2$, for some $m \in K^{\times}$ but $y \notin K$. Then $A = Ke_1 + Ke_2$, where $e_1 = (1 + m^{-1}y)/2$, $e_2 = 1 - e_1$ are idempotents of A, and $y = me_1 - me_2$. The automorphism α of A defined by $e_1{}^{\alpha} = e_2$ fixes K element-wise and induces an automorphism α of $SL(4, A)$. Let

$$\sigma = \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{bmatrix},$$

and let Ω be the inner automorphism of $SL(4, A)$ induced by σ. Let Φ be the automorphism $X = {}^tX^{-1}$. Finally, let $\pi = \alpha\Omega\Phi$.

LEMMA 3.3. *L is the fixed point of α, and $L \cong SL(4, K)$.*

Proof. We use the same argument as in the proof of Lemma 2.5 with the aid of Lemma 3.2.

Remark. It was pointed out to the author by Professor Hering that there is a uniform argument which will prove Lemma 2.1 and Lemma 2.2 in all characteristics as expected. The idea is the following: Consider the 3-dimensional special linear group S acting on the projective plane. Let the images of the subspaces of V, which are represented by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, and $(0, 0, 1)$ respectively, be A and B respectively. Suppose $\xi \in S$ does not fix both A and B. Then $\langle X_{12}(t), X_{21}(t), \xi \rangle = S$.

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