ON THE RATING OF A SPECIAL STOP LOSS COVER

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INTRODUCTION

Stop Loss reinsurance has attracted the interest of ASTIN members for years. May I recall the paper of Borch [1] in which he demonstrates some optimality qualities of the stop loss reinsurance from the ceding company's point of view, the contribution of Kahn [2] and the paper of Pesonen [3]. I also mention the paper of Esscher [4] and Verbeek's contribution [5]. Going back to the pre-ASTIN days we find a paper of Dubois [6].

The rating problems have been dealt with by several authors. Let me recall the rating formula worked out by a group of Dutch Actuaries some 20 years ago. This was based on the assumption that the mean and the standard deviation were known. Based on Chebycheff's inequality an approximation formula was worked out which, of course, was heavily on the safe side.

Even younger members of ASTIN are probably familiar with the studies made in the early sixties by a group of Swedish Actuaries, the results of which were presented by Bohman at the Actuarial Congress in London in 1964. Partly based on this, Bühlmann worked out some tables which he used for rating purposes.

My present contribution to the subject may not justify the above reviews, particularly as I will deal with a very special retention situation which a practical underwriter will rightly not accept, namely a stop-loss point as low as equal to the mean value of the distribution.

My excuse for this is that the formula deduced is very handy and that it is of value to the underwriter to know the stop loss risk rate also at this low level.

Let us denote the aggregate annual claims amount for a certain portfolio z and its distribution function $F(z)$ and define

$$E = m = \int_0^\infty x \, dF(x)$$

$$V = \sigma^2 = \int_0^\infty (x - E)^2 \, dF(x)$$

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and the stop loss risk premium when the retention is A

$$c(A) = \int_A^\infty (x - A)\, dF(x).$$

We will study the special case

$$e(E) = \int_E^\infty (x - E)\, dF(x) = \int_0^E (E - x)\, dF(x).$$

CALCULATION OF e (E) FOR VARIOUS DISTRIBUTIONS

1. $F(x)$ is generated by a Poisson process with the parameter λ. All claims are of equal size s. We have

$$E = \lambda \cdot s$$

and

$$V = \lambda \cdot s^2.$$

Further

$$e(E) = \int_E^\infty (x - E)\, dF(x) = \int_0^E (E - x)\, dF(x)$$

$$= \sum_{v=0}^{[\lambda]} (E - vs)\, P_\lambda(v) = E \cdot \sum_{v=0}^{[\lambda]} (\mathrm{I} - v/\lambda)\, P_\lambda(v) =$$

$$= E \left(\sum_{v=0}^{[\lambda]} P_\lambda(v) - \sum_{v=1}^{[\lambda]} P_\lambda(v - \mathrm{I}) \right) = E \left(\sum_{v=0}^{[\lambda]} P_\lambda(v) - \sum_{v=0}^{[\lambda]-1} P_\lambda(v) \right) =$$

$$= E \cdot P_\lambda([\lambda])$$

where $[\lambda]$ is the integer part of λ.

It is useful for the following if we replace in the formula

$$P_\lambda([\lambda]) = \frac{e^{-\lambda}\, \lambda^{[\lambda]}}{[\lambda]!}$$

the factorial by the Γ-function.

Thus

$$P_\lambda(\lambda) = \frac{e^{-\lambda}\, \lambda^{[\lambda]}}{\Gamma(\lambda + \mathrm{I})}$$

As seen in the following table the error is small.

Comparison of $P_\lambda([\lambda])$ with $\dfrac{e^{-\lambda}\lambda^\lambda}{\Gamma(\lambda+1)}$

λ	$P_\lambda\,([\lambda])\,/\,\dfrac{e^{-\lambda}\lambda^\lambda}{\Gamma(\lambda+1)}$
1.5	1.085
2.5	1.051
3.5	1.036
4.5	1.028
5.5	1.023
6.5	1.019
7.5	1.017
8.5	1.015
9.5	1.013
10.5	1.012

2. $F(x)$ is generated by a Poisson-Pareto process. In another paper by G. Benktander "A Motor Excess Rating Problem: Flat Rate with Refund", it has been shown that the formula for the stop loss premium

$$e(E) \gtrsim E \cdot P_\lambda(\lambda)$$

represents a remarkably good approximation.

The λ to be used here should not be equal to the Poisson Parameter (the expected number of claims n) but smaller. A good value is

$$\lambda = \frac{E^2}{V} \sim \frac{n(1 + 1/k)^2}{4}$$

(See G. Benktander "The Calculation of a Fluctuation Loading for an Excess of Loss Cover", ASTIN Bulletin, Volume VIII, Part 3.)

The results just obtained or referred to lead us to calculate $e(E)$ directly for some distributions which could describe the total claims amount and compare it with $E \cdot P_\lambda(\lambda)$.

3. The exponential distribution

$$f(x) = (1/a)\,e^{-x/a}$$

$$E = a \qquad V = a^2 \qquad \lambda = 1$$

and

$$e(E) = \int_a^\infty (1/a)\,(x-a)\,e^{-x/a}\,dx = a\int_1^\infty (y-1)\,e^{-y}\,dy = a\cdot e^{-1} = E\cdot P_1(1)$$

For the exponential distribution the formula is thus exact.

4. The Gamma distribution

$$f(x) = \frac{c^\gamma}{\Gamma(\gamma)}\,e^{-cx}\,x^{\gamma-1}$$

$$E = \frac{\gamma}{c} \qquad V = \frac{\gamma}{c^2} \qquad \lambda = \frac{E^2}{V} = \gamma$$

$$e(E) = \frac{\gamma^\gamma}{c\Gamma(\gamma)}\,e^{-\gamma} = \frac{\gamma}{c}\cdot\frac{\gamma^\gamma\,e^{-\gamma}}{\Gamma(\gamma+1)} = E\cdot P_\lambda(\lambda)$$

Also in this case the formula is exact which is not surprising considering the close connection between the Gamma- and the Poisson-distribution.

5. The normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{(x-m)^2}{2\sigma^2}}$$

$$E = m \qquad V = \sigma^2 \qquad \lambda = \frac{m^2}{\sigma^2}$$

$$e(E) = \frac{1}{\sqrt{2\pi}}\int_m^\infty \frac{(x-m)}{\sigma}\,e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx = \frac{\sigma}{\sqrt{2\pi}}\int_0^\infty ye^{-y^2/2}\,dy = \frac{\sigma}{\sqrt{2\pi}}$$

as

$$\sigma = \frac{m}{\sqrt{\lambda}}$$

we get

$$e(E) = m\cdot\frac{1}{\sqrt{2\pi\lambda}} = E\cdot\frac{1}{\sqrt{2\pi\lambda}}$$

We thus have to compare $\frac{1}{\sqrt{2\pi\lambda}}$ with $P_\lambda(\lambda)$.

λ	$\frac{1}{\sqrt{2\pi\lambda}}$	$P_0(\lambda)$
1	0.399	0.368
2	0.282	0.271
3	0.230	0.224
4	0.199	0.195
5	0.178	0.175
6	0.163	0.161
7	0.151	0.149
8	0.141	0.140
9	0.133	0.132
10	0.126	0.125
20	0.089	0.089

The approximation is very good and converges towards the exact value. Using the Stirling-formula

$$\lambda! = \Gamma(\lambda + 1) = e^{-\lambda}\lambda^\lambda\sqrt{2\pi\lambda}\left(1 + \frac{1}{12\lambda} + \ldots\ldots\right)$$

we get

$$P_\lambda(\lambda) = \frac{e^{-\lambda}\lambda^\lambda}{\Gamma(\lambda+1)} = \frac{1}{\sqrt{2\pi\lambda}\left(1 + \frac{1}{12\lambda} + \ldots\right)} \to \frac{1}{\sqrt{2\pi\lambda}}$$

6. The Log-normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi x}}e^{-\frac{1}{2}\left(\frac{\ln x - m}{\sigma}\right)^2}$$

$$E = e^{m+\sigma^2/2} \qquad V = e^{2m+\sigma^2}(e^{\sigma^2} - 1)$$

$$\lambda = \frac{E^2}{V} = \frac{1}{e^{\sigma^2} - 1}$$

or

$$\sigma = \sqrt{\ln(1 + 1/\lambda)}.$$

The coefficient of variation is

$$\frac{\sqrt{V}}{E} = \frac{1}{\sqrt{\lambda}}$$

In practical applications the main interest should be concentrated on the λ-interval 1 to 100.

The corresponding interval for the dispersion of $\ln x$, σ, is

$$\sqrt{\ln 2} \text{ to } \sqrt{\ln 1.01} = 0.833 \text{ to } 0.1.$$

After some calculations we get

$$c(E) = e^{m + \sigma^2/2} \{\varphi(\sigma/2) - \varphi(-\sigma/2)\} = E \cdot [\varphi(\sigma/2) - \varphi(-\sigma/2)].$$

λ	$\dfrac{\sigma}{2}$	$\varphi\left(\dfrac{\sigma}{2}\right) - \varphi\left(-\dfrac{\sigma}{2}\right)$	$P_0(\lambda)$
1	0.416	0.323	0.368
2	0.318	0.250	0.271
3	0.268	0.212	0.224
4	0.236	0.187	0.195
5	0.213	0.169	0.176
6	0.196	0.156	0.161
7	0.183	0.145	0.149
8	0.172	0.136	0.140
9	0.162	0.129	0.132
10	0.154	0.123	0.125
20	0.110	0.088	0.089
30	0.091	0.072	0.073
40	0.079	0.063	0.063

The approximation is, as can be seen, good, slightly on the safe side and converging towards the exact value when λ increases. This is not astonishing because

$$\varphi\left(\frac{\sigma}{2}\right) - \varphi\left(-\frac{\sigma}{2}\right) = \frac{2}{\sqrt{2\pi}} \int_0^{\sigma/2} e^{-x^2/2}\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{\sigma}{2} e^{-\frac{\theta^2\sigma^2}{8}} \qquad 0 < \theta < 1$$

$$= \frac{2}{\sqrt{2\pi}} \frac{\sigma}{2} \left(1 - \frac{\theta^2\sigma^2}{8} + \ldots\right).$$

As

$$\sigma = \sqrt{\ln(1 + 1/\lambda)} \simeq \sqrt{1/\lambda},$$

we get

$$\frac{1}{\sqrt{2\pi\lambda}} \left(1 - \frac{\theta^2}{8}\frac{1}{\lambda} + \ldots\right) \to P_\lambda(\lambda).$$

7. Pareto

$$f(x) = \alpha \cdot a^\alpha x^{-\alpha-1} \quad x \geq a > 0$$

$$E = a \cdot \frac{\alpha}{\alpha - 1} \qquad V = a^2 \frac{\alpha}{(\alpha - 1)^2 (\alpha - 2)} \qquad \lambda = \alpha(\alpha - 2)$$

$$c(E) = \frac{a}{\alpha - 1} \left(1 - \left(\frac{\alpha - 1}{\alpha}\right)^{(\alpha-1)}\right) = \frac{E}{\alpha} \left(1 - \left(\frac{\alpha - 1}{\alpha}\right)^{(\alpha-1)}\right)$$

α	λ	$\frac{1}{\alpha}\left(1 - \left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\right)$	$P_\lambda([\lambda])$
2.25	0.56	0.231	0.570
2.5	1.25	0.214	0.358
2.75	2.06	0.199	0.270
3.00	3.00	0.185	0.224
3.25	4.06	0.173	0.195
3.50	5.25	0.162	0.174
3.75	6.56	0.153	0.157
4.00	8.00	0.144	0.140

The correspondence is not as good as in other examples above. It has, however, to be kept in mind that the (unlimited) Pareto distribution does not represent a good description of the total claims amount.

8. $F(x)$ is generated by a Poisson process with fluctuating basic probabilities according to a Gamma-structure function (resulting in a Negative Binomial distribution).

All claims are of equal size s.

$$f(\nu s) = \frac{\Gamma(h + \nu)}{\Gamma(\nu + 1)\,\Gamma(h)} \left(\frac{h}{h + \lambda}\right)^h \left(\frac{\lambda}{h + \lambda}\right)^\nu \quad \nu = 0, 1, \ldots.$$

$$E = \lambda \cdot s$$

$$V = \lambda \cdot s^2 + \frac{\lambda^2 s^2}{h} = \lambda s^2 + \frac{E^2}{h}$$

We transform this distribution in a Poisson distribution determining its parameter λ' in the same way as above.

$$\lambda' = \frac{E^2}{V} = \frac{\lambda^2 s^2}{\lambda s^2 + \frac{\lambda^2 s^2}{h}}$$

$$1/\lambda' = 1/\lambda + 1/h$$

$$c(E) = E \cdot \sum_{v=[\lambda]+1}^{\infty} (v/\lambda - 1) f(v) = E \cdot \sum_{v=0}^{[\lambda]} (1 - v/\lambda) f(v)$$

is approximated by $E \cdot P_{\lambda'}([\lambda'])$.

The approximation is good, even for small h (= large variation in the basic probability).

λ Neg. Binom.	h	$\lambda' = \dfrac{\lambda h}{\lambda + h}$	$\dfrac{e(E)}{E}$	$P_\lambda.([\lambda'])$
1	15	0.937	0.380	0.392
2	15	1.705	0.288	0.302
4	15	3.158	0.220	0.223
8	15	5.217	0.173	0.175
1	25	0.962	0.375	0.382
2	25	1.852	0.281	0.291
4	25	3.448	0.210	0.217
8	25	6.061	0.160	0.161
1	50	0.980	0.372	0.375
2	50	1.923	0.276	0.281
4	50	3.704	0.203	0.209
8	50	6.897	0.150	0.151

CONCLUSION

We have seen that for a large group of distributions the risk premium of a special stop loss cover (retention equal to the expected value) can be approximately calculated by a handy formula.

$$c(E) = E \cdot P_\lambda([\lambda])$$

with

$$\lambda = E^2/V$$

E = Expected value of the distribution
$V = \sigma^2$ = Variance.

In 5. we have seen that

$$E \cdot P_\lambda(\lambda) = \frac{E}{\sqrt{2\pi\lambda}} \cdot \frac{1}{\left(1 + \dfrac{1}{12\lambda} + \dots\right)}$$

$$= \frac{\sigma}{\sqrt{2\pi}} \frac{1}{\left(1 + \frac{1}{12}\frac{\sigma^2}{E^2} + \ldots\right)} < \frac{\sigma}{\sqrt{2\pi}}$$

$$\lim_{\lambda \to \infty} E \cdot P_\lambda(\lambda) = \frac{\sigma}{\sqrt{2\pi}}$$

Thus the convenient approximation $e(E) = \sigma/\sqrt{2\pi}$ which is exact in case of a normal distribution is more on the safe side than $P_\lambda(\lambda)$. How does the approximation $e(E) = \sigma/\sqrt{2\pi}$ fit generalized Poisson distribution functions?

If we assume the existence of all moments of the claim size distribution function and that the expected number of claims λ is large enough so that all terms of order $o(\lambda^{-\frac{1}{2}})$ and higher order in the Edgeworth expansion can be neglected, then $\sigma/\sqrt{2\pi}$ is a good approximation for the risk premium of the special stop loss cover.

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