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PRESENTING CYCLOTOMIC q-SCHUR ALGEBRAS

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Abstract. We give a presentation of cyclotomic q-Schur algebras by generators and defining relations. As an application, we give an algorithm for computing decomposition numbers of cyclotomic q-Schur algebras.

Contents

§0. Introduction

Let $\mathscr{H}_{n,r}$ be the Ariki-Koike algebra associated to a complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ introduced by Ariki and Koike in [AK]. The cyclotomic q-Schur algebra $\mathscr{S}_{n,r}$ associated to $\mathscr{H}_{n,r}$, introduced in [DJM], is defined as the endomorphism algebra of a certain $\mathscr{H}_{n,r}$-module. The main aim of this article is to give a presentation of cyclotomic q-Schur algebras by generators and defining relations.

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In the case where $r = 1$, $\mathscr{H}_{n,1}$ is the Iwahori-Hecke algebra of the symmetric group $\mathfrak{S}_n$, and $\mathscr{S}_{n,1}$ is the q-Schur algebra of type A. In this case, $\mathscr{S}_{n,1}$ can be realized as a quotient algebra of the quantum group $U_q = U_q(\mathfrak{gl}_m)$ via the Schur-Weyl duality between $\mathscr{H}_{n,1}$ and U_q given in [J]. We note that the Schur-Weyl duality holds not only over $\mathbb{Q}(q)$ but also over $\mathbb{Z}[q, q^{-1}]$ (see [Du]). By using the surjection from U_q to $\mathscr{S}_{n,1}$, Doty and Giaquinto [DG] gave a presentation of $\mathscr{S}_{n,1}$ by generators and defining relations. They also gave a presentation of $\mathscr{S}_{n,1}$ which is compatible with Lusztig's modified form of U_q. After that, Doty [Do] realized the generalized q-Schur algebra (in the sense of Donkin) as a quotient algebra of a quantum group (also Lusztig's modified form) associated to any Cartan matrix of finite type.

In the case where $r > 1$, a Schur-Weyl duality between $\mathscr{H}_{n,r}$ and $U_q(\mathfrak{g})$ over $\mathcal{K} = \mathbb{Q}(q, \gamma_1, \dots, \gamma_r)$ was obtained by Sakamoto and Shoji [SakS], where $\mathfrak{g} = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ is a Levi subalgebra of a parabolic subalgebra of $\mathfrak{gl}_m$. However, this Schur-Weyl duality does not hold over $\mathbb{Z}[q, q^{-1}, \gamma_1, \dots, \gamma_r]$. In fact, Sakamoto-Shoji's Schur-Weyl duality should be understood as a Schur-Weyl duality between the modified Ariki-Koike algebra $\mathscr{H}^0_{n,r}$, introduced in [S1], and $U_q(\mathfrak{g})$, rather than the duality between $\mathscr{H}_{n,r}$ and $U_q(\mathfrak{g})$. The image of $U_q(\mathfrak{g})$ in the Schur-Weyl duality is isomorphic to the modified cyclotomic q-Schur algebra $\overline{\mathscr{S}}^0_{n,r}$ associated to $\mathscr{H}^0_{n,r}$ introduced in [SawS]. $\mathscr{H}^0_{n,r}$ and $\overline{\mathscr{S}}^0_{n,r}$ are defined over any integral domain R with parameters satisfying certain conditions. In particular, we have $\mathscr{H}_{n,r} \cong \mathscr{H}^0_{n,r}$ over $\mathcal{K}$ though $\overline{\mathscr{S}}^0_{n,r} \not\cong \mathscr{S}_{n,r}$. (Note that $\mathscr{H}_{n,r} \not\cong \mathscr{H}^0_{n,r}$ over R in general.) Some relations between $\mathscr{S}_{n,r}$ and $\overline{\mathscr{S}}^0_{n,r}$ were studied in [SawS] and [Saw]. They showed that $\overline{\mathscr{S}}^0_{n,r}$ turns out to be a subquotient algebra of $\mathscr{S}_{n,r}$, and $\overline{\mathscr{S}}^0_{n,r} \cong \bigoplus_{\substack{(n_1,\dots,n_r)\\ n_1+\cdots+n_r=n}} \mathscr{S}_{n_1,1} \otimes \cdots \otimes \mathscr{S}_{n_r,1}$, where each component $\mathscr{S}_{n_k,1}$ is a q-Schur algebra of type A which is a quotient algebra of the corresponding Levi component $U_q(\mathfrak{gl}_{m_k})$ of $U_q(\mathfrak{gl}_m)$.

In [SW], we have generalized the results in [SawS] and [Saw] as follows. Let $\mathbf{p} = (r_1, \dots, r_g) \in \mathbb{Z}^g_{>0}$ be such that $r_1 + \cdots + r_g = r$. We define a subquotient algebra $\overline{\mathscr{S}}^{\mathbf{p}}_{n,r}$ of $\mathscr{S}_{n,r}$ with respect to $\mathbf{p}$ by using a cellular basis of $\mathscr{S}_{n,r}$ given in [DJM]. Then we have $\overline{\mathscr{S}}^{\mathbf{p}}_{n,r} \cong \bigoplus_{\substack{(n_1,\dots,n_g)\\ n_1+\cdots+n_g=n}} \mathscr{S}_{n_1,r_1} \otimes \cdots \otimes \mathscr{S}_{n_g,r_g}$. The case of $\mathbf{p} = (1, \dots, 1)$ is the one discussed in [SawS], and $\overline{\mathscr{S}}^{(r)}_{n,r}$ (the case of $\mathbf{p} = (r)$) is just $\mathscr{S}_{n,r}$. These structures suggest to us that $\overline{\mathscr{S}}^{\mathbf{p}}_{n,r}$ is a quotient algebra of a certain algebra $\widetilde{U}_q(\mathfrak{g}^{\mathbf{p}})$ with respect to the Levi subalgebra

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$\mathfrak{g}^{\mathbf{p}} = \mathfrak{gl}_{m_1+\cdots+m_{r_1}} \oplus \cdots \oplus \mathfrak{gl}_{m_{r_1+\cdots+r_{g-1}+1}+\cdots+m_r}$ of $\mathfrak{gl}_m$. In particular, $\mathscr{S}_{n,r}$ should be a quotient algebra of a certain algebra $\widetilde{U}_q(\mathfrak{gl}_m)$. (Note that $\widetilde{U}_q(\mathfrak{gl}_m)$ (also $\widetilde{U}_q(\mathfrak{g}^{\mathbf{p}})$) is not a quantum group.) This is a motivation for this article.

On the other hand, in [DR2] Du and Rui defined (upper and lower) Borel subalgebras $\mathscr{S}^{\geq 0}_{n,r}$ and $\mathscr{S}^{\leq 0}_{n,r}$ of $\mathscr{S}_{n,r}$, and they showed that $\mathscr{S}_{n,r} = \mathscr{S}^{\leq 0}_{n,r} \cdot \mathscr{S}^{\geq 0}_{n,r}$. Moreover, they showed that the Borel subalgebra $\mathscr{S}^{\geq 0}_{n,r}$ (resp., $\mathscr{S}^{\leq 0}_{n,r}$) is isomorphic to the Borel subalgebra $\mathscr{S}^{\geq 0}_{m,1}$ (resp., $\mathscr{S}^{\leq 0}_{m,1}$) of a q-Schur algebra $\mathscr{S}_{m,1}$ of type A with an appropriate rank. In fact, the Borel subalgebra $\mathscr{S}^{\geq 0}_{m,1}$ (resp., $\mathscr{S}^{\leq 0}_{m,1}$) of $\mathscr{S}_{m,1}$ is a quotient algebra of an upper (resp., lower) Borel subalgebra of $U_q(\mathfrak{gl}_m)$. These structures imply that $\mathscr{S}_{n,r}$ is presented by generators of $U_q(\mathfrak{gl}_m)$ with certain defining relations which are different from the defining relations of $U_q(\mathfrak{gl}_m)$. A main idea here is to find presentations of $\mathscr{S}_{n,r}$ by generators and relations.

This article is organized as follows. In Section 1, we introduce a certain algebra $\widetilde{U}_q = \widetilde{U}_q(\mathfrak{gl}_m)$ associated to the Cartan data of $\mathfrak{gl}_m$. The quantum group $U_q(\mathfrak{gl}_m)$ turns out to be a quotient algebra of $\widetilde{U}_q$. We also prepare several notions for representations of $\widetilde{U}_q$ similar to the case of quantum groups, for example, weight modules, highest-weight modules, and Verma modules. In Section 2, we define various finite-dimensional quotient algebras $\mathcal{S}_q$ of $\widetilde{U}_q$ which are constructions inspired by the generalized q-Schur algebras defined in [Do]. In fact, both the q-Schur algebra $\mathscr{S}_{n,1}$ of type A and the cyclotomic q-Schur algebra $\mathscr{S}_{n,r}$ are examples of these finite-dimensional quotient algebras of $\widetilde{U}_q$. We also give a method to study representations of $\mathcal{S}_q$ analogous to the theory of cellular algebras in [GL]. In some cases, $\mathcal{S}_q$ turns out to be a quasi-hereditary cellular algebra. In Section 3, we develop an argument of specialization of $\mathcal{S}_q$ to an arbitrary ring and parameters by taking divided powers. We note that the arguments in Sections 1–3 can be applied to any Cartan matrix of finite type (see Remarks 3.16(ii)).

After recalling some known results on q-Schur algebras and cyclotomic q-Schur algebras in Sections 4 and 5, we define a surjective homomorphism $\widetilde{\rho}$ from $\widetilde{U}_q$ to $\mathscr{S}_{n,r}$ in Section 6. By using the surjection $\widetilde{\rho}$ combined with the results in Sections 1–3, we give two presentations of $\mathscr{S}_{n,r}$ in Section 7 (see Theorem 7.16).

Finally, we give an algorithm to compute the decomposition numbers of cyclotomic q-Schur algebras in Section 8.

§1. The algebra $\widetilde{U}_q$

In this section, we introduce a new algebra $\widetilde{U}_q = \widetilde{U}_q(\mathfrak{gl}_m)$ associated to the Cartan data of $\mathfrak{gl}_m$. Then we study some representations of $\widetilde{U}_q$. The definition of $\widetilde{U}_q$ is motivated by some structures of the cyclotomic q-Schur algebras $\mathscr{S}_{n,r}$ given in [SW] and [DR2] (see the introduction). Then $\widetilde{U}_q$ will be used to obtain a presentation of $\mathscr{S}_{n,r}$ in Sections 6 and 7.

1.1.
Let $P = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i$ be the weight lattice of $\mathfrak{gl}_m$, and let $P^\vee = \bigoplus_{i=1}^m \mathbb{Z}h_i$ be the dual-weight lattice with the natural pairing $\langle\, , \rangle : P \times P^\vee \to \mathbb{Z}$ such that $\langle \varepsilon_i, h_j\rangle = \delta_{ij}$. Set $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \dots, m-1$; then $\Pi = \{\alpha_i \mid 1 \le i \le m-1\}$ is the set of simple roots, and $Q = \bigoplus_{i=1}^{m-1} \mathbb{Z}\alpha_i$ is the root lattice of $\mathfrak{gl}_m$. Put $Q^+ = \bigoplus_{i=1}^{m-1} \mathbb{Z}_{\ge 0}\alpha_i$. We define a partial order $\ge$ on P by $\lambda \ge \mu$ if $\lambda - \mu \in Q^+$.

1.2.
The quantum group $U_q = U_q(\mathfrak{gl}_m)$ is the associative algebra over $\mathbb{Q}(q)$, where q is an indeterminate, with 1 generated by e_i, f_i $(1 \le i \le m-1)$ and $K_i^\pm$ $(1 \le i \le m)$ with the following defining relations (we denote K_i^+ by K_i simply):

$$(1.2.1) \qquad K_iK_j = K_jK_i, \qquad K_iK_i^- = K_i^-K_i = 1,$$

$$(1.2.2) \qquad K_ie_jK_i^- = q^{\langle \alpha_j, h_i\rangle} e_j,$$

$$(1.2.3) \qquad K_if_jK_i^- = q^{-\langle \alpha_j, h_i\rangle} f_j,$$

$$(1.2.4) \qquad e_if_j - f_je_i = \delta_{ij}\frac{K_iK_{i+1}^- - K_i^-K_{i+1}}{q-q^{-1}},$$

$$(1.2.5) \qquad e_{i\pm1}e_i^2 - (q+q^{-1})e_ie_{i\pm1}e_i + e_i^2e_{i\pm1} = 0,$$
$$e_ie_j = e_je_i \quad (|i-j| \ge 2),$$

$$(1.2.6) \qquad f_{i\pm1}f_i^2 - (q+q^{-1})f_if_{i\pm1}f_i + f_i^2f_{i\pm1} = 0,$$
$$f_if_j = f_jf_i \quad (|i-j| \ge 2).$$

Let U_q^+ (resp., U_q^-) be the subalgebra of U_q generated by e_i (resp., f_i) for $i = 1, \dots, m-1$, and let U_q^0 be the subalgebra of U_q generated by $K_i^\pm$ for $i = 1, \dots, m$. It is well known that U_q has the triangular decomposition

$$U_q \cong U_q^- \otimes U_q^0 \otimes U_q^+ \quad \text{as vector spaces.}$$

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Let $U_q^{\geq 0}$ (resp., $U_q^{\leq 0}$) be the subalgebra of U_q generated by e_i (resp., f_i) for $1 \leq i \leq m-1$ and $K_i^{\pm}$ for $1 \leq i \leq m$. We call $U_q^{\geq 0}$ (resp., $U_q^{\leq 0}$) a *Borel subalgebra* of U_q. The following lemma is well known.

LEMMA 1.3. *We have the following.*

(i) U_q^+ *(resp.,* U_q^-*) is isomorphic to the algebra defined by generators* e_i *(resp.,* f_i*)* $(1 \leq i \leq m-1)$ *with a defining relation (1.2.5) (resp., (1.2.6)).*

(ii) U_q^0 *is isomorphic to* $\mathbb{Q}(q)[K_1^{\pm}, \ldots, K_m^{\pm}]$.

(iii) $U_q^{\geq 0}$ *is isomorphic to the algebra defined by generators* e_i $(1 \leq i \leq m-1)$ *and* $K_i^{\pm}$ $(1 \leq i \leq m)$ *with defining relations (1.2.1), (1.2.2), and (1.2.5).*

(iv) $U_q^{\leq 0}$ *is isomorphic to the algebra defined by generators* f_i $(1 \leq i \leq m-1)$ *and* $K_i^{\pm}$ $(1 \leq i \leq m)$ *with defining relations (1.2.1), (1.2.3), and (1.2.6).*

1.4.

Put $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$. We define the $\mathcal{Z}$-form of U_q as follows. For any integer $k \in \mathbb{Z}$, put

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}.$$

For any positive integer $t \in \mathbb{Z}_{>0}$, put $[t]! = [t][t-1]\cdots[1]$ and set $[0]! = 1$. For any integer k and any positive integer t, put

$$\begin{bmatrix} k \\ t \end{bmatrix} = \frac{[k][k-1]\cdots[k-t+1]}{[t][t-1]\cdots[1]} = \frac{[k]!}{[t]![k-t]!}.$$

For $k \in \mathbb{Z}_{\geq 0}$ and $i = 1, \ldots, m-1$, put

$$e_i^{(k)} = \frac{e_i^k}{[k]!}, \qquad f_i^{(k)} = \frac{f_i^k}{[k]!}.$$

For $t \in \mathbb{Z}_{\geq 0}$, $c \in \mathbb{Z}$, and $i = 1, \ldots, m$, put

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K_i q^{c-s+1} - K_i^{-1} q^{-c+s-1}}{q^s - q^{-s}}.$$

Let ${}_{\mathcal{Z}}U_q$ be the $\mathcal{Z}$-subalgebra of U_q generated by all $e_i^{(k)}, f_i^{(k)}, K_i^{\pm}$, and $\left[\begin{smallmatrix} K_i;0 \\ t \end{smallmatrix}\right]$. We also define the $\mathcal{Z}$-subalgebra ${}_{\mathcal{Z}}U_q^{\geq 0}$ (resp., ${}_{\mathcal{Z}}U_q^{\leq 0}$) of U_q generated by all $e_i^{(k)}$ (resp., $f_i^{(k)}$), $K_i^{\pm}$, and $\left[\begin{smallmatrix} K_i;0 \\ t \end{smallmatrix}\right]$.

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1.5.

Let $\mathcal{A} = \mathcal{Z}[\gamma_1, \dots, \gamma_r]$ be the polynomial ring over $\mathcal{Z}$ with indeterminate elements $\gamma_1, \dots, \gamma_r$, where r is an arbitrary nonnegative integer (put $\mathcal{A} = \mathcal{Z}$ when $r = 0$), and let $\mathcal{K} = \mathbb{Q}(q, \gamma_1, \dots, \gamma_r)$ be the quotient field of $\mathcal{A}$. We define the associative algebra $\widetilde{U}_q = \widetilde{U}_q(\mathfrak{gl}_m)$ over $\mathcal{K}$ with the unit element 1 by the following generators and defining relations.

Generators e_i, f_i $(1 \le i \le m-1)$, $K_i^{\pm}$ $(1 \le i \le m)$, τ_i $(1 \le i \le m-1)$.

Defining relations

(1.5.1) $$K_iK_j = K_jK_i, \qquad K_iK_i^- = K_i^-K_i = 1,$$

(1.5.2) $$K_ie_jK_i^- = q^{\langle \alpha_j, h_i \rangle} e_j,$$

(1.5.3) $$K_if_jK_i^- = q^{-\langle \alpha_j, h_i \rangle} f_j,$$

(1.5.4) $$K_i\tau_jK_i^- = \tau_j,$$

(1.5.5) $$e_if_j - f_je_i = \delta_{ij}\tau_i,$$

(1.5.6) $$e_{i\pm1}e_i^2 - (q+q^{-1})e_ie_{i\pm1}e_i + e_i^2e_{i\pm1} = 0,$$
$$e_ie_j = e_je_i \quad (|i-j| \ge 2),$$

(1.5.7) $$f_{i\pm1}f_i^2 - (q+q^{-1})f_if_{i\pm1}f_i + f_i^2f_{i\pm1} = 0,$$
$$f_if_j = f_jf_i \quad (|i-j| \ge 2).$$

Set $\deg e_i = \alpha_i$, $\deg f_i = -\alpha_i$, $\deg K_i^{\pm} = 0$, and $\deg \tau_i = 0$. Since all the defining relations of $\widetilde{U}_q$ are homogeneous under this degree, $\widetilde{U}_q$ is a Q-graded algebra, and $\widetilde{U}_q$ has the following root space decomposition:

$$\widetilde{U}_q = \bigoplus_{\alpha \in Q} (\widetilde{U}_q)_\alpha,$$

where $(\widetilde{U}_q)_\alpha = \{u \in \widetilde{U}_q \mid K_iuK_i^- = q^{\langle \alpha, h_i \rangle}u \text{ for } 1 \le i \le m\}$. For $u \in \widetilde{U}_q$, we denote by $\deg(u) = \alpha$ if $u \in (\widetilde{U}_q)_\alpha$.

The following proposition is clear from definitions.

PROPOSITION 1.6. *Let $\widetilde{I}$ be the two-sided ideal of $\widetilde{U}_q$ generated by*

$$\tau_i - \frac{K_iK_{i+1}^- - K_i^-K_{i+1}}{q-q^{-1}} \qquad \textit{for } i = 1, \dots, m-1.$$

Then we have the following isomorphism of algebras:

$$\widetilde{U}_q/\widetilde{I} \cong \mathcal{K} \otimes_{\mathbb{Q}(q)} U_q.$$

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REMARK 1.7. We note that the parameters $\gamma_1, \dots, \gamma_r$ do not appear in the definition of $\widetilde{U}_q$. However, we will use these parameters later when we consider some representations of $\widetilde{U}_q$ or some quotient algebras of $\widetilde{U}_q$.

1.8.

Let $\widetilde{U}_q^+$ (resp., $\widetilde{U}_q^-$) be the subalgebra of $\widetilde{U}_q$ generated by e_i (resp., f_i) for $i = 1, \dots, m-1$, and let $\widetilde{U}_q^0$ be the subalgebra of $\widetilde{U}_q$ generated by $K_i^{\pm}$ for $i = 1, \dots, m$. We also define a Borel subalgebra of $\widetilde{U}_q$ as follows. Let $\widetilde{U}_q^{\geq 0}$ (resp., $\widetilde{U}_q^{\leq 0}$) be the subalgebra of $\widetilde{U}_q$ generated by $\widetilde{U}_q^+$ (resp., $\widetilde{U}_q^-$) and $\widetilde{U}_q^0$. Lemma 1.3 and Proposition 1.6 imply the following corollary.

COROLLARY 1.9. *The following isomorphisms of algebras exist:*

$$\widetilde{U}_q^{\pm} \cong \mathcal{K} \otimes_{\mathbb{Q}(q)} U_q^{\pm}, \qquad \widetilde{U}_q^0 \cong \mathcal{K} \otimes_{\mathbb{Q}(q)} U_q^0,$$
$$\widetilde{U}_q^{\geq 0} \cong \mathcal{K} \otimes_{\mathbb{Q}(q)} U_q^{\geq 0}, \qquad \widetilde{U}_q^{\leq 0} \cong \mathcal{K} \otimes_{\mathbb{Q}(q)} U_q^{\leq 0}.$$

Proof. We show only the isomorphism for the Borel subalgebra $\widetilde{U}_q^{\geq 0}$. The other isomorphisms can be shown in a similar way. By Lemma 1.3, we have a surjective homomorphism of algebras $\mathcal{K} \otimes_{\mathbb{Q}(q)} U_q^{\geq 0} \to \widetilde{U}_q^{\geq 0}$. On the other hand, by restricting the surjection $\widetilde{U}_q \to \mathcal{K} \otimes_{\mathbb{Q}(q)} U_q$ in Proposition 1.6 to $\widetilde{U}_q^{\geq 0}$, we have a surjection $\widetilde{U}_q^{\geq 0} \to \mathcal{K} \otimes_{\mathbb{Q}(q)} U_q^{\geq 0}$. Thus, we have $\widetilde{U}_q^{\geq 0} \cong \mathcal{K} \otimes_{\mathbb{Q}(q)} U_q^{\geq 0}$. ∎

1.10.

For $\eta = (\eta_1, \dots, \eta_{m-1})$ such that $\eta_i \in \widetilde{U}_q^- \widetilde{U}_q^0 \widetilde{U}_q^+$ with $\deg(\eta_i) = 0$, let $\widehat{\mathcal{O}}^\eta$ be the category consisting of $\widetilde{U}_q$-modules satisfying the following conditions (a) and (b).

(a) $M \in \widehat{\mathcal{O}}^\eta$ has the weight-space decomposition

$$M = \bigoplus_{\mu \in P} M_\mu,$$

where $M_\mu = \{v \in M \mid K_i \cdot v = q^{\langle \mu, h_i \rangle} v \text{ for } 1 \leq i \leq m\}$.

(b) For $M \in \widehat{\mathcal{O}}^\eta$ and $i = 1, \dots, m-1$, it holds that $(\tau_i - \eta_i) \cdot M = 0$.

Let $\widehat{\mathcal{O}}^\eta_{\text{tri}}$ be the full subcategory of $\widehat{\mathcal{O}}^\eta$ satisfying the following additional condition.

(c) For each $u \in \widetilde{U}_q$, there exists an element $x \in \widetilde{U}_q^- \widetilde{U}_q^0 \widetilde{U}_q^+$ such that

$$(u - x) \cdot M = 0 \quad \text{for any } M \in \widehat{\mathcal{O}}^\eta_{\text{tri}}.$$

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By this definition, in $\widehat{\mathcal{O}}^\eta_{\mathrm{tri}}$ the action of $\widetilde{U}_q$ has a triangular decomposition.

Finally, let $\mathcal{O}^\eta$ be the full subcategory of $\widehat{\mathcal{O}}^\eta$ satisfying the following additional conditions.

(d) For any $M \in \mathcal{O}^\eta$, the dimension of M is finite.

(e) For any $M \in \mathcal{O}^\eta$, we have

$$M_\mu = 0 \quad \text{unless } \mu \in P_{\geq 0},$$

where $P_{\geq 0} = \bigoplus_{i=1}^m \mathbb{Z}_{\geq 0}\varepsilon_i$.

As is shown later, $\mathcal{O}^\eta$ is a full subcategory of $\widehat{\mathcal{O}}^\eta_{\mathrm{tri}}$. Moreover, we will construct all simple objects of $\mathcal{O}^\eta$ through some quotient algebras of $\widetilde{U}_q$ (see Theorem 2.20).

REMARKS 1.11. (i) If $\eta_i \in \widetilde{U}_q^0$ for all $i = 1, \dots, m-1$, we have $\widehat{\mathcal{O}}^\eta = \widehat{\mathcal{O}}^\eta_{\mathrm{tri}}$.

(ii) Let $\widetilde{I}^\eta$ be the two-sided ideal of $\widetilde{U}_q$ generated by $(\tau_i - \eta_i)$, and put $\widetilde{U}_q^\eta = \widetilde{U}_q/\widetilde{I}^\eta$. Then, we can regard a $\widetilde{U}_q^\eta$-module as a $\widetilde{U}_q$-module through the natural surjection. Clearly, any $\widetilde{U}_q^\eta$-module equipped with the weight-space decomposition is contained in $\widehat{\mathcal{O}}^\eta$. On the other hand, a $\widetilde{U}_q$-module M contained in $\widehat{\mathcal{O}}^\eta$ is regarded as a $\widetilde{U}_q^\eta$-module since we have $\widetilde{I}^\eta \cdot M = 0$ by condition (b). Thus, the category $\widehat{\mathcal{O}}^\eta$ coincides with the category consisting of $\widetilde{U}_q^\eta$-modules which have weight-space decompositions.

(iii) When $\mathcal{K} = \mathbb{Q}(q)$ and $\eta_i = (K_i K_{i+1}^- - K_i^- K_{i+1})/(q - q^{-1})$ for any $i = 1, \dots, m-1$, $\widehat{\mathcal{O}}^\eta$ coincides with the category of U_q-modules having weight-space decompositions.

1.12.

Next, we introduce a notion of highest-weight modules. Let η be as in Section 1.10. We say that a $\widetilde{U}_q$-module $M^\eta(\lambda)$ is a *highest-weight module* of *highest weight* $\lambda \in P$ associated to η if there exists an element $v_\lambda \in M^\eta(\lambda)$ satisfying the following conditions:

(1.12.1) $\quad u \cdot v_\lambda = 0 \quad$ for any $u \in \widetilde{U}_q$ such that

$$\deg(u) = \sum_{i=1}^{m-1} d_i\alpha_i \text{ with } d_i > 0 \text{ for some } i,$$

(1.12.2) $\quad K_i \cdot v_\lambda = q^{\langle \lambda, h_i \rangle} v_\lambda \quad$ for $i = 1, \dots, m$,

(1.12.3) $\quad \widetilde{U}_q \cdot v_\lambda = M^\eta(\lambda)$,

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(1.12.4) $(\tau_i - \eta_i)\cdot M^{\eta}(\lambda) = 0 \quad \text{for } i = 1,\dots,m-1.$

We call the above element v_λ a *highest-weight vector* of $M^{\eta}(\lambda)$.

REMARKS 1.13. (i) Note that, since we take $\eta_i \in \widetilde{U}_q^-\widetilde{U}_q^0\widetilde{U}_q^+$ such that $\deg(\eta_i) = 0$, then (1.12.1), (1.12.2), and (1.12.4) imply that $\tau_i \cdot v_\lambda \in \mathcal{K}\cdot v_\lambda$.

(ii) Each highest-weight module $M^{\eta}(\lambda)$ is contained in $\widehat{\mathcal{O}}^{\eta}$.

(iii) If a highest-weight module $M^{\eta}(\lambda)$ is contained in $\widehat{\mathcal{O}}^{\eta}_{\text{tri}}$, we can replace (1.12.1) with

(1.13.1) $$e_i \cdot v_\lambda = 0 \quad \text{for } i = 1,\dots,m-1.$$

(iv) For a $\widetilde{U}_q^{\eta}$-module M, if there exists an element $v_\lambda \in M$ for some $\lambda \in P$ satisfying conditions (1.12.1)–(1.12.3), M is a highest-weight module of highest weight $\lambda \in P$ associated to η. In particular, if $\eta_i = (K_iK_{i+1}^- - K_i^-K_{i+1})/(q-q^{-1})$ for any $i = 1,\dots,m-1$ (i.e., $\widetilde{U}_q^{\eta} \cong U_q$), the definition of a highest-weight module in Section 1.12 coincides with the usual definition of a highest-weight module of $U_q(\mathfrak{gl}_m)$.

LEMMA 1.14. *If a highest-weight module $M^{\eta}(\lambda)$ is contained in $\widehat{\mathcal{O}}^{\eta}_{\text{tri}}$, we have the following.*

(i) *The dimension of the weight space $M^{\eta}(\lambda)_\lambda$ with highest weight λ is equal to* 1.

(ii) *$M^{\eta}(\lambda)$ has a unique maximal submodule.*

Proof. Item (i) is clear from definitions. By (i) and (1.12.3), a proper $\widetilde{U}_q$-submodule of $M^{\eta}(\lambda)$ does not have a weight λ. Thus, the sum of all proper $\widetilde{U}_q$-submodules of $M^{\eta}(\lambda)$ does not have the weight λ, and this is the unique maximal submodule of $M^{\eta}(\lambda)$. □

REMARK 1.15. When a highest-weight module $M^{\eta}(\lambda)$ with a highest-weight vector v_λ is *not* contained in $\widehat{\mathcal{O}}^{\eta}_{\text{tri}}$, it may occur that $u\cdot v_\lambda \notin \mathcal{K}v_\lambda$ and that $u\cdot v_\lambda$ has the weight λ for some $u \in \widetilde{U}_q$ such that $\deg(u) = 0$.

1.16.

Let $J^{\eta}(\lambda)$ be the left ideal of $\widetilde{U}_q$ generated by

$$u \in \widetilde{U}_q \qquad \text{such that } \deg(u) = \sum_{i=1}^{m-1} d_i\alpha_i \text{ with } d_i > 0 \text{ for some } i,$$

$$K_i - q^{\langle\lambda,h_i\rangle}1 \quad \text{for } i = 1,\dots,m,$$

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$$(\tau_i - \eta_i)\cdot u \qquad \text{for } i = 1, \dots, m-1 \text{ and } u \in \widetilde{U}_q.$$

Put $V^\eta(\lambda) = \widetilde{U}_q/J^\eta(\lambda)$; then one sees that $V^\eta(\lambda)$ is a highest-weight module of highest weight λ associated to η with highest-weight vector $1 + J^\eta(\lambda)$. We call $V^\eta(\lambda)$ a *Verma module* of $\widetilde{U}_q$. We have the following lemma.

LEMMA 1.17. *Any highest-weight module $M^\eta(\lambda)$ of highest weight λ associated to η is a homomorphic image of $V^\eta(\lambda)$.*

Proof. Let $M^\eta(\lambda)$ be a highest-weight module of a highest weight λ associated to η with a highest-weight vector v_λ. We regard $\widetilde{U}_q$ as a $\widetilde{U}_q$-module by left multiplications. Then, we have a natural surjective homomorphism of $\widetilde{U}_q$-modules $\widetilde{U}_q \to M^\eta(\lambda)$ such that $1 \mapsto v_\lambda$. Moreover, one can check that $J^\eta(\lambda)$ is included in the kernel of this homomorphism. Thus, this homomorphism induces the surjective homomorphism from $V^\eta(\lambda)$ to $M^\eta(\lambda)$. □

1.18.

Finally, we define an $\mathcal{A}$-form of $\widetilde{U}_q$ as follows. We use the same notations as in Section 1.4. Let ${}_{\mathcal{A}}\widetilde{U}_q$ be the $\mathcal{A}$-subalgebra of $\widetilde{U}_q$ generated by all $e_i^{(k)}, f_i^{(k)}, K_i^\pm, \tau_i$, and $\left[\begin{smallmatrix} K_i;0 \\ t \end{smallmatrix}\right]$. We also define the $\mathcal{A}$-subalgebra ${}_{\mathcal{A}}\widetilde{U}_q^{\geq 0}$ (resp., ${}_{\mathcal{A}}\widetilde{U}_q^{\leq 0}$) of $\widetilde{U}_q$ generated by all $e_i^{(k)}$ (resp., $f_i^{(k)}$), $K_i^\pm$, and $\left[\begin{smallmatrix} K_i;0 \\ t \end{smallmatrix}\right]$. Then, an isomorphism ${}_{\mathcal{A}}\widetilde{U}_q^{\geq 0} \cong \mathcal{A} \otimes_{\mathbb{Z}} {}_{\mathbb{Z}}U_q^{\geq 0}$ (resp., ${}_{\mathcal{A}}\widetilde{U}_q^{\leq 0} \cong \mathcal{A} \otimes_{\mathbb{Z}} {}_{\mathbb{Z}}U_q^{\leq 0}$) follows from Corollary 1.9.

§2. The algebra $\mathcal{S}_q$

In this section, we define various finite-dimensional quotient algebras $\mathcal{S}_q$ of $\widetilde{U}_q$. This definition is inspired by the presentation of generalized q-Schur algebras given in [Do]. Then we study the representation theory of $\mathcal{S}_q$ which has properties similar to those of the theory of cellular algebras and of standardly based algebras introduced by [GL] and [DR1], respectively. The results in this section will be applied to obtain a presentation of cyclotomic q-Schur algebras in Section 7.

2.1.

Recall that $P = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i$ is the weight lattice of $\mathfrak{gl}_m$. We can identify P with a set of m-tuple of integers $\mathbb{Z}^m$ by the correspondence

$$P \ni \lambda = \sum_{i=1}^m \lambda_i \varepsilon_i \mapsto (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m.$$

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Under this identification, we use the notation $\lambda = (\lambda_1, \dots, \lambda_m)$ for $\lambda \in P$. Let Λ be a finite subset of $P_{\geq 0} = \bigoplus_{i=1}^{m} \mathbb{Z}_{\geq 0}\varepsilon_i$.

We define the associative algebra $\widetilde{\mathcal{S}}_q = \widetilde{\mathcal{S}}_q(\Lambda)$ over $\mathcal{K}$ with 1 by following generators and defining relations.

Generators E_i, F_i $(1 \leq i \leq m-1)$, 1_λ $(\lambda \in \Lambda)$, τ_i^λ $(1 \leq i \leq m-1, \lambda \in \Lambda)$.

Defining relations

$$1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda, \qquad \sum_{\lambda \in \Lambda} 1_\lambda = 1; \tag{2.1.1}$$

$$\tau_i^\lambda 1_\mu = 1_\mu \tau_i^\lambda = \delta_{\lambda\mu} \tau_i^\lambda; \tag{2.1.2}$$

$$E_i 1_\lambda = \begin{cases} 1_{\lambda+\alpha_i} E_i & \text{if } \lambda + \alpha_i \in \Lambda, \\ 0 & \text{otherwise;} \end{cases} \tag{2.1.3}$$

$$F_i 1_\lambda = \begin{cases} 1_{\lambda-\alpha_i} F_i & \text{if } \lambda - \alpha_i \in \Lambda, \\ 0 & \text{otherwise;} \end{cases} \tag{2.1.4}$$

$$1_\lambda E_i = \begin{cases} E_i 1_{\lambda-\alpha_i} & \text{if } \lambda - \alpha_i \in \Lambda, \\ 0 & \text{otherwise;} \end{cases} \tag{2.1.5}$$

$$1_\lambda F_i = \begin{cases} F_i 1_{\lambda+\alpha_i} & \text{if } \lambda + \alpha_i \in \Lambda, \\ 0 & \text{otherwise;} \end{cases} \tag{2.1.6}$$

$$E_i F_j - F_j E_i = \delta_{ij} \Big(\sum_{\lambda \in \Lambda} \tau_i^\lambda \Big); \tag{2.1.7}$$

$$\begin{aligned} &E_{i\pm1} E_i^2 - (q + q^{-1}) E_i E_{i\pm1} E_i + E_i^2 E_{i\pm1} = 0, \\ &E_i E_j = E_j E_i \quad (|i-j| \geq 2); \end{aligned} \tag{2.1.8}$$

$$\begin{aligned} &F_{i\pm1} F_i^2 - (q + q^{-1}) F_i F_{i\pm1} F_i + F_i^2 F_{i\pm1} = 0, \\ &F_i F_j = F_j F_i \quad (|i-j| \geq 2). \end{aligned} \tag{2.1.9}$$

We can prove the following proposition in a way similar to the proof in [Do, Proposition 3.4].

PROPOSITION 2.2. *There exists a surjective homomorphism of algebras*

$$\widetilde{\Psi} : \widetilde{U}_q \to \widetilde{\mathcal{S}}_q$$

such that $\widetilde{\Psi}(e_i) = E_i, \widetilde{\Psi}(f_i) = F_i, \widetilde{\Psi}(K_i^\pm) = \sum_{\lambda \in \Lambda} q^{\pm\lambda_i} 1_\lambda, \widetilde{\Psi}(\tau_i) = \sum_{\lambda \in \Lambda} \tau_i^\lambda$.

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Proof. In order to show that $\widetilde{\Psi}$ is well defined, we should check the defining relations of $\widetilde{U}_q$ in the images of $\widetilde{\Psi}$, and we obtain them by direct calculations. Note that $\tau_i^\lambda = (\sum_{\mu\in\Lambda}\tau_i^\mu)1_\lambda = \widetilde{\Psi}(\tau_i)1_\lambda$ by (2.1.2). Thus, in order to prove that $\widetilde{\Psi}$ is surjective, it is enough to show that, for all $\lambda\in\Lambda$, 1_λ is a linear combination of $\widetilde{\Psi}(K_i)$. This will be proved in Lemma 2.3. ∎

We define a partial order $\succeq$ on $P_{\geq 0}$ by $\lambda\succ\mu$ if $\lambda\neq\mu$ and $\lambda_i\geq\mu_i$ for any $i=1,\dots,m$. For $\lambda=(\lambda_1,\dots,\lambda_m)\in\Lambda$, put

$$K_\lambda = \begin{bmatrix} K_1;0 \\ \lambda_1 \end{bmatrix}\begin{bmatrix} K_2;0 \\ \lambda_2 \end{bmatrix}\cdots\begin{bmatrix} K_m;0 \\ \lambda_m \end{bmatrix}. \tag{2.2.1}$$

Then we have the following lemma.

LEMMA 2.3.

(i) $\widetilde{\Psi}\left(\left[\begin{smallmatrix} K_i;0 \\ t \end{smallmatrix}\right]\right)$ *($1\leq i\leq m, t\in\mathbb{Z}_{\geq 0}$) is a linear combination of elements in* $\{1_\lambda \mid \lambda\in\Lambda\}$ *with coefficients in* $\mathcal{Z}$.

(ii) *For* $\lambda\in\Lambda$, *we have*

$$1_\lambda = \widetilde{\Psi}(K_\lambda) + \sum_{\substack{\mu\in\Lambda\\ \mu\succ\lambda}} r_\mu\widetilde{\Psi}(K_\mu) \quad (r_\mu\in\mathcal{Z}).$$

Proof. In this proof, we denote $\widetilde{\Psi}(K_i^\pm)$ by $K_i^\pm$ simply. Thus, we have $K_i^\pm = \sum_{\lambda\in\Lambda} q^{\pm\lambda_i}1_\lambda$. For $1\leq i\leq m, t\in\mathbb{Z}_{\geq 0}$ and $\lambda\in\Lambda$, we have

$$\begin{aligned} \begin{bmatrix} K_i;0 \\ t \end{bmatrix} 1_\lambda &= \prod_{s=1}^t \frac{K_i q^{-s+1} - K_i^- q^{s-1}}{q^s - q^{-s}} 1_\lambda \\ &= \prod_{s=1}^t \frac{q^{\lambda_i - s+1} - q^{-(\lambda_i-s+1)}}{q^s-q^{-s}} 1_\lambda \\ &= \prod_{s=1}^t \frac{[\lambda_i - s+1]}{[s]}1_\lambda \\ &= \frac{[\lambda_i][\lambda_i-1]\cdots[\lambda_i-t+1]}{[1][2]\cdots[t]}1_\lambda \\ &= \begin{cases} \begin{bmatrix} \lambda_i \\ t \end{bmatrix} 1_\lambda & \text{if } t\leq\lambda_i, \\ 0 & \text{if } t>\lambda_i. \end{cases} \end{aligned} \tag{2.3.1}$$

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Since $1 = \sum_{\lambda \in \Lambda} 1_\lambda$ and $\left[{\lambda_i \atop t} \right] \in \mathcal{Z}$, we have (i). By the definition of K_λ and (2.3.1), we have

$$K_\lambda = K_\lambda \Big(\sum_{\mu \in \Lambda} 1_\mu \Big) = 1_\lambda + \sum_{\substack{\mu \in \Lambda \\ \mu \succ \lambda}} \Big(\prod_{i=1}^{m} \begin{bmatrix} \mu_i \\ \lambda_i \end{bmatrix} 1_\mu \Big). \tag{2.3.2}$$

Since Λ is a finite set, there exists a maximal element $\lambda \in \Lambda$ with respect to the order $\succeq$. Thus, we have $1_\lambda = K_\lambda$ when λ is a maximal element of Λ by (2.3.2). By induction on Λ together with (2.3.2), we have (ii). □

REMARK 2.4. For $\lambda = (\lambda_1, \dots, \lambda_m) \in P_{\geq 0}$, set $|\lambda| = \sum_{i=1}^{m} \lambda_i$. If $\Lambda = \{\lambda \in P_{\geq 0} \mid |\lambda| = n\}$ for some $n \in \mathbb{Z}_{>0}$, we have $\mu \not\succ \lambda$ for any $\lambda, \mu \in \Lambda$ since $|\mu| > |\lambda|$ if $\mu \succ \lambda$. Thus, we have $1_\lambda = \widetilde{\Psi}(K_\lambda)$ for any $\lambda \in \Lambda$ by Lemma 2.3.

2.5.
Let $\widetilde{\mathcal{S}}_q^+$ (resp., $\widetilde{\mathcal{S}}_q^-$) be the subalgebra of $\widetilde{\mathcal{S}}_q$ generated by E_i (resp., F_i) for $1 \leq i \leq m-1$, and let $\widetilde{\mathcal{S}}_q^0$ be the subalgebra of $\widetilde{\mathcal{S}}_q$ generated by 1_λ for $\lambda \in \Lambda$. By Lemma 2.3, it is clear that $\widetilde{\mathcal{S}}_q^0$ (resp., $\widetilde{\mathcal{S}}_q^\pm$) coincides with the image of $\widetilde{U}_q^0$ (resp., $\widetilde{U}_q^\pm$) under the surjection $\widetilde{\Psi}$ in Proposition 2.2.

We consider the Q-grading on $\widetilde{\mathcal{S}}_q$ arising from the grading on $\widetilde{U}_q$; namely, we set $\deg E_i = \alpha_i, \deg F_i = -\alpha_i, \deg 1_\lambda = 0, \deg \tau_i^\lambda = 0$.

For each $\lambda \in \Lambda$ and $i = 1, \dots, m-1$, we take an element η_i^λ of $\widetilde{\mathcal{S}}_q^- \widetilde{\mathcal{S}}_q^+ \cdot 1_\lambda$ such that $\deg(\eta_i^\lambda) = 0$. By the condition $\deg(\eta_i^\lambda) = 0$ together with (2.1.3)–(2.1.6), we have $\eta_i^\lambda \in 1_\lambda \cdot \widetilde{\mathcal{S}}_q^- \widetilde{\mathcal{S}}_q^+ \cdot 1_\lambda$. Moreover, again by (2.1.3)–(2.1.6), we have $\eta_i^\lambda \in \widetilde{\mathcal{S}}_q^- \widetilde{\mathcal{S}}_q^0 \widetilde{\mathcal{S}}_q^+$. Put $\eta_\Lambda = \{\eta_i^\lambda \mid 1 \leq i \leq m-1, \lambda \in \Lambda\}$. Let $\widetilde{\mathcal{I}}^{\eta_\Lambda}$ be the two-sided ideal of $\widetilde{\mathcal{S}}_q$ generated by all $\tau_i^\lambda - \eta_i^\lambda$ $(1 \leq i \leq m-1, \lambda \in \Lambda)$. We define the quotient algebra $\mathcal{S}_q$ of $\widetilde{\mathcal{S}}_q$ by

$$\mathcal{S}_q = \mathcal{S}_q^{\eta_\Lambda} = \widetilde{\mathcal{S}}_q / \widetilde{\mathcal{I}}^{\eta_\Lambda}.$$

Let $\mathcal{S}_q^0$ (resp., $\mathcal{S}_q^\pm$) be the image of $\widetilde{\mathcal{S}}_q^0$ (resp., $\widetilde{\mathcal{S}}_q^\pm$) under the natural surjection $\widetilde{\mathcal{S}}_q \to \mathcal{S}_q$. Under the map $\widetilde{\mathcal{S}}_q \to \mathcal{S}_q$, we denote the image of E_i (resp., F_i, 1_λ) by the same symbol E_i (resp., F_i, 1_λ) again, and the image of τ_i^λ by η_i^λ. We denote the composition of $\widetilde{\Psi}$ and the natural surjection $\widetilde{\mathcal{S}}_q \to \mathcal{S}_q$ by $\Psi : \widetilde{U}_q \to \mathcal{S}_q$. Thus, we have $\Psi(e_i) = E_i$, $\Psi(f_i) = F_i$, $\Psi(K_i^\pm) = \sum_{\lambda \in \Lambda} q^{\pm \lambda_i} 1_\lambda$, and $\Psi(\tau_i) = \sum_{\lambda \in \Lambda} \eta_i^\lambda$.

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PROPOSITION 2.6. *The algebra $\mathcal{S}_q$ has a triangular decomposition*

$$\mathcal{S}_q = \mathcal{S}_q^- \mathcal{S}_q^0 \mathcal{S}_q^+.$$

Moreover, the algebra $\mathcal{S}_q$ is finite-dimensional.

Proof. First, we show the following claim.

CLAIM A. *For $1 \le i, j_1, \dots, j_l \le m-1$, we have*

$$E_i F_{j_1} \cdots F_{j_l} = \sum_{k=1}^{m-1} a_k E_k + b,$$

where $a_k \in \mathcal{S}_q$ and $b \in \mathcal{S}_q^- \mathcal{S}_q^0$.

We prove this claim by induction on l. When $l = 1$, we have

$$E_i F_{j_1} = \begin{cases} F_{j_1} E_i + \sum_{\lambda \in \Lambda} \eta_i^\lambda & \text{if } i = j_1, \\ F_{j_1} E_i & \text{otherwise.} \end{cases}$$

Since $\eta_i^\lambda \in \mathcal{S}_q^- \mathcal{S}_q^0 \mathcal{S}_q^+$, we obtain the claim. When $l \ge 2$, we have

$$E_i F_{j_1} \cdots F_{j_l} = \begin{cases} F_{j_1} E_i F_{j_2} \cdots F_{j_l} + \big(\sum_{\lambda \in \Lambda} \eta_i^\lambda\big) F_{j_2} \cdots F_{j_l} & \text{if } i = j_1, \\ F_{j_1} E_i F_{j_2} \cdots F_{j_l} & \text{otherwise.} \end{cases}$$

Note that $\eta_i^\lambda \in \mathcal{S}_q^- \mathcal{S}_q^0 \mathcal{S}_q^+$ and $\deg(\eta_i^\lambda) = 0$. Applying the induction hypothesis to the right-hand side of this formula, we obtain the claim.

For any $u \in \mathcal{S}_q$, we have $u = u \cdot 1 = \sum_{\lambda \in \Lambda} u \cdot 1_\lambda$. Thus, in order to prove the first assertion of the proposition, it suffices to show the following claim.

CLAIM B. *We have $u \cdot 1_\lambda \in \mathcal{S}_q^- \mathcal{S}_q^+ \cdot 1_\lambda$ for any $u \in \mathcal{S}_q$ and $\lambda \in \Lambda$.*

Indeed, this claim implies that $u \in \mathcal{S}_q^- \mathcal{S}_q^0 \mathcal{S}_q^+$ for any $u \in \mathcal{S}_q$ by relation (2.1.3). Hence, we show Claim B by backward induction on Λ with respect to order $\ge$. By Claim A combined with relations (2.1.1) and (2.1.3)–(2.1.6), for any $u \in \mathcal{S}_q$ and $\lambda \in \Lambda$, we have

$$u \cdot 1_\lambda = \sum_{k=1}^{m-1} a_k E_k 1_\lambda + b \cdot 1_\lambda \quad (a_k \in \mathcal{S}_q, b \in \mathcal{S}_q^-). \tag{2.6.1}$$

Clearly, $b \cdot 1_\lambda \in \mathcal{S}_q^- \mathcal{S}_q^+ \cdot 1_\lambda$. On the other hand, we have $a_k E_k 1_\lambda = a_k 1_{\lambda+\alpha_k} E_k$ by (2.1.3), where we set $1_{\lambda+\alpha_k} = 0$ if $\lambda + \alpha_k \notin \Lambda$.

First, we assume that λ is a maximal element of Λ. Then, for any $k = 1,\dots,m-1$, we have $\lambda+\alpha_k \notin \Lambda$ since $\lambda+\alpha_k \geq \lambda$ in P and λ is maximal in Λ. Thus, we have $1_{\lambda+\alpha_k} = 0$ for $k = 1,\dots,m-1$. In this case, we have $u\cdot 1_\lambda = b\cdot 1_\lambda \in \mathcal{S}_q^- \mathcal{S}_q^+ \cdot 1_\lambda$.

Next, we assume that λ is not maximal in Λ and that $\lambda+\alpha_k \in \Lambda$. In this case, by the induction hypothesis, we have $a_k 1_{\lambda+\alpha_k} \in \mathcal{S}_q^- \mathcal{S}_q^+ \cdot 1_{\lambda+\alpha_k}$. Thus, we have $a_k 1_{\lambda+\alpha_k} E_k = a_k E_k 1_\lambda \in \mathcal{S}_q^- \mathcal{S}_q^+ \cdot 1_\lambda$. Combined with (2.6.1), we obtain Claim B; thus, the first assertion of the proposition is proved.

Recall that $\mathcal{S}_q^0$ is the subalgebra of $\mathcal{S}_q$ generated by $\{1_\lambda \mid \lambda \in \Lambda\}$ and that $\{1_\lambda \neq 0 \mid \lambda \in \Lambda\}$ is a set of pairwise orthogonal idempotents. Thus, $\{1_\lambda \neq 0 \mid \lambda \in \Lambda\}$ gives a $\mathcal{K}$-basis of $\mathcal{S}_q^0$.

On the other hand, the set $\{E_{i_1}E_{i_2}\cdots E_{i_l} \mid 1 \leq i_1,\dots,i_l \leq m-1, l \geq 0\}$ gives a spanning set of $\mathcal{S}_q^+$ over $\mathcal{K}$. Since

$$\begin{aligned} E_{i_1}\cdots E_{i_l} &= \sum_{\lambda\in\Lambda} (E_{i_1}\cdots E_{i_l} 1_\lambda) \\ &= \sum_{\lambda\in\Lambda} (1_{\lambda+\alpha_{i_1}+\cdots+\alpha_{i_l}} E_{i_1}\cdots E_{i_l}), \end{aligned}$$

we have $E_{i_1}\cdots E_{i_l} = 0$ if the integer l is sufficient large. This implies that $\mathcal{S}_q^+$ is finitely generated over $\mathcal{K}$. Similarly, we see that $\mathcal{S}_q^-$ is finitely generated over $\mathcal{K}$. Combined with the triangular decomposition, we conclude that $\mathcal{S}_q$ is finite-dimensional. ∎

The next result follows from the proof of Proposition 2.6.

Corollary 2.7. $\{1_\lambda \neq 0 \mid \lambda \in \Lambda\}$ *gives a* $\mathcal{K}$*-basis of* $\mathcal{S}_q^0$.

2.8.

For each $\lambda \in \Lambda$, we define the following subspaces of $\mathcal{S}_q$:

$$\begin{aligned} \mathcal{S}_q(\geq\lambda) &= \{x 1_\mu y \mid x \in \mathcal{S}_q^-, y \in \mathcal{S}_q^+, \mu \in \Lambda \text{ such that } \mu \geq \lambda\}, \\ \mathcal{S}_q(>\lambda) &= \{x 1_\mu y \mid x \in \mathcal{S}_q^-, y \in \mathcal{S}_q^+, \mu \in \Lambda \text{ such that } \mu > \lambda\}. \end{aligned}$$

By using the triangular decomposition and the defining relations of $\mathcal{S}_q$, one can easily check the following lemma.

Lemma 2.9. *For* $\lambda \in \Lambda$*, both* $\mathcal{S}_q(\geq\lambda)$ *and* $\mathcal{S}_q(>\lambda)$ *are two-sided ideals of* $\mathcal{S}_q$.

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2.10.

Thanks to Lemma 2.9, for $\lambda \in \Lambda$, $\mathcal{S}_q(\geq \lambda)/\mathcal{S}_q(> \lambda)$ turns out to be an $(\mathcal{S}_q, \mathcal{S}_q)$-bimodule by multiplications. In general, it happens that $\mathcal{S}_q(\geq \lambda) = \mathcal{S}_q(> \lambda)$. So, we take a subset $\Lambda^+ = \{\lambda \in \Lambda \mid \mathcal{S}_q(\geq \lambda) \neq \mathcal{S}_q(> \lambda)\}$ of Λ. It is clear that

(2.10.1) $$\lambda \in \Lambda^+ \quad \text{if and only if } 1_\lambda \notin \mathcal{S}_q(> \lambda).$$

For $\lambda \in \Lambda^+$, we define a subspace $\Delta(\lambda)$ of $\mathcal{S}_q(\geq \lambda)/\mathcal{S}_q(> \lambda)$ by

$$\Delta(\lambda) = \mathcal{S}_q^- \cdot 1_\lambda + \mathcal{S}_q(> \lambda).$$

Note that $E_k 1_\lambda = 1_{\lambda+\alpha_k} E_k \in \mathcal{S}_q(> \lambda)$ for $k = 1, \dots, m-1$. Together with the triangular decomposition, $\Delta(\lambda)$ turns out to be a left $\mathcal{S}_q$-submodule of $\mathcal{S}_q(\geq \lambda)/\mathcal{S}_q(> \lambda)$. Similarly, we can define a right $\mathcal{S}_q$-submodule $\Delta^\sharp(\lambda)$ of $\mathcal{S}_q(\geq \lambda)/\mathcal{S}_q(> \lambda)$ by

$$\Delta^\sharp(\lambda) = 1_\lambda \cdot \mathcal{S}_q^+ + \mathcal{S}_q(> \lambda).$$

For $x \in \mathcal{S}_q^-, y \in \mathcal{S}_q^+$, we denote the coset of $\mathcal{S}_q(\geq \lambda)/\mathcal{S}_q(> \lambda)$ containing $x1_\lambda y$ by $\overline{x1_\lambda y}$. Then, we denote an element of $\Delta(\lambda)$ (resp., $\Delta^\sharp(\lambda)$) by $\overline{x1_\lambda}$ $(x \in \mathcal{S}_q^-)$ (resp., $\overline{1_\lambda y}$ $(y \in \mathcal{S}_q^+)$). It is clear that $\Delta(\lambda) = \mathcal{S}_q \cdot \overline{1_\lambda}$ and $\Delta^\sharp(\lambda) = \overline{1_\lambda} \cdot \mathcal{S}_q$. We can check the following lemma immediately from the definitions.

LEMMA 2.11. *For $\lambda \in \Lambda^+$, there exists a surjective homomorphism of $(\mathcal{S}_q, \mathcal{S}_q)$-bimodules*

$$\Delta(\lambda) \otimes_{\mathcal{K}} \Delta^\sharp(\lambda) \to \mathcal{S}_q(\geq \lambda)/\mathcal{S}_q(> \lambda)$$

such that $\overline{x1_\lambda} \otimes \overline{1_\lambda y} \mapsto \overline{x1_\lambda y}$ for $x \in \mathcal{S}_q^-, y \in \mathcal{S}_q^+$.

2.12.

As will be shown later, if the surjection in Lemma 2.11 gives an isomorphism for any $\lambda \in \Lambda^+$ and if $\mathcal{S}_q$ has a certain involution ι, $\mathcal{S}_q$ turns out to be a quasi-hereditary cellular algebra, and $\Delta(\lambda)$ $(\lambda \in \Lambda^+)$ is a left cell (standard) module of $\mathcal{S}_q$. In such a case, we can apply a general theory of (quasi-hereditary) cellular algebras. However, in general, we do not know whether $\Delta(\lambda) \otimes_{\mathcal{K}} \Delta^\sharp(\lambda)$ is isomorphic to $\mathcal{S}_q(\geq \lambda)/\mathcal{S}_q(> \lambda)$. (In fact, it happens that $\Delta(\lambda) \otimes_{\mathcal{K}} \Delta^\sharp(\lambda)$ is not isomorphic to $\mathcal{S}_q(\geq \lambda)/\mathcal{S}_q(> \lambda)$; see Appendix C.) And we do not know whether $\mathcal{S}_q$ has such an involution. Nevertheless, we develop a certain representation theory of $\mathcal{S}_q$ which is almost similar to the theory of standardly based algebras in the sense of [DR1], and also similar to the theory of cellular algebras (see, e.g., [GL], [M, Chapter 2]).

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2.13.

For $y \in \mathcal{S}_q^+, x \in \mathcal{S}_q^-$, and $\lambda \in \Lambda^+$, we have $1_\lambda yx1_\lambda = 1_\lambda 1_{\lambda+\alpha} yx$ if $\deg(yx) = \alpha$. Thus, we have $1_\lambda yx 1_\lambda = 0$ if $\deg(yx) = \alpha \neq 0$. On the other hand, if $\deg(yx) = 0$, we can write

$$(2.13.1)\quad 1_\lambda yx1_\lambda = r_0 1_\lambda + \sum_{\substack{Y \in \mathcal{S}_q^+, X \in \mathcal{S}_q^- \\ \deg(Y) = -\deg(X) \neq 0}} r_{XY} 1_\lambda XY 1_\lambda \quad (r_0, r_{XY} \in \mathcal{K})$$

by investigating the degrees through the triangular decomposition. These imply, for $y \in \mathcal{S}_q^+$, $x \in \mathcal{S}_q^-$, and $\lambda \in \Lambda^+$, that we have

$$1_\lambda yx1_\lambda \equiv r_{yx} 1_\lambda \mod \mathcal{S}_q(> \lambda) \quad (r_{yx} \in \mathcal{K}).$$

By using this formula, for $\lambda \in \Lambda^+$, we can define a bilinear form $\langle\,,\rangle : \Delta^\sharp(\lambda) \times \Delta(\lambda) \to \mathcal{K}$ such that

$$(2.13.2)\quad \langle \overline{1_\lambda y}, \overline{x1_\lambda} \rangle 1_\lambda \equiv 1_\lambda yx1_\lambda \mod \mathcal{S}_q(> \lambda) \quad \text{for } y \in \mathcal{S}_q^+, x \in \mathcal{S}_q^-.$$

For $\alpha \in Q^+$, put

$$\Upsilon_\alpha = \big\{ (i_1, i_2, \dots, i_k) \,\big|\, 1 \leq i_1, i_2, \dots, i_k \leq m-1 \\ \text{such that } \alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_k} = \alpha \big\}.$$

From the definition, for $(i_1, \dots, i_k) \in \Upsilon_\alpha, (j_1, \dots, j_l) \in \Upsilon_\beta$ $(\alpha, \beta \in Q^+)$, we have

$$(2.13.3)\quad \langle \overline{1_\lambda E_{i_1} \cdots E_{i_k}}, \overline{F_{j_1} \cdots F_{j_l} 1_\lambda} \rangle = 0 \quad \text{if } \alpha \neq \beta.$$

We have the following lemma.

LEMMA 2.14. *For $\lambda \in \Lambda^+$, we have the following formulas.*

(i) $\langle \overline{y} \cdot u, \overline{x} \rangle = \langle \overline{y}, u \cdot \overline{x} \rangle$ *for* $\overline{x} \in \Delta(\lambda), \overline{y} \in \Delta^\sharp(\lambda), u \in \mathcal{S}_q$.

(ii) $(F_{i_1} \cdots F_{i_k} 1_\lambda E_{j_1} \cdots E_{j_l}) \cdot \overline{x} = \langle \overline{1_\lambda E_{j_1} \cdots E_{j_l}}, \overline{x} \rangle \overline{F_{i_1} \cdots F_{i_k} 1_\lambda}$ *for* $\overline{x} \in \Delta(\lambda)$ *and* $F_{i_1} \cdots F_{i_k} 1_\lambda E_{j_1} \cdots E_{j_l} \in \mathcal{S}_q(\geq \lambda)$.

Proof. (i) For $x \in \mathcal{S}_q^-$, $y \in \mathcal{S}_q^+$, and $u \in \mathcal{S}_q$, we have

$$\langle \overline{1_\lambda y} \cdot u, \overline{x1_\lambda} \rangle 1_\lambda \equiv 1_\lambda yux1_\lambda \\ \equiv \langle \overline{1_\lambda y}, u \cdot \overline{x1_\lambda} \rangle 1_\lambda \mod \mathcal{S}_q(> \lambda).$$

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(ii) For $x \in \mathcal{S}_q^-$ and $F_{i_1} \cdots F_{i_k} 1_\lambda E_{j_1} \cdots E_{j_l} \in \mathcal{S}_q(\geq \lambda)$, we have

$$\begin{aligned}(F_{i_1} \cdots F_{i_k} 1_\lambda E_{j_1} \cdots E_{j_l}) \cdot \overline{x 1_\lambda} &= \overline{F_{i_1} \cdots F_{i_k} (1_\lambda E_{j_1} \cdots E_{j_l} x 1_\lambda)} \\ &= \overline{F_{i_1} \cdots F_{i_k} \langle \overline{1_\lambda E_{j_1} \cdots E_{j_l}}, \overline{x 1_\lambda} \rangle 1_\lambda} \\ &= \langle \overline{1_\lambda E_{j_1} \cdots E_{j_l}}, \overline{x 1_\lambda} \rangle \overline{F_{i_1} \cdots F_{i_k} 1_\lambda}. \qquad \square\end{aligned}$$

2.15.
For $\lambda \in \Lambda^+$, let

$$\operatorname{rad} \Delta(\lambda) = \{ \overline{x} \in \Delta(\lambda) \mid \langle \overline{y}, \overline{x} \rangle = 0 \text{ for any } \overline{y} \in \Delta^\sharp(\lambda) \},$$
$$\operatorname{rad} \Delta^\sharp(\lambda) = \{ \overline{y} \in \Delta^\sharp(\lambda) \mid \langle \overline{y}, \overline{x} \rangle = 0 \text{ for any } \overline{x} \in \Delta(\lambda) \}.$$

By Lemma 2.14(i), $\operatorname{rad} \Delta(\lambda)$ (resp., $\operatorname{rad} \Delta^\sharp(\lambda)$) is a left (resp., right) $\mathcal{S}_q$-submodule of $\Delta(\lambda)$ (resp., $\Delta^\sharp(\lambda)$). Put $L(\lambda) = \Delta(\lambda)/\operatorname{rad} \Delta(\lambda)$, and put $L^\sharp(\lambda) = \Delta^\sharp(\lambda)/\operatorname{rad} \Delta^\sharp(\lambda)$ We have the following theorem. This theorem is proven in a way similar to proofs in the general theory of standardly based algebras or cellular algebras (see [DR1], [GL], [M, Chapter 2]).

THEOREM 2.16. *We have the following.*

(i) *For $\lambda \in \Lambda^+$, $\operatorname{rad} \Delta(\lambda)$ (resp., $\operatorname{rad} \Delta^\sharp(\lambda)$) is a unique proper maximal $\mathcal{S}_q$-submodule of $\Delta(\lambda)$ (resp., $\Delta^\sharp(\lambda)$). Thus, $L(\lambda)$ (resp., $L^\sharp(\lambda)$) is a left (resp., right) absolutely simple $\mathcal{S}_q$-module.*

(ii) *For $\lambda, \mu \in \Lambda^+$, if $L(\mu)$ (resp., $L^\sharp(\mu)$) is a composition factor of $\Delta(\lambda)$ (resp., $\Delta^\sharp(\lambda)$), we have $\lambda \geq \mu$. Thus, $L(\lambda) \cong L(\mu)$ (resp., $L^\sharp(\lambda) \cong L^\sharp(\mu)$) if and only if $\lambda = \mu$. Moreover, the multiplicity of $L(\lambda)$ (resp., $L^\sharp(\lambda)$) in $\Delta(\lambda)$ (resp., $\Delta^\sharp(\lambda)$) is equal to 1.*

(iii) *$\{L(\lambda) \mid \lambda \in \Lambda^+\}$ (resp., $\{L^\sharp(\lambda) \mid \lambda \in \Lambda^+\}$) gives a complete set of nonisomorphic left (resp., right) simple $\mathcal{S}_q$-modules.*

(iv) *$\mathcal{S}_q$ is semisimple if and only if $\Delta(\lambda) \cong L(\lambda)$ and $\Delta^\sharp(\lambda) \cong L^\sharp(\lambda)$ for any $\lambda \in \Lambda^+$.*

Proof. We prove only the assertions for left $\mathcal{S}_q$-modules. The proof is similar for right $\mathcal{S}_q$-modules. (i) It is clear that $\langle \overline{1_\lambda}, \overline{1_\lambda} \rangle = 1$. Thus, we have $\Delta(\lambda) \supsetneqq \operatorname{rad} \Delta(\lambda)$. For $\overline{x} \in \Delta(\lambda) \setminus \operatorname{rad} \Delta(\lambda)$, there exists an element $\overline{y} \in \Delta^\sharp(\lambda)$ such that $\langle \overline{y}, \overline{x} \rangle \neq 0$. Since $\langle\, , \,\rangle$ is a bilinear form over a field $\mathcal{K}$, we can suppose that $\langle \overline{y}, \overline{x} \rangle = 1$. Let

$$\overline{y} = \sum_{\substack{(j_1, \ldots, j_l) \in \Upsilon_\alpha \\ \alpha \in Q^+}} r_{(j_1, \ldots, j_l)} \overline{1_\lambda E_{j_1} \cdots E_{j_l}}.$$

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For $\overline{t} = \overline{F_{i_1} \cdots F_{i_k} 1_\lambda} \in \Delta(\lambda)$, put

$$y_{\overline{t}} = F_{i_1} \cdots F_{i_k} 1_\lambda \Big(\sum_{\substack{(j_1,\dots,j_l) \in \Upsilon_\alpha \\ \alpha \in Q^+}} r_{(j_1,\dots,j_l)} E_{j_l} \cdots E_{j_1} \Big) \in \mathcal{S}_q.$$

Then, we have

$$\begin{aligned} y_{\overline{t}} \cdot \overline{x} &= \sum r_{(j_1,\dots,j_l)} (F_{i_1} \cdots F_{i_k} 1_\lambda E_{j_l} \cdots E_{j_1}) \cdot \overline{x} \\ &= \sum r_{(j_1,\dots,j_l)} \langle \overline{1_\lambda E_{j_1} \cdots E_{j_l}}, \overline{x} \rangle \overline{F_{i_1} \cdots F_{i_k} 1_\lambda} \quad (\because \text{Lemma 2.14(ii)}) \\ &= \langle \overline{y}, \overline{x} \rangle \overline{F_{i_1} \cdots F_{i_k} 1_\lambda} \\ &= \overline{F_{i_1} \cdots F_{i_k} 1_\lambda}. \end{aligned}$$

This implies that $\Delta(\lambda)$ is generated by $\overline{x}$ as an $\mathcal{S}_q$-module. Since this fact holds for any $\overline{x} \in \Delta(\lambda) \setminus \operatorname{rad} \Delta(\lambda)$, $\operatorname{rad} \Delta(\lambda)$ is the unique maximal proper submodule of $\Delta(\lambda)$.

(ii) For $\lambda \in \Lambda^+$, we have $1_\lambda \cdot L(\lambda) \neq 0$ since $\overline{1_\lambda} \notin \operatorname{rad} \Delta(\lambda)$. On the other hand, one sees easily that $1_\mu \cdot \Delta(\lambda) = 0$ for any $\mu \in \Lambda$ such that $\mu \not\leq \lambda$. Thus, if $L(\mu)$ is a composition factor of $\Delta(\lambda)$, we have $1_\mu \cdot \Delta(\lambda) \neq 0$ and $\mu \leq \lambda$. Moreover, one sees that $1_\lambda \cdot \operatorname{rad} \Delta(\lambda) = 0$. (Note that $\overline{1_\lambda} \notin \operatorname{rad} \Delta(\lambda)$.) This implies that $L(\lambda)$ does not appear in $\operatorname{rad} \Delta(\lambda)$ as a composition factor. Thus, we have (ii).

(iii) Let $\{\lambda_{\langle 1 \rangle}, \lambda_{\langle 2 \rangle}, \dots, \lambda_{\langle z \rangle}\}$ be such that $i < j$ if $\lambda_{\langle i \rangle} > \lambda_{\langle j \rangle}$. Put $\mathcal{S}_q(\lambda_{\langle i \rangle}) = \sum_{j \leq i} \mathcal{S}_q^- 1_{\lambda_{\langle j \rangle}} \mathcal{S}_q^+$; then $\mathcal{S}_q(\lambda_{\langle i \rangle})$ turns out to be a two-sided ideal of $\mathcal{S}_q$. Thus, we have the following filtration of two-sided ideals:

(2.16.1) $\quad \mathcal{S}_q = \mathcal{S}_q(\lambda_{\langle z \rangle}) \supset \mathcal{S}_q(\lambda_{\langle z-1 \rangle}) \supset \cdots \supset \mathcal{S}_q(\lambda_{\langle 1 \rangle}) \supset \mathcal{S}_q(\lambda_{\langle 0 \rangle}) = 0.$

One sees easily that $\mathcal{S}_q(\lambda_{\langle i \rangle}) / \mathcal{S}_q(\lambda_{\langle i-1 \rangle}) \cong \mathcal{S}_q(\geq \lambda_{\langle i \rangle}) / \mathcal{S}_q(> \lambda_{\langle i \rangle})$ as $(\mathcal{S}_q, \mathcal{S}_q)$-bimodules for $\lambda_{\langle i \rangle} \in \Lambda$. Moreover, one can check that

$$\begin{aligned} \mathcal{S}_q(\lambda_{\langle i \rangle}) \neq \mathcal{S}_q(\lambda_{\langle i-1 \rangle}) \quad &\text{if and only if } 1_{\lambda_{\langle i \rangle}} \notin \mathcal{S}_q(> \lambda_{\langle i \rangle}) \\ &\text{if and only if } \lambda_{\langle i \rangle} \in \Lambda^+. \end{aligned}$$

Let $\Lambda^+ = \{\lambda_{\langle c_1 \rangle}, \dots, \lambda_{\langle c_{z'} \rangle}\}$ such that $i < j$ if $c_i < c_j$. Then, we have the following filtration of two-sided ideals:

(2.16.2) $\quad \mathcal{S}_q = \mathcal{S}_q(\lambda_{\langle c_{z'} \rangle}) \supsetneqq \mathcal{S}_q(\lambda_{\langle c_{z'-1} \rangle}) \supsetneqq \cdots \supsetneqq \mathcal{S}_q(\lambda_{\langle c_1 \rangle}) \supsetneqq \mathcal{S}_q(\lambda_{\langle c_0 \rangle}) = 0,$

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such that $\mathcal{S}_q(\lambda_{\langle c_i\rangle})/\mathcal{S}_q(\lambda_{\langle c_{i-1}\rangle}) \cong \mathcal{S}_q(\geq \lambda_{\langle c_i\rangle})/\mathcal{S}_q(> \lambda_{\langle c_i\rangle})$ as $(\mathcal{S}_q, \mathcal{S}_q)$-bimodules.

By the filtration of $\mathcal{S}_q$ in (2.16.2) and the surjective homomorphism of $(\mathcal{S}_q, \mathcal{S}_q)$-bimodules $\Delta(\lambda) \otimes_{\mathcal{K}} \Delta^\sharp(\lambda) \to \mathcal{S}_q(\geq \lambda)/\mathcal{S}_q(> \lambda)$ for $\lambda \in \Lambda^+$ in Lemma 2.11, any composition factor of $\mathcal{S}_q$ is a composition factor of $\Delta(\lambda)$ for some $\lambda \in \Lambda^+$. Thus, it is enough to show that any composition factor of $\Delta(\lambda)$ ($\lambda \in \Lambda^+$) is isomorphic to $L(\mu)$ for some $\mu \in \Lambda^+$. We prove it by using an induction on Λ^+.

Let $\lambda \in \Lambda^+$ be a minimal element with respect to order $\geq$. We take $\overline{x} = \sum r_{(i_1,\dots,i_k)} \overline{F_{i_1} \cdots F_{i_k} 1_\lambda} \in \operatorname{rad} \Delta(\lambda)$. Put $x = \sum r_{(i_1,\dots,i_k)} F_{i_1} \cdots F_{i_k} 1_\lambda \in \mathcal{S}_q(\geq \lambda)$. For $\mu \in \Lambda^+$ such that $\lambda \neq \mu$, we have $\mathcal{S}_q(\geq \mu) \cdot x \in \mathcal{S}_q(\geq \lambda) \cap \mathcal{S}_q(\geq \mu) \subset \mathcal{S}_q(> \lambda)$ since both of $\mathcal{S}_q(\geq \lambda)$ and $\mathcal{S}_q(\geq \mu)$ are two-sided ideals of $\mathcal{S}_q$ and λ is a minimal element of Λ^+. This implies that $\mathcal{S}_q(\geq \mu) \cdot \overline{x} = 0$ for any $\mu \in \Lambda^+$ such that $\mu \neq \lambda$. On the other hand, for any $F_{y_1} \cdots F_{y_b} 1_\lambda E_{x_1} \cdots E_{x_a} \in \mathcal{S}_q(\geq \lambda)$, we have

$$(F_{y_1} \cdots F_{y_b} 1_\lambda E_{x_1} \cdots E_{x_a}) \cdot \overline{x} = \langle \overline{1_\lambda E_{x_1} \cdots E_{x_a}}, \overline{x} \rangle \overline{F_{y_1} \cdots F_{y_b} 1_\lambda} = 0,$$

where the first equation follows Lemma 2.14(ii), and the second equation follows $\overline{x} \in \operatorname{rad} \Delta(\lambda)$. This implies that $\mathcal{S}_q(\geq \lambda) \cdot \overline{x} = 0$. Together with the above arguments, we have $\mathcal{S}_q \cdot \overline{x} = 0$. In particular, we have $\overline{x} = 1 \cdot \overline{x} = 0$. This means that $\operatorname{rad} \Delta(\lambda) = 0$, and we have $\Delta(\lambda) = L(\lambda)$.

Next, we suppose that $\lambda \in \Lambda^+$ is not minimal. Put

$$\mathcal{S}_q(\not< \lambda) = \sum_{\substack{\mu \in \Lambda \\ \mu \not< \lambda}} \mathcal{S}_q^- 1_\mu \mathcal{S}_q^+ \qquad \text{and} \qquad \mathcal{S}_q(\not\leq \lambda) = \sum_{\substack{\mu \in \Lambda \\ \mu \not\leq \lambda}} \mathcal{S}_q^- 1_\mu \mathcal{S}_q^+.$$

One sees that $\mathcal{S}_q(\not< \lambda)$ and $\mathcal{S}_q(\not\leq \lambda)$ are two-sided ideals of $\mathcal{S}_q$. It is clear that $\mathcal{S}_q(\not\leq \lambda) \cdot \Delta(\lambda) = 0$. Moreover, we see that $\mathcal{S}_q(\geq \lambda) \cdot \operatorname{rad} \Delta(\lambda) = 0$ in a way similar to the above arguments. Thus, we have $\mathcal{S}_q(\not< \lambda) \cdot \operatorname{rad} \Delta(\lambda) = 0$. This implies that the action of $\mathcal{S}_q$ on $\operatorname{rad} \Delta(\lambda)$ induces the action of $\mathcal{S}_q/\mathcal{S}_q(\not< \lambda)$ on $\operatorname{rad} \Delta(\lambda)$. Thus, any composition factor of $\operatorname{rad} \Delta(\lambda)$ is a composition factor of $\mathcal{S}_q/\mathcal{S}_q(\not< \lambda)$. Moreover, we can take a total order of Λ such that $\mathcal{S}_q(\not< \lambda) = \mathcal{S}_q(\lambda_{\langle k\rangle})$ for some k and that $\lambda_{\langle j\rangle} < \lambda$ for any $j = k+1, \dots, z$. Thus, by Lemma 2.11, any composition factor of $\mathcal{S}_q/\mathcal{S}_q(\not< \lambda)$ is a composition factor of $\Delta(\mu)$ for some $\mu \in \Lambda^+$ such that $\mu < \lambda$. By the induction hypothesis, we see that any composition factor of $\Delta(\mu)$ such that $\mu < \lambda$ is isomorphic to $L(\nu)$ for some $\nu \in \Lambda^+$. It follows that any composition factor of $\operatorname{rad} \Delta(\lambda)$

is isomorphic to $L(\nu)$ for some $\nu \in \Lambda^+$. Since $\Delta(\lambda)/\operatorname{rad}\Delta(\lambda) = L(\lambda)$, we obtain (iii).

(iv) Suppose that $\mathcal{S}_q$ is semisimple; then $L(\lambda)$ and $L(\mu)$ ($\lambda \neq \mu \in \Lambda^+$) belong to different blocks of $\mathcal{S}_q$. On the other hand, $\Delta(\lambda)$ is indecomposable since $\Delta(\lambda)$ has a unique top. Thus, all the composition factors of $\Delta(\lambda)$ belong to the same block. This means that $\Delta(\lambda)$ has only $L(\lambda)$ as a composition factor, and we have $\Delta(\lambda) = L(\lambda)$ for any $\lambda \in \Lambda^+$ by (ii). We have $\Delta^\sharp(\lambda) = L^\sharp(\lambda)$ for any $\lambda \in \Lambda^+$ in a similar way.

Next, we suppose that $\Delta(\lambda) \cong L(\lambda)$ and $\Delta^\sharp(\lambda) \cong L^\sharp(\lambda)$ for any $\lambda \in \Lambda^+$. Then, the surjective homomorphism of $(\mathcal{S}_q, \mathcal{S}_q)$-bimodules $\Delta(\lambda) \otimes_{\mathcal{K}} \Delta^\sharp(\lambda) \to \mathcal{S}_q(\geq \lambda)/\mathcal{S}_q(> \lambda)$ in Lemma 2.11 must be an isomorphism. Thus, the filtration (2.16.2) implies that

$$\dim_{\mathcal{K}} \mathcal{S}_q = \sum_{\lambda \in \Lambda^+} \left(\dim_{\mathcal{K}} \Delta(\lambda)\right)^2.$$

($\dim_{\mathcal{K}} L(\lambda) = \dim_{\mathcal{K}} L^\sharp(\lambda)$ will be proved in Lemma 3.8.) This implies that $\mathcal{S}_q$ is semisimple. ∎

2.17.

Let $\mathcal{S}_q^{\geq 0}$ (resp., $\mathcal{S}_q^{\leq 0}$) be the subalgebra of $\mathcal{S}_q$ generated by $\mathcal{S}_q^+$ (resp., $\mathcal{S}_q^-$) and $\mathcal{S}_q^0$. Thus, $\mathcal{S}_q^{\geq 0}$ (resp., $\mathcal{S}_q^{\leq 0}$) is generated by E_i (resp., F_i) for $i = 1, \ldots, m-1$ and 1_λ for $\lambda \in \Lambda$. For $\lambda \in \Lambda$ such that $1_\lambda \neq 0$ in $\mathcal{S}_q$, let $\theta_\lambda = \mathcal{K} v_\lambda$ be the 1-dimensional vector space with a basis v_λ. We define a left action of $\mathcal{S}_q^{\geq 0}$ on θ_λ by

$$1_\mu \cdot v_\lambda = \delta_{\lambda\mu} v_\lambda, \qquad E_i \cdot v_\lambda = 0 \quad \text{for } \mu \in \Lambda \text{ and } i = 1, \ldots, m-1.$$

One can check that this action is well defined for $\lambda \in \Lambda$ such that $1_\lambda \neq 0$. Similarly, we define a right action of $\mathcal{S}_q^{\leq 0}$ on θ_λ by

$$v_\lambda \cdot 1_\mu = \delta_{\lambda\mu} v_\lambda, \qquad v_\lambda \cdot F_i = 0 \quad \text{for } \mu \in \Lambda \text{ and } i = 1, \ldots, m-1.$$

We have the following theorem. (A similar theorem for cyclotomic q-Schur algebras has been obtained by [DR2]. The proof given here is similar to the proof given in [DR2].)

THEOREM 2.18. *We have the following.*

(i) $\{1_\lambda \mid \lambda \in \Lambda$ *such that* $1_\lambda \neq 0\}$ *is a complete set of primitive idempotents in* $\mathcal{S}_q^{\geq 0}$ *and* $\mathcal{S}_q^{\leq 0}$.

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(ii) *$\{\theta_\lambda \mid \lambda \in \Lambda$ such that $1_\lambda \neq 0\}$ is a complete set of nonisomorphic simple left $\mathcal{S}_q^{\geq 0}$-modules and of nonisomorphic simple right $\mathcal{S}_q^{\leq 0}$-modules.*

(iii) *For $\lambda \in \Lambda$ such that $1_\lambda \neq 0$, we have the following isomorphism of left $\mathcal{S}_q$-modules:*

$$\mathcal{S}_q \otimes_{\mathcal{S}_q^{\geq 0}} \theta_\lambda \cong \begin{cases} \Delta(\lambda) & \text{if } \lambda \in \Lambda^+, \\ 0 & \text{otherwise.} \end{cases}$$

(iv) *For $\lambda \in \Lambda$ such that $1_\lambda \neq 0$, we have the following isomorphism of right $\mathcal{S}_q$-modules:*

$$\theta_\lambda \otimes_{\mathcal{S}_q^{\leq 0}} \mathcal{S}_q \cong \begin{cases} \Delta^\sharp(\lambda) & \text{if } \lambda \in \Lambda^+, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We show the theorem only for $\mathcal{S}_q^{\geq 0}$. The proof is similar for $\mathcal{S}_q^{\leq 0}$. Note that

$$1_\lambda E_{i_1} \cdots E_{i_k} 1_\lambda = 1_\lambda 1_{\lambda+\alpha_{i_1}+\cdots+\alpha_{i_k}} E_{i_1} \cdots E_{i_k} = 0$$

for $1 \leq i_1, \dots, i_k \leq m-1, k \geq 1$. Thus, for $\lambda \in \Lambda$ such that $1_\lambda \neq 0$, we have $1_\lambda \mathcal{S}_q^{\geq 0} 1_\lambda = \mathcal{K} 1_\lambda$. This implies that 1_λ is a primitive idempotent of $\mathcal{S}_q^{\geq 0}$ since $1_\lambda \mathcal{S}_q^{\geq 0} 1_\lambda \cong \operatorname{End}_{\mathcal{S}_q^{\geq 0}}(\mathcal{S}_q^{\geq 0} 1_\lambda)$, and $\dim_{\mathcal{K}} \operatorname{End}_{\mathcal{S}_q^{\geq 0}}(\mathcal{S}_q^{\geq 0} 1_\lambda) \geq 2$ if 1_λ is not primitive. Moreover, we have $1 = \sum_{\lambda \in \Lambda} 1_\lambda$, and so $\{1_\lambda \mid \lambda \in \Lambda$ such that $1_\lambda \neq 0\}$ is the complete set of primitive idempotents in $\mathcal{S}_q^{\geq 0}$. This proves (i). In addition, we deduce that, for $\lambda \in \Lambda$ such that $1_\lambda \neq 0$, $\Theta_\lambda = \mathcal{S}_q^{\geq 0} 1_\lambda$ is a principal indecomposable $\mathcal{S}_q^{\geq 0}$-module. By investigating the degrees, $\mathcal{S}_q^{\geq 0} \cdot (x 1_\lambda)$ is a proper $\mathcal{S}_q^{\geq 0}$-submodule of Θ_λ for any $x \in \mathcal{S}_q^+$ such that $x \neq 1$. This implies that $\Theta_\lambda / \operatorname{Rad} \Theta_\lambda \cong \theta_\lambda$.

Next, we prove (iii). If $\lambda \notin \Lambda^+$, we can write $1_\lambda = \sum_{x \in \mathcal{S}_q^-, y \in \mathcal{S}_q^+, \mu > \lambda} r_{x,y,\mu} \times x 1_\mu y$ in $\mathcal{S}_q$. Thus, we have

$$1 \otimes \theta_\lambda = \sum_{\nu \in \Lambda} 1_\nu \otimes \theta_\lambda = 1_\lambda \otimes \theta_\lambda = \sum_{x \in \mathcal{S}_q^-, y \in \mathcal{S}_q^+, \mu > \lambda} r_{x,y,\mu} x 1_\mu y \otimes \theta_\lambda = 0.$$

This implies that $\mathcal{S}_q \otimes_{\mathcal{S}_q^{\geq 0}} \theta_\lambda = \mathcal{S}_q \cdot (1 \otimes \theta_\lambda) = 0$. Hence, we suppose that $\lambda \in \Lambda^+$. Note that $\Delta(\lambda)$ is generated by an element $\overline{1_\lambda}$ and that $\mathcal{S}_q \otimes_{\mathcal{S}_q^{\geq 0}} \theta_\lambda$ is generated by $1 \otimes v_\lambda$ as $\mathcal{S}_q$-modules. We define a map $f_\lambda : \Delta(\lambda) \to \mathcal{S}_q \otimes_{\mathcal{S}_q^{\geq 0}} \theta_\lambda$ by $\overline{u \cdot 1_\lambda} \mapsto u \otimes v_\lambda$ for $u \in \mathcal{S}_q$. One can check that f_λ gives a well-defined

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$\mathcal{S}_q$-homomorphism. On the other hand, we define the map $\widetilde{g}_\lambda : \mathcal{S}_q \times \theta_\lambda \to \Delta(\lambda)$ by $(u, rv_\lambda) \mapsto r\overline{u \cdot 1_\lambda}$ for $u \in \mathcal{S}_q, r \in \mathcal{K}$. One can check that $\widetilde{g}_\lambda$ gives a well-defined $\mathcal{S}_q^{\geq 0}$-balanced map. Thus, $\widetilde{g}_\lambda$ induces an $\mathcal{S}_q$-homomorphism $g_\lambda : \mathcal{S}_q \otimes_{\mathcal{S}_q^{\geq 0}} \theta_\lambda \to \Delta(\lambda)$ such that $u \otimes v_\lambda \mapsto \overline{u \cdot 1_\lambda}$. Thus, (iii) is proved. ∎

2.19.

Given that $\eta_\Lambda = \{\eta_i^\lambda \mid 1 \leq i \leq m-1, \lambda \in \Lambda\}$, where $\eta_i^\lambda \in \widetilde{\mathcal{S}}_q^- \widetilde{\mathcal{S}}_q^+ 1_\lambda$ is such that $\deg(\eta_i^\lambda) = 0$, we take $\eta_i \in \widetilde{U}_q^- \widetilde{U}_q^0 \widetilde{U}_q^+$ $(1 \leq i \leq m-1)$ such that $\widetilde{\Psi}(\eta_i) = \sum_{\lambda \in \Lambda} \eta_i^\lambda$, and we put $\eta = (\eta_1, \ldots, \eta_{m-1})$.

On the other hand, given that $\eta = (\eta_1, \ldots, \eta_{m-1})$, where $\eta_i \in \widetilde{U}_q^- \widetilde{U}_q^0 \widetilde{U}_q^+$ is such that $\deg(\eta_i) = 0$, and given that $\Lambda \subset P$, set $\eta_i^\lambda = \widetilde{\Psi}(\eta_i) 1_\lambda$ $(1 \leq i \leq m-1, \lambda \in \Lambda)$, and put $\eta_\Lambda = \{\eta_i^\lambda \mid 1 \leq i \leq m-1, \lambda \in \Lambda\}$.

Under this correspondence, we have the following theorem.

THEOREM 2.20. *We have the following.*

(i) *Let $\mathcal{S}_q^{\eta_\Lambda}$-mod be the category of finite-dimensional left $\mathcal{S}_q^{\eta_\Lambda}$-modules. Then $\mathcal{S}_q^{\eta_\Lambda}$-mod is a full subcategory of $\mathcal{O}^\eta$. In particular, when we regard a $\mathcal{S}_q^{\eta_\Lambda}$-module as a $\widetilde{U}_q$-module through the surjection $\Psi : \widetilde{U}_q \to \mathcal{S}_q^{\eta_\Lambda}$, $\Delta(\lambda)$ $(\lambda \in \Lambda^+)$ is a highest-weight module, and $L(\lambda)$ $(\lambda \in \Lambda^+)$ is a simple highest-weight module with a highest weight λ associated to η.*

(ii) *For each $M \in \mathcal{O}^\eta$, if the set of weights λ such that $M_\lambda \neq 0$ is contained in Λ, then we have $M \in \mathcal{S}_q^{\eta_\Lambda}$-mod, where we regard the $\mathcal{S}_q^{\eta_\Lambda}$-mod as a full subcategory of $\mathcal{O}^\eta$ by (i). In particular, any simple object of $\mathcal{O}^\eta$ is obtained as in Theorem 2.16 through the quotient algebra $\mathcal{S}_q^{\eta_\Lambda}$ for a suitable $\Lambda \subset P_{\geq 0}$, where the choice of Λ depends on the simple object of $\mathcal{O}^\eta$.*

(iii) *$\mathcal{O}^\eta$ is a full subcategory of $\widehat{\mathcal{O}}^\eta_{\mathrm{tri}}$.*

Proof. (i) is clear through the surjection $\Phi : \widetilde{U}_q \to \mathcal{S}_q^{\eta_\Lambda}$ and by the definitions of $\Delta(\lambda)$ and $L(\lambda)$.

We prove (ii). For $M \in \mathcal{O}^\eta$, put $\Lambda_M = \{\lambda \in P_{\geq 0} \mid M_\lambda \neq 0\}$. (Note that $M_\lambda = 0$ unless $\lambda \in P_{\geq 0}$ by condition (e) in the definition of $\mathcal{O}^\eta$.) Since the dimension of M is finite, Λ_M is a finite set. We take a finite subset Λ of $P_{\geq 0}$ such that $\Lambda_M \subset \Lambda$. Then, we can define an action of $\mathcal{S}_q^{\eta_\Lambda}$ on M as follows:

$$E_i \cdot m = e_i \cdot m \quad \text{for } 1 \leq i \leq m-1, m \in M,$$

$$F_i \cdot m = f_i \cdot m \quad \text{for } 1 \leq i \leq m-1, m \in M,$$

$$1_\lambda \cdot m = \delta_{\lambda\mu} m \quad \text{for } \lambda \in \Lambda, m \in M_\mu.$$

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One can check that this action is well defined by using the defining relations of $\widetilde{U}_q$ and the definition of $\mathcal{O}^\eta$. We denote this $\mathcal{S}_q^{\eta_\Lambda}$-module by M^Λ. When we regard M^Λ as a $\widetilde{U}_q$-module through the surjection Ψ, M^Λ coincides with M. This implies that $M \in \mathcal{S}_q^{\eta_\Lambda}$-mod. Now, the last assertion of (ii) is clear.

Since $\mathcal{S}_q^{\eta_\Lambda}$ has the triangular decomposition compatible with that of $\widetilde{U}_q$, (iii) follows from (ii). ∎

2.21.

We define an algebra antiautomorphism $\iota : \widetilde{\mathcal{S}}_q \to \widetilde{\mathcal{S}}_q$ by $\iota(E_i) = F_i$, $\iota(F_i) = E_i$, $\iota(1_\lambda) = 1_\lambda$, and $\iota(\tau_i^\lambda) = \tau_i^\lambda$ for $i = 1, \dots, m-1$ and $\lambda \in \Lambda$. We can easily check that ι is well defined. We consider the following conditions.

(C-1) $$\iota(\eta_i^\lambda) = \eta_i^\lambda \quad \text{for any } i = 1, \dots, m-1 \text{ and } \lambda \in \Lambda.$$

(C-2) $$\Delta(\lambda) \otimes_{\mathcal{K}} \Delta^\sharp(\lambda) \cong \mathcal{S}_q(\geq \lambda)\mathcal{S}_q(> \lambda) \quad \text{as } (\mathcal{S}_q, \mathcal{S}_q)\text{-bimodules for any } \lambda \in \Lambda^+.$$

Thanks to condition (C-1), ι induces a well-defined algebra antiautomorphism on $\mathcal{S}_q$. In view of Lemma 2.11, condition (C-2) is equivalent to the following condition:

(C′-2) $$\sum_{x \in \mathcal{S}_q^-, y \in \mathcal{S}_q^+} r_{xy} x 1_\lambda y \in \mathcal{S}_q(> \lambda) \Rightarrow \sum_{x \in \mathcal{S}_q^-, y \in \mathcal{S}_q^+} r_{xy} \overline{x 1_\lambda} \otimes \overline{1_\lambda y} = 0 \in \Delta(\lambda) \otimes_{\mathcal{K}} \Delta^\sharp(\lambda).$$

It is clear that

$$u \in \mathcal{S}_q(\geq \lambda) \quad \text{if and only if } \iota(u) \in \mathcal{S}_q(\geq \lambda),$$
$$u \in \mathcal{S}_q(> \lambda) \quad \text{if and only if } \iota(u) \in \mathcal{S}_q(> \lambda).$$

This implies that $\Delta(\lambda) \ni \overline{x} \mapsto \overline{\iota(x)} \in \Delta^\sharp(\lambda)$ gives an isomorphism of $\mathcal{K}$-vector spaces. We consider the filtration of $\mathcal{S}_q$ in (2.16.2). Recall that

$$\mathcal{S}_q(\lambda_{\langle c_i \rangle})/\mathcal{S}_q(\lambda_{\langle c_{i-1} \rangle}) \cong \mathcal{S}_q(\geq \lambda_{\langle c_i \rangle})/\mathcal{S}_q(> \lambda_{\langle c_i \rangle}) \quad \text{as } (\mathcal{S}_q, \mathcal{S}_q)\text{-bimodules.}$$

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Under conditions (C-1) and (C-2), we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{S}_q(\lambda_{\langle c_i\rangle})/\mathcal{S}_q(\lambda_{\langle c_{i-1}\rangle}) & \cong & \Delta(\lambda_{\langle c_i\rangle})\otimes_{\mathcal{K}}\Delta^{\sharp}(\lambda_{\langle c_i\rangle}) \\ \downarrow \iota & & \downarrow \overline{x}\otimes\overline{y}\mapsto\overline{\iota(y)}\otimes\overline{\iota(x)} \\ \mathcal{S}_q(\lambda_{\langle c_i\rangle})/\mathcal{S}_q(\lambda_{\langle c_{i-1}\rangle}) & \cong & \Delta(\lambda_{\langle c_i\rangle})\otimes_{\mathcal{K}}\Delta^{\sharp}(\lambda_{\langle c_i\rangle}) \end{array}$$

This implies that $\mathcal{S}_q(\lambda_{\langle c_i\rangle})/\mathcal{S}_q(\lambda_{\langle c_{i-1}\rangle})$ is a cell ideal of $\mathcal{S}_q/\mathcal{S}_q(\lambda_{\langle c_{i-1}\rangle})$ in the sense of [KX]. Thus, $\mathcal{S}_q$ turns out to be a cellular algebra (see [KX, Definition 3.2]), and $\Delta(\lambda)$ $(\lambda\in\Lambda^+)$ gives a cell module of $\mathcal{S}_q$. Moreover, we already know that $\{L(\lambda)\mid\lambda\in\Lambda^+\}$ gives a complete set of nonisomorphic simple $\mathcal{S}_q$-modules. Thus, we have the following theorem.

THEOREM 2.22. *If $\mathcal{S}_q$ satisfies conditions (C-1) and (C-2), then $\mathcal{S}_q$ is a quasi-hereditary cellular algebra.*

§3. Specialization to an arbitrary ring

In this section, we define an $\mathcal{A}$-form ${}_{\mathcal{A}}\mathcal{S}_q$ of $\mathcal{S}_q$, and we consider a specialization ${}_R\mathcal{S}_q$ of ${}_{\mathcal{A}}\mathcal{S}_q$ to an arbitrary ring R. Recall that $\mathcal{S}_q$ depends on the choice of $\{\eta_i^{\lambda}\mid 1\le i\le m-1,\lambda\in\Lambda\}$. In this section, we will assume some conditions on this set so that, in the case where R is a field, we obtain the properties of ${}_R\mathcal{S}_q$ which are similar to those obtained in the preceding section and are compatible with the case where $R=\mathcal{K}$.

3.1.

Put $E_i^{(k)}=E_i^k/[k]!$, $F_i^{(k)}=F_i^k/[k]!$. Let ${}_{\mathcal{A}}\mathcal{S}_q$ be the $\mathcal{A}$-subalgebra of $\mathcal{S}_q$ generated by $E_i^{(k)}$, $F_i^{(k)}$ $(1\le i\le m-1,k\ge 1)$, and 1_λ $(\lambda\in\Lambda)$. Note that, by Lemma 2.3, we have $\Psi({}_{\mathcal{A}}\widetilde{U}_q)={}_{\mathcal{A}}\mathcal{S}_q$.

Let ${}_{\mathcal{A}}\mathcal{S}_q^+$ (resp., ${}_{\mathcal{A}}\mathcal{S}_q^-$) be the $\mathcal{A}$-subalgebra of ${}_{\mathcal{A}}\mathcal{S}_q$ generated by $E_i^{(k)}$ (resp., $F_i^{(k)}$) for $1\le i\le m-1$, $k\ge 0$, and let ${}_{\mathcal{A}}\mathcal{S}_q^0$ be the $\mathcal{A}$-subalgebra of ${}_{\mathcal{A}}\mathcal{S}_q$ generated by 1_λ for $\lambda\in\Lambda$. As shown in Section 2, $\mathcal{S}_q$ has the triangular decomposition $\mathcal{S}_q=\mathcal{S}_q^-\mathcal{S}_q^0\mathcal{S}_q^+$ over $\mathcal{K}$. However, it may happen that such relations break over $\mathcal{A}$. Hence, in order for the triangular decomposition to hold over $\mathcal{A}$, we impose the following condition:

((A-1)) $\quad E_i^{(k)}F_i^{(l)}\in {}_{\mathcal{A}}\mathcal{S}_q^-\,{}_{\mathcal{A}}\mathcal{S}_q^0\,{}_{\mathcal{A}}\mathcal{S}_q^+\quad$ for $1\le i\le m-1, k,l\ge 1$.

Indeed, under this assumption, we can prove the following proposition by replacing E_i, F_j $(1 \le i, j \le m-1)$ with the divided powers $E_i^{(k)}$, $F_j^{(l)}$ $(1 \le i, j \le m-1, k, l \ge 1)$ in the proof of Proposition 2.6.

PROPOSITION 3.2. *Suppose that condition (A-1) holds. Then* ${}_{\mathcal{A}}\mathcal{S}_q$ *has a triangular decomposition*

$${}_{\mathcal{A}}\mathcal{S}_q = {}_{\mathcal{A}}\mathcal{S}_q^- \, {}_{\mathcal{A}}\mathcal{S}_q^0 \, {}_{\mathcal{A}}\mathcal{S}_q^+.$$

Moreover, ${}_{\mathcal{A}}\mathcal{S}_q$ *is finitely generated over* $\mathcal{A}$*.*

In the rest of this section, we always assume condition (A-1).

3.3.

Let R be an arbitrary ring, and we take $\xi_0, \xi_1, \dots, \xi_r \in R$, where ξ_0 is invertible in R. We regard R as an $\mathcal{A}$-module by the homomorphism of rings $\pi : \mathcal{A} \to R$ such that $q \mapsto \xi_0, \gamma_i \mapsto \xi_i$ $(1 \le i \le r)$. Then, we obtain the specialized algebra $R \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{S}_q$ of ${}_{\mathcal{A}}\mathcal{S}_q$ through the homomorphism π. We denote it by ${}_R\mathcal{S}_q$, and we denote $1 \otimes x \in R \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{S}_q$ simply by x if it does not cause any confusion. Let ${}_R\mathcal{S}_q^+$ (resp., ${}_R\mathcal{S}_q^-$) be the R-subalgebra of ${}_R\mathcal{S}_q$ generated by $1 \otimes E_i^{(k)}$ (resp., $1 \otimes F_i^{(k)}$) for $1 \le i \le m-1$, $k \ge 0$, and let ${}_R\mathcal{S}_q^0$ be the R-subalgebra of ${}_R\mathcal{S}_q$ generated by $1 \otimes 1_\lambda$ for $\lambda \in \Lambda$. By Proposition 3.2, we have the triangular decomposition

$${}_R\mathcal{S}_q = {}_R\mathcal{S}_q^- \, {}_R\mathcal{S}_q^0 \, {}_R\mathcal{S}_q^+.$$

Thanks to the triangular decomposition, we have the following results which are similar to the case over $\mathcal{K}$. For $\lambda \in \Lambda$, let

$${}_R\mathcal{S}_q(\ge \lambda) = \{x 1_\mu y \mid x \in {}_R\mathcal{S}_q^-, y \in {}_R\mathcal{S}_q^+, \mu \in \Lambda \text{ such that } \mu \ge \lambda\},$$
$${}_R\mathcal{S}_q(> \lambda) = \{x 1_\mu y \mid x \in {}_R\mathcal{S}_q^-, y \in {}_R\mathcal{S}_q^+, \mu \in \Lambda \text{ such that } \mu > \lambda\}.$$

Then, ${}_R\mathcal{S}_q(\ge \lambda)$ and ${}_R\mathcal{S}_q(> \lambda)$ are two-sided ideals of ${}_R\mathcal{S}_q$. Put

$${}_R\Lambda^+ = \{\lambda \in \Lambda \mid {}_R\mathcal{S}_q(\ge \lambda) \ne {}_R\mathcal{S}_q(> \lambda)\} = \{\lambda \in \Lambda \mid 1_\lambda \notin {}_R\mathcal{S}_q(> \lambda)\}.$$

For $\lambda \in {}_R\Lambda^+$, we define a left (resp., right) ${}_R\mathcal{S}_q$-submodule ${}_R\Delta(\lambda)$ (resp., ${}_R\Delta^\sharp(\lambda)$) of ${}_R\mathcal{S}_q(\ge \lambda)/{}_R\mathcal{S}_q(> \lambda)$ by

$${}_R\Delta(\lambda) = {}_R\mathcal{S}_q^- \cdot 1_\lambda + {}_R\mathcal{S}_q(> \lambda), \qquad {}_R\Delta^\sharp(\lambda) = 1_\lambda \cdot {}_R\mathcal{S}_q^+ + {}_R\mathcal{S}_q(> \lambda).$$

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Let ${}_R\mathcal{S}_q^{\geq 0}$ (resp., ${}_R\mathcal{S}_q^{\leq 0}$) be the subalgebra of ${}_R\mathcal{S}_q$ generated by ${}_R\mathcal{S}_q^{+}$ (resp., ${}_R\mathcal{S}_q^{-}$) and ${}_R\mathcal{S}_q^{0}$. For $\lambda \in \Lambda$ such that $1_\lambda \neq 0$ in ${}_R\mathcal{S}_q$, let $\theta_\lambda = Rv_\lambda$ be the free R-module with a basis v_λ. We define the left action of ${}_R\mathcal{S}_q^{\geq 0}$ on θ_λ by

$$1_\mu \cdot v_\lambda = \delta_{\lambda\mu} v_\lambda, \qquad E_i^{(k)} \cdot v_\lambda = 0 \quad \text{for } \mu \in \Lambda, i = 1, \dots, m-1 \text{ and } k \geq 1.$$

Similarly, we define a right action of ${}_R\mathcal{S}_q^{\leq 0}$ on θ_λ by

$$v_\lambda \cdot 1_\mu = \delta_{\lambda\mu} v_\lambda, \qquad v_\lambda \cdot F_i^{(k)} = 0 \quad \text{for } \mu \in \Lambda, i = 1, \dots, m-1 \text{ and } k \geq 1.$$

We have the following theorem which is shown in a way similar to the proof of Theorem 2.18.

THEOREM 3.4. *We have the following.*

(i) $\{1_\lambda \mid \lambda \in \Lambda$ *such that* $1_\lambda \neq 0\}$ *is a complete set of primitive idempotents in* ${}_R\mathcal{S}_q^{\geq 0}$ *and* ${}_R\mathcal{S}_q^{\leq 0}$.

(ii) $\{\theta_\lambda \mid \lambda \in \Lambda$ *such that* $1_\lambda \neq 0\}$ *is a complete set of nonisomorphic simple left* ${}_R\mathcal{S}_q^{\geq 0}$*-modules and of nonisomorphic simple right* ${}_R\mathcal{S}_q^{\leq 0}$*-modules.*

(iii) *For* $\lambda \in \Lambda$ *such that* $1_\lambda \neq 0$*, we have the following isomorphism of left (resp., right)* ${}_R\mathcal{S}_q$*-modules:*

$${}_R\mathcal{S}_q \otimes_{{}_R\mathcal{S}_q^{\geq 0}} \theta_\lambda \cong \begin{cases} {}_R\Delta(\lambda) & \text{if } \lambda \in {}_R\Lambda^+, \\ 0 & \text{otherwise;} \end{cases}$$

$$\theta_\lambda \otimes_{{}_R\mathcal{S}_q^{\leq 0}} {}_R\mathcal{S}_q \cong \begin{cases} {}_R\Delta^\sharp(\lambda) & \text{if } \lambda \in {}_R\Lambda^+, \\ 0 & \text{otherwise.} \end{cases}$$

3.5.

For $\lambda \in {}_R\Lambda^+$, we can define a bilinear form $\langle\, ,\, \rangle : {}_R\Delta^\sharp(\lambda) \times {}_R\Delta(\lambda) \to R$ such that

$$\langle \overline{1_\lambda y}, \overline{x 1_\lambda} \rangle 1_\lambda \equiv 1_\lambda y x 1_\lambda \mod {}_R\mathcal{S}_q(> \lambda) \quad \text{for } x \in {}_R\mathcal{S}_q^{-}, y \in {}_R\mathcal{S}_q^{+}.$$

Put $\operatorname{rad} {}_R\Delta(\lambda) = \{\overline{x} \in {}_R\Delta(\lambda) \mid \langle \overline{y}, \overline{x} \rangle = 0 \text{ for any } \overline{y} \in {}_R\Delta^\sharp(\lambda)\}$, and put ${}_R L(\lambda) = {}_R\Delta(\lambda) / \operatorname{rad} {}_R\Delta(\lambda)$. Similarly, put $\operatorname{rad} {}_R\Delta^\sharp(\lambda) = \{\overline{y} \in {}_R\Delta^\sharp(\lambda) \mid \langle \overline{y}, \overline{x} \rangle = 0 \text{ for any } \overline{x} \in {}_R\Delta(\lambda)\}$, and put ${}_R L^\sharp(\lambda) = {}_R\Delta^\sharp(\lambda) / \operatorname{rad} {}_R\Delta^\sharp(\lambda)$. Then, one can prove the following theorem by replacing E_i, F_j $(1 \leq i, j \leq m-1)$ with the divided powers $E_i^{(k)}$, $F_j^{(l)}$ $(1 \leq i, j \leq m-1, k, l \geq 1)$ in the proof of Theorem 2.16.

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THEOREM 3.6. *Suppose that R is a field. Then we have the following.*

(i) *For $\lambda \in {}_R\Lambda^+$, $\mathrm{rad}\, {}_R\Delta(\lambda)$, (resp., $\mathrm{rad}\, {}_R\Delta^\sharp(\lambda)$) is a unique proper maximal submodule of ${}_R\Delta(\lambda)$ (resp., ${}_R\Delta^\sharp(\lambda)$). Thus, ${}_RL(\lambda)$ (resp., ${}_RL^\sharp(\lambda)$) is an absolutely simple left (resp., right) ${}_R\mathcal{S}_q$-module.*

(ii) *For $\lambda, \mu \in {}_R\Lambda^+$, if ${}_RL(\mu)$ (resp., ${}_RL^\sharp(\mu)$) is a composition factor of ${}_R\Delta(\lambda)$ (resp., ${}_R\Delta^\sharp(\lambda)$), we have $\lambda \geq \mu$. Thus, ${}_RL(\lambda) \cong {}_RL(\mu)$ if and only if $\lambda = \mu$. Moreover, the multiplicity of ${}_RL(\lambda)$ (resp., ${}_RL^\sharp(\lambda)$) in ${}_R\Delta(\lambda)$ (resp., ${}_R\Delta^\sharp(\lambda)$) is equal to 1.*

(iii) *$\{{}_RL(\lambda) \mid \lambda \in {}_R\Lambda^+\}$ (resp., $\{{}_RL^\sharp(\lambda) \mid \lambda \in {}_R\Lambda^+\}$) gives a complete set of nonisomorphic left (resp., right) simple ${}_R\mathcal{S}_q$-modules.*

(iv) *${}_R\mathcal{S}_q$ is semisimple if and only if ${}_R\Delta(\lambda) \cong {}_RL(\lambda)$ and ${}_R\Delta^\sharp(\lambda) \cong {}_RL^\sharp(\lambda)$ for any $\lambda \in \Lambda^+$.*

3.7.

Throughout the rest of this section, we assume that R is a field. Since $\mathrm{rad}\, {}_R\Delta^\sharp(\lambda) \times \mathrm{rad}\, {}_R\Delta(\lambda)$ is included in the kernel of the bilinear form $\langle\, ,\, \rangle :$ ${}_R\Delta^\sharp(\lambda) \times {}_R\Delta(\lambda) \to R$, $\langle\, ,\, \rangle$ induces a bilinear form on ${}_RL^\sharp(\lambda) \times {}_RL(\lambda)$. Clearly, this bilinear form is nondegenerate on ${}_RL^\sharp(\lambda) \times {}_RL(\lambda)$. We regard $\mathrm{Hom}_R({}_RL^\sharp(\lambda), R)$ as a left ${}_R\mathcal{S}_q$-module by the standard way. Thanks to the associativity of the bilinear form $\langle\, ,\, \rangle$ (see Lemma 2.14(i)), the R-homomorphism $G : {}_RL(\lambda) \to \mathrm{Hom}_R({}_RL^\sharp(\lambda), R)$ given by $\overline{x} \mapsto \langle -, \overline{x} \rangle$ turns out to be an ${}_R\mathcal{S}_q$-homomorphism. Since $\langle\, ,\, \rangle$ is nondegenerate on ${}_RL^\sharp(\lambda) \times {}_RL(\lambda)$, the homomorphism G is nonzero. Hence, G is an isomorphism of left ${}_R\mathcal{S}_q$-modules since both of ${}_RL(\lambda)$ and $\mathrm{Hom}_R({}_RL^\sharp(\lambda), R)$ are simple. Thus, we have the following lemma (a similar argument holds for ${}_RL^\sharp(\lambda)$).

LEMMA 3.8. *Suppose that R is a field. For $\lambda \in {}_R\Lambda^+$, we have the following isomorphisms:*

(i) *${}_RL(\lambda) \cong \mathrm{Hom}_R({}_RL^\sharp(\lambda), R)$ as left ${}_R\mathcal{S}_q$-modules,*

(ii) *${}_RL^\sharp(\lambda) \cong \mathrm{Hom}_R({}_RL(\lambda), R)$ as right ${}_R\mathcal{S}_q$-modules.*

In particular, we have $\dim_R {}_RL(\lambda) = \dim_R {}_RL^\sharp(\lambda)$.

3.9.

For $\lambda \in {}_R\Lambda^+$, let ${}_RP(\lambda)$ be the projective cover of ${}_RL(\lambda)$. For $\lambda, \mu \in {}_R\Lambda^+$, we denote the multiplicity of ${}_RL(\mu)$ in the composition series of ${}_RP(\lambda)$ by $[{}_RP(\lambda) : {}_RL(\mu)]$. Similarly, we denote the multiplicity of ${}_RL(\mu)$ (resp., ${}_RL^\sharp(\mu)$) in the composition series of ${}_R\Delta(\lambda)$ (resp., ${}_R\Delta^\sharp(\lambda)$) by $[{}_R\Delta(\lambda) : {}_RL(\mu)]$ (resp., $[{}_R\Delta^\sharp(\lambda) : {}_RL^\sharp(\lambda)]$). We have the following relation concerning these multiplicities.

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LEMMA 3.10. *Suppose that R is a field. For $\lambda, \mu \in {}_R\Lambda^+$, we have*

$$[{}_RP(\lambda) : {}_RL(\mu)] \leqq \sum_{\nu \in {}_R\Lambda^+} [{}_R\Delta(\nu) : {}_RL(\mu)][{}_R\Delta^\sharp(\nu) : {}_RL^\sharp(\lambda)].$$

Proof. In the proof, we omit the subscript R as we always consider the objects over R. Let $\Lambda^+ = \{\lambda_{\langle 1\rangle}, \dots, \lambda_{\langle z\rangle}\}$ be such that $i < j$ if $\lambda_{\langle i\rangle} > \lambda_{\langle j\rangle}$. Then we have the following filtrations of two-sided ideals:

$$\mathcal{S}_q = \mathcal{S}_q(\lambda_{\langle z\rangle}) \supsetneqq \mathcal{S}_q(\lambda_{\langle z-1\rangle}) \supsetneqq \cdots \supsetneqq \mathcal{S}_q(\lambda_{\langle 1\rangle}) \supsetneqq \mathcal{S}_q(\lambda_{\langle 0\rangle}) = 0, \tag{3.10.1}$$

such that $\mathcal{S}_q(\lambda_{\langle i\rangle})/\mathcal{S}_q(\lambda_{\langle i-1\rangle}) \cong \mathcal{S}_q(\geq \lambda_{\langle i\rangle})/\mathcal{S}_q(> \lambda_{\langle i\rangle})$ as $\mathcal{S}_q$-bimodules. Since $P(\lambda)$ is a left projective $\mathcal{S}_q$-module, the filtration (3.10.1) implies the existence of left $\mathcal{S}_q$-modules M_j $(0 \leq j \leq z)$ which give the following filtration:

$$P(\lambda) = M_z \supset M_{z-1} \supset \cdots \supset M_1 \supset M_0 = 0,$$

such that $M_i/M_{i-1} \cong (\mathcal{S}_q(\geq \lambda_{\langle i\rangle})/\mathcal{S}_q(> \lambda_{\langle i\rangle})) \otimes_{\mathcal{S}_q} P(\lambda)$. This implies that

$$[P(\lambda) : L(\mu)] = \sum_{\nu \in \Lambda^+} \big[\big(\mathcal{S}_q(\geq \nu)/\mathcal{S}_q(> \nu)\big) \otimes_{\mathcal{S}_q} P(\lambda) : L(\mu)\big]. \tag{3.10.2}$$

Since there exists a surjection $\Delta(\nu) \otimes_R \Delta^\sharp(\nu) \to \mathcal{S}_q(\geq \nu)/\mathcal{S}_q(> \nu)$ of $\mathcal{S}_q$-bimodules, (3.10.2) implies that

$$[P(\lambda) : L(\mu)] \leqq \sum_{\nu \in \Lambda^+} [\Delta(\nu) \otimes_R \Delta^\sharp(\nu) \otimes_{\mathcal{S}_q} P(\lambda) : L(\mu)].$$

Thus, it suffices to prove the following equality:

$$[\Delta(\nu) \otimes_R \Delta^\sharp(\nu) \otimes_{\mathcal{S}_q} P(\lambda) : L(\mu)] = [\Delta(\nu) : L(\mu)][\Delta^\sharp(\nu) : L^\sharp(\lambda)].$$

Since

$$[\Delta(\nu) \otimes_R \Delta^\sharp(\nu) \otimes_{\mathcal{S}_q} P(\lambda) : L(\mu)] = [\Delta(\nu) : L(\mu)] \cdot \dim_R\big(\Delta^\sharp(\nu) \otimes_{\mathcal{S}_q} P(\lambda)\big),$$

it is enough to show that $\dim_R(\Delta^\sharp(\nu) \otimes_{R\mathcal{S}_q} P(\lambda)) = [\Delta^\sharp(\nu) : L^\sharp(\lambda)]$. By the general theory of finite-dimensional algebras over a field, we have

$$\begin{aligned}
&\dim_R\big(\Delta^\sharp(\nu) \otimes_{\mathcal{S}_q} P(\lambda)\big) \\
&\quad = \dim_R\big(\mathrm{Hom}_R((\Delta^\sharp(\nu) \otimes_{\mathcal{S}_q} P(\lambda), R))\big)
\end{aligned}$$

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$$
\begin{aligned}
&= \dim_R\bigl(\mathrm{Hom}_{\mathcal{S}_q}(P(\lambda), \mathrm{Hom}_R(\Delta^\sharp(\nu), R))\bigr)\\
&= \bigl[\mathrm{Hom}_R\bigl(\Delta^\sharp(\nu), R\bigr) : L(\lambda)\bigr]\\
&= \bigl[\mathrm{Hom}_R\bigl(\Delta^\sharp(\nu), R\bigr) : \mathrm{Hom}_R\bigl(L^\sharp(\lambda), R\bigr)\bigr] \quad \text{(Lemma 3.8)}\\
&= \bigl[\Delta^\sharp(\nu) : L^\sharp(\lambda)\bigr].
\end{aligned}
$$

The lemma is then proved. □

3.11.

For $\lambda \in {}_R\Lambda^+$, ${}_R\Delta(\lambda)$ is an indecomposable ${}_R\mathcal{S}_q$-module since ${}_R\Delta(\lambda)$ has a unique top. Thus, all the composition factors of ${}_R\Delta(\lambda)$ belong to the same block of ${}_R\mathcal{S}_q$.

For $\lambda, \mu \in {}_R\Lambda^+$, we denote by $\lambda \sim \mu$ if there exists a sequence $\lambda = \lambda_0, \lambda_1, \dots, \lambda_k = \mu$ ($\lambda_i \in {}_R\Lambda^+$) such that ${}_R\Delta(\lambda_{i-1})$ and ${}_R\Delta(\lambda_i)$ ($1 \le i \le k$) have a common composition factor. Clearly, $\sim$ gives an equivalent relation on ${}_R\Lambda^+$, and ${}_R\Delta(\lambda)$ and ${}_R\Delta(\mu)$ belong to the same block if $\lambda \sim \mu$. If ${}_R\mathcal{S}_q$ satisfies condition (C-1), one can prove that the converse is also true. To prove it, we prepare the following lemma.

Lemma 3.12. *Suppose that R is a field. If ${}_R\mathcal{S}_q$ satisfies condition (C-1), we have*

$$[{}_R\Delta(\lambda) : {}_RL(\mu)] = [{}_R\Delta^\sharp(\lambda) : {}_RL^\sharp(\mu)].$$

Proof. Thanks to (C-1), we can define an isomorphism of R-modules $\iota : {}_R\Delta(\lambda) \to {}_R\Delta^\sharp(\lambda)$ via $\overline{x} \mapsto \overline{\iota(x)}$. For $y \in {}_R\mathcal{S}_q^+$ and $x \in {}_R\mathcal{S}_q^-$, we have

$$\langle \overline{1_\lambda y}, \overline{x 1_\lambda}\rangle 1_\lambda \equiv 1_\lambda y x 1_\lambda = 1_\lambda \iota(x)\iota(y) 1_\lambda \equiv \langle \overline{1_\lambda \iota(x)}, \overline{\iota(y) 1_\lambda}\rangle 1_\lambda \mod {}_R\mathcal{S}_q(> \lambda).$$

Thus, we have $\langle \overline{y}, \overline{x}\rangle = \langle \overline{\iota(x)}, \overline{\iota(y)}\rangle$ for any $\overline{x} \in {}_R\Delta(\lambda)$ and $\overline{y} \in \Delta^\sharp(\lambda)$. This implies that $\mathrm{rad}\, {}_R\Delta^\sharp(\lambda) = \{\overline{\iota(x)} \mid \overline{x} \in \mathrm{rad}\, {}_R\Delta(\lambda)\}$. Therefore, $\iota : {}_R\Delta(\lambda) \to {}_R\Delta^\sharp(\lambda)$ induces an R-isomorphism ${}_RL(\lambda) \to {}_RL^\sharp(\lambda)$. Let ${}_R\Delta(\lambda) = M_0 \supsetneqq M_1 \supsetneqq \cdots \supsetneqq M_k \supsetneqq 0$ be a composition series of ${}_R\Delta(\lambda)$ such that $M_{i-1}/M_i \cong {}_RL(\mu_i)$. By investigating the action of ${}_R\mathcal{S}_q$, we see that $\iota({}_R\Delta(\lambda)) = \iota(M_0) \supsetneqq \iota(M_1) \supsetneqq \cdots \supsetneqq \iota(M_k) \supsetneqq 0$ gives a composition series of ${}_R\Delta^\sharp(\lambda)$ such that $\iota(M_{i-1})/\iota(M_i) \cong {}_RL^\sharp(\mu_i)$. This implies the lemma. □

We have the following theorem.

Theorem 3.13. *Suppose that R is a field. If ${}_R\mathcal{S}_q$ satisfies condition (C-1), then $\lambda \sim \mu$ if and only if ${}_R\Delta(\lambda)$ and ${}_R\Delta(\mu)$ belong to the same block of ${}_R\mathcal{S}_q$ for $\lambda, \mu \in {}_R\Lambda^+$.*

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Proof. Because we have already seen the "only if" part, we prove the "if" part. Assume that ${}_R\Delta(\lambda)$ and ${}_R\Delta(\mu)$ belong to the same block. Then ${}_RP(\lambda)$ and ${}_RP(\mu)$ belong to the same block. Thus, there exists a sequence $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_k = \mu$ $(\lambda_i \in {}_R\Lambda^+)$ such that ${}_RP(\lambda_{i-1})$ and ${}_RP(\lambda_i)$ $(1 \leq i \leq k)$ have a common composition factor ${}_RL(\mu_i)$. By Lemma 3.10, there exists $\nu_i, \nu_i' \in {}_R\Lambda^+$ $(1 \leq i \leq k)$ such that $[{}_R\Delta(\nu_i) : {}_RL(\mu_i)] \neq 0$, $[{}_R\Delta^\sharp(\nu_i) : {}_RL^\sharp(\lambda_{i-1})] \neq 0$, $[{}_R\Delta(\nu_i') : {}_RL(\mu_i)] \neq 0$, and $[{}_R\Delta^\sharp(\nu_i') : {}_RL^\sharp(\lambda_i)] \neq 0$. Combined with Lemma 3.12, we have

$$\lambda_{i-1} \sim \nu_i \sim \mu_i \sim \nu_i' \sim \lambda_i$$

for each $1 \leq i \leq k$. Thus, we have $\lambda \sim \mu$. □

3.14.

Finally, we consider the following condition:

(A-2) For any $\lambda \in {}_\mathcal{A}\Lambda^+$, ${}_\mathcal{A}\Delta(\lambda)$ is a free $\mathcal{A}$-module, and

$${}_\mathcal{A}\Delta(\lambda) \otimes_\mathcal{A} {}_\mathcal{A}\Delta^\sharp(\lambda) \cong {}_\mathcal{A}\mathcal{S}_q(\geq \lambda)/{}_\mathcal{A}\mathcal{S}_q(> \lambda)$$

as $({}_\mathcal{A}\mathcal{S}_q, {}_\mathcal{A}\mathcal{S}_q)$-bimodules.

We have the following theorem.

THEOREM 3.14. *Suppose that conditions (A-1), (A-2), and (C-1) hold. Then, for an arbitrary ring R and parameters $\xi_0, \xi_1, \ldots, \xi_r \in R$, ${}_R\mathcal{S}_q$ is a cellular algebra with respect to the poset Λ^+. In particular, when R is a field, ${}_R\mathcal{S}_q$ is a quasi-hereditary cellular algebra.*

Proof. Thanks to (C-1), the map ${}_\mathcal{A}\Delta(\lambda) \ni \overline{x} \mapsto \overline{\iota(x)} \in {}_\mathcal{A}\Delta^\sharp(\lambda)$ gives an isomorphism of $\mathcal{A}$-modules. Thus, (A-2) implies that ${}_\mathcal{A}\Delta^\sharp(\lambda)$ is a free $\mathcal{A}$-module. Now, we can prove that ${}_\mathcal{A}\mathcal{S}_q$ is a cellular algebra with respect to the poset ${}_\mathcal{A}\Lambda^+$ in a way similar to that of the case over $\mathcal{K}$ (see Theorem 2.22), and that ${}_\mathcal{A}\Delta(\lambda)$ $(\lambda \in {}_\mathcal{A}\Lambda^+)$ is a (left) cell module of ${}_\mathcal{A}\mathcal{S}_q$. Thus, for any ring R, ${}_R\mathcal{S}_q$ is a cellular algebra with respect to the poset ${}_\mathcal{A}\Lambda^+$, and $R \otimes_\mathcal{A} {}_\mathcal{A}\Delta(\lambda)$ $(\lambda \in {}_\mathcal{A}\Lambda^+)$ is a cell module of ${}_R\mathcal{S}_q$.

From now on, we assume that R is a field. It is clear that $1 \otimes 1_\lambda \in {}_R\mathcal{S}_q(> \lambda)$ if $1_\lambda \in {}_\mathcal{A}\mathcal{S}_q(> \lambda)$. This implies that ${}_R\Lambda^+ \subset {}_\mathcal{A}\Lambda^+$. Since $R \otimes_\mathcal{A} {}_\mathcal{A}\Delta(\lambda)$ has an element $1 \otimes \overline{1_\lambda}$, we have that $\mathrm{rad}(R \otimes_\mathcal{A} {}_\mathcal{A}\Delta(\lambda)) \neq R \otimes_\mathcal{A} {}_\mathcal{A}\Delta(\lambda)$ for any $\lambda \in {}_\mathcal{A}\Lambda^+$. This implies that ${}_R\mathcal{S}_q$ is quasi-hereditary and that the number of isomorphism classes of simple ${}_R\mathcal{S}_q$-modules is equal to ${}_\mathcal{A}\Lambda^+$ by the general theory of cellular algebras. On the other hand, we know that the number of isomorphism classes of simple ${}_R\mathcal{S}_q$-modules is equal to ${}_R\Lambda^+$ by Theorem 3.6. Thus, we have ${}_R\Lambda^+ = {}_\mathcal{A}\Lambda^+$. In particular, we have ${}_\mathcal{A}\Lambda^+ = \Lambda^+$ when $R = \mathcal{K}$. ∎

Remarks 3.16. (i) Let ${}_\mathcal{A}\widetilde{\mathcal{S}}_q^\natural = {}_\mathcal{A}\widetilde{\mathcal{S}}_q^\natural(\Lambda)$ be the $\mathcal{A}$-subalgebra of $\widetilde{\mathcal{S}}_q$ generated by $E_i, F_i, 1_\lambda, \tau_i^\lambda$ for $1 \le i \le m-1, \lambda \in \Lambda$. Clearly, ${}_\mathcal{A}\widetilde{\mathcal{S}}_q^\natural$ is isomorphic to the associative algebra over $\mathcal{A}$ defined by generators $E_i, F_i, 1_\lambda, \tau_i^\lambda$ and defining relations (2.1.1)–(2.1.9). Moreover, ${}_\mathcal{A}\widetilde{\mathcal{S}}_q^\natural$ is a homomorphic image of ${}_\mathcal{A}\widetilde{U}_q^\natural$, where ${}_\mathcal{A}\widetilde{U}_q^\natural$ is the $\mathcal{A}$-subalgebra of $\widetilde{U}_q$ generated by all $e_i, f_i, \tau_i, K_j^\pm, \left[\begin{smallmatrix} K_j;0 \\ t \end{smallmatrix}\right]$. For ${}_\mathcal{A}\widetilde{\mathcal{S}}_q^\natural$, we can take η_Λ, and we can define the quotient algebra ${}_\mathcal{A}\mathcal{S}_q^\natural = {}_\mathcal{A}\mathcal{S}_q^{\natural\eta_\Lambda}$ as the case of $\mathcal{S}_q^{\eta_\Lambda}$. (In this case, condition (A-1) for ${}_\mathcal{A}\mathcal{S}_q^\natural$ to have the triangular decomposition is unnecessary since we do not take a divided power.) For an arbitrary ring R and parameters $\xi_0, \xi_1, \dots, \xi_r$, we take the specialized algebra ${}_R\mathcal{S}_q^\natural = R \otimes_\mathcal{A} {}_\mathcal{A}\mathcal{S}_q^\natural$. Then, for ${}_R\mathcal{S}_q^\natural$, one can apply similar arguments as in the case of ${}_R\mathcal{S}_q$. In particular, results similar to those of Theorems 3.4, 3.6, 3.13, and 3.15 hold for ${}_R\mathcal{S}_q^\natural$. However, ${}_R\mathcal{S}_q^\natural$ is different from ${}_R\mathcal{S}_q$ in general.

(ii) For any Cartan matrix of finite type, one can define the algebra $\widetilde{U}_q$ and its quotient algebra $\mathcal{S}_q$ associated to a given Cartan matrix in a similar way. Namely, for the given Cartan data, we define $\widetilde{U}_q$ by replacing the "commutative relations between e_i and f_i" in the defining relations of the corresponding quantum group with "formal generators τ_i" which commute with Cartan parts. Then we specialize τ_i to various elements η_i^λ to define finite-dimensional quotient algebras $\mathcal{S}_q$ as in Section 2. In this case, we should take a weight lattice P whose rank is equal to the rank of the root lattice, and we take a finite subset Λ of P to define the quotient algebra $\widetilde{\mathcal{S}}_q$ without taking a subset of P such as $P_{\ge 0}$. We should use an argument similar to that in the proof of [Do, Lemma 3.2] instead of Lemma 2.3 in order to prove a statement similar to that in Proposition 2.2. We also remove condition (e) from the definition of $\mathcal{O}^\eta$. Then, we have all statements in Sections 2 and 3 corresponding to the given Cartan matrix.

§4. Review of q-Schur algebras of type A

In this section, we recall the definition of the q-Schur algebra $\mathscr{S}_{n,1}$ of type A and review some known results concerning the presentations of $\mathscr{S}_{n,1}$ given in [DG] and the Borel subalgebras of $\mathscr{S}_{n,1}$. The Borel subalgebras will play

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an important role in Sections 6 and 7 to obtain presentations of cyclotomic q-Schur algebras.

4.1.

Let n, m be positive integers, and let $\Lambda_{n,1}$ be the set of compositions of n with m parts, namely,

$$\Lambda_{n,1} = \{\mu = (\mu_1, \dots, \mu_m) \in \mathbb{Z}_{\geq 0}^m \mid \mu_1 + \cdots + \mu_m = n\}.$$

The set $\Lambda_{n,1}$ depends on the choice of the integer m. However, we fix the integer m for the set $\Lambda_{n,1}$ throughout this article. Hence, we omit m from the notation of the set $\Lambda_{n,1}$. We regard $\Lambda_{n,1}$ as a subset of P by the injective map from $\Lambda_{n,1}$ to P given by $\mu = (\mu_1, \dots, \mu_m) \mapsto \sum_{i=1}^m \mu_i \varepsilon_i$. Thus, for $\mu = (\mu_1, \dots, \mu_m) \in \Lambda_{n,1}$ and α_i $(1 \leq i \leq m-1)$, we have

$$\mu \pm \alpha_i = (\mu_1, \dots, \mu_{i-1}, \mu_i \pm 1, \mu_{i+1} \mp 1, \mu_{i+2}, \dots, \mu_m).$$

For $\mu \in \Lambda_{n,1}$, the diagram of μ is the set $[\mu] = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq j \leq \mu_i, 1 \leq i \leq m\}$, and a μ-tableau is a bijection $\mathfrak{t} : [\mu] \to \{1, 2, \dots, n\}$. Let $\mathfrak{t}^\mu$ be the μ-tableau in which the integers $1, 2, \dots, n$, are attached in the order from left to right and from top to bottom in $[\mu]$. The symmetric group $\mathfrak{S}_n$ acts on the set of μ-tableaux from the right by permuting the integers attached in $[\mu]$. For $\mu, \nu \in \Lambda_{n,1}$, a μ-tableau of type ν is a map $T : [\mu] \to \{1, \dots, m\}$ such that $\nu_i = \sharp\{x \in [\mu] \mid T(x) = i\}$. For μ, ν and μ-tableau $\mathfrak{t}$, let $\nu(\mathfrak{t})$ be a μ-tableau of type ν obtained by replacing each entry i in $\mathfrak{t}$ by k if the i appear in the kth row of $\mathfrak{t}^\nu$.

For $\mu \in \Lambda_{n,1}$, let $\mathfrak{S}_\mu$ be the Young subgroup of $\mathfrak{S}_n$ corresponding to μ, and let $\mathscr{D}_\mu$ be the set of distinguished representatives of right $\mathfrak{S}_\mu$-cosets. For $\mu, \nu \in \Lambda_{n,1}$, $\mathscr{D}_{\mu\nu} = \mathscr{D}_\mu \cap \mathscr{D}_\nu^{-1}$ is the set of distinguished representatives of $\mathfrak{S}_\mu$-$\mathfrak{S}_\nu$ double cosets.

4.2.

Let R be an integral domain, and let q be an invertible element in R. The Iwahori-Hecke algebra ${}_R\mathscr{H}_n$ of the symmetric group $\mathfrak{S}_n$ is the associative algebra over R generated by $T_1, \dots, T_{n-1}$ with the following defining relations:

$$\begin{aligned} &(T_i - q)(T_i + q^{-1}) && (1 \leq i \leq n-1), \\ &T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} && (1 \leq i \leq n-2), \\ &T_i T_j = T_j T_i && (|i-j| \geq 2). \end{aligned}$$

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For $w \in \mathfrak{S}_n$, we denote by $\ell(w)$ the length of w, and we denote by T_w the standard basis of ${}_R\mathscr{H}_n$ corresponding to w. We define an antiautomorphism $*: {}_R\mathscr{H}_n \ni x \mapsto x^* \in {}_R\mathscr{H}_n$ by $T_i^* = T_i$ for $i = 1, \dots, n-1$. Thus, we have $T_w^* = T_{w^{-1}}$ for $w \in \mathfrak{S}_n$. For $\mu \in \Lambda_{n,1}$, set $x_\mu = \sum_{w \in \mathfrak{S}_\mu} q^{\ell(w)} T_w$, and we define the right ${}_R\mathscr{H}_n$-module $M^\mu = x_\mu \cdot {}_R\mathscr{H}_n$. The q-Schur algebra ${}_R\mathscr{S}_{n,1}$ associated to ${}_R\mathscr{H}_n$ is defined by

$$ {}_R\mathscr{S}_{n,1} = \operatorname{End}_{{}_R\mathscr{H}_n}\Big(\bigoplus_{\mu \in \Lambda_{n,1}} M^\mu \Big). $$

The following lemma is well known (see, e.g., [M, (4.6)]).

Lemma 4.3. *For $\mu, \nu \in \Lambda_{n,1}$ and $d \in \mathscr{D}_{\mu\nu}$, let $T = \nu(\mathfrak{t}^\mu \cdot d)$, $S = \mu(\mathfrak{t}^\nu \cdot d^{-1})$. Then we have*

$$ \sum_{\substack{y \in \mathscr{D}_\nu \\ \mu(\mathfrak{t}^\nu \cdot y) = S}} q^{\ell(y)} T_y^* x_\nu = \sum_{w \in \mathfrak{S}_\mu d \mathfrak{S}_\nu} q^{\ell(w)} T_w = \sum_{\substack{x \in \mathscr{D}_\mu \\ \nu(\mathfrak{t}^\mu \cdot x) = T}} q^{\ell(x)} x_\mu T_x. $$

Thanks to this lemma, for $\mu, \nu \in \Lambda_{n,1}$ and $d \in \mathscr{D}_{\mu\nu}$, we can define an ${}_R\mathscr{H}_n$-module homomorphism $\psi^d_{\mu,\nu} : M^\nu \to M^\mu$ by

$$ \begin{aligned} \psi^d_{\mu,\nu}(x_\nu \cdot h) &= \Big(\sum_{\substack{y \in \mathscr{D}_\nu \\ \mu(\mathfrak{t}^\nu \cdot y) = S}} q^{\ell(y)} T_y^* x_\nu \Big) \cdot h \\ &= \Big(\sum_{\substack{x \in \mathscr{D}_\mu \\ \nu(\mathfrak{t}^\mu \cdot x) = T}} q^{\ell(x)} x_\mu T_x \Big) \cdot h \quad (h \in {}_R\mathscr{H}_n). \end{aligned} $$

We extend this homomorphism to an element of ${}_R\mathscr{S}_{n,1}$ by $\psi^d_{\mu,\nu}(m_\tau) = 0$ for $m_\tau \in M^\tau$ with $\tau \in \Lambda_{n,1}$ such that $\tau \neq \nu$. It is known that $\{\psi^d_{\mu,\nu} \mid \mu, \nu \in \Lambda_{n,1}, d \in \mathscr{D}_{\mu\nu}\}$ gives a free R-basis of ${}_R\mathscr{S}_{n,1}$ (see [M, Theorem 4.7]).

4.4.

Next, we define the Borel subalgebras of ${}_R\mathscr{S}_{n,1}$ following [DR2]. Let $I(m,n) = \{\mathbf{i} = (i_1, \dots, i_n) \mid 1 \le i_k \le m \text{ for } 1 \le k \le n\}$. $\mathfrak{S}_n$ acts on $I(m,n)$ from the right by $\mathbf{i} \cdot w = (i_{w(1)}, \dots, i_{w(n)})$ for $\mathbf{i} = (i_1, \dots, i_n) \in I(m,n)$ and $w \in \mathfrak{S}_n$. We define a partial order $\succeq$ on $I(m,n)$ by

$$ (i_1, \dots, i_n) \succeq (j_1, \dots, j_n) \quad \text{if and only if } i_k \ge j_k \text{ for all } k = 1, \dots, n. $$

For $\lambda \in \Lambda_{n,1}$, put

$$\mathbf{i}_\lambda = (\underbrace{1,\dots,1}_{\lambda_1 \text{ terms}}, \underbrace{2,\dots,2}_{\lambda_2 \text{ terms}}, \dots, \underbrace{m,\dots,m}_{\lambda_m \text{ terms}}).$$

For $\mu \in \Lambda_{n,1}$, we set

$$\Omega^{\succeq}(\mu) = \{(\lambda, d) \mid \lambda \in \Lambda_{n,1}, d \in \mathscr{D}_{\lambda\mu} \text{ such that } \mathbf{i}_\lambda \cdot d \succeq \mathbf{i}_\mu\},$$
$$\Omega^{\preceq}(\mu) = \{(\lambda, d) \mid \lambda \in \Lambda_{n,1}, d \in \mathscr{D}_{\lambda\mu} \text{ such that } \mathbf{i}_\mu \cdot d \preceq \mathbf{i}_\lambda\}.$$

Let ${}_R\mathscr{S}^{\leq 0}_{n,1}$ be the free R-submodule of ${}_R\mathscr{S}_{n,1}$ spanned by $\{\psi^d_{\lambda,\mu} \mid (\lambda, d) \in \Omega^{\succeq}(\mu), \mu \in \Lambda_{n,1}\}$, and let ${}_R\mathscr{S}^{\geq 0}_{n,1}$ be the free R-submodule of ${}_R\mathscr{S}_{n,1}$ spanned by $\{\psi^d_{\mu,\lambda} \mid (\lambda, d) \in \Omega^{\preceq}(\mu), \mu \in \Lambda_{n,1}\}$. By [DR2, Theorem 2.3], ${}_R\mathscr{S}^{\leq 0}_{n,1}$ (resp., ${}_R\mathscr{S}^{\geq 0}_{n,1}$) becomes a subalgebra of ${}_R\mathscr{S}_{n,1}$.

4.5.

We denote ${}_{\mathbb{Q}(q)}\mathscr{S}_{n,1}$ (resp., ${}_{\mathbb{Q}(q)}\mathscr{S}^{\leq 0}_{n,1}, {}_{\mathbb{Q}(q)}\mathscr{S}^{\geq 0}_{n,1}$) simply by $\mathscr{S}$ (resp., $\mathscr{S}^{\leq 0}_{n,1}$, $\mathscr{S}^{\geq 0}_{n,1}$). The following theorem comes from the works of several authors.

THEOREM 4.6. (see [J], [Du], [PW], [DR2], [DP])

(i) *There exists a surjective homomorphism of algebras*

$$\rho : U_q \to \mathscr{S}_{n,1}.$$

(ii) *By restricting ρ to $U_q^{\geq 0}$ (resp., $U_q^{\leq 0}$), we have the surjective homomorphisms*

$$\rho\,|_{U_q^{\geq 0}} : U_q^{\geq 0} \to \mathscr{S}^{\geq 0}_{n,1}, \qquad \rho\,|_{U_q^{\leq 0}} : U_q^{\leq 0} \to \mathscr{S}^{\leq 0}_{n,1}.$$

(iii) *By restricting ρ to ${}_{\mathcal{Z}}U_q$, we have the surjective homomorphism*

$$\rho\,|_{\mathcal{Z}U_q} : {}_{\mathcal{Z}}U_q \to {}_{\mathcal{Z}}\mathscr{S}_{n,1}.$$

(iv) *By restricting ρ to ${}_{\mathcal{Z}}U_q^{\geq 0}$ (resp., ${}_{\mathcal{Z}}U_q^{\leq 0}$), we have the surjective homomorphisms*

$$\rho\,|_{\mathcal{Z}U_q^{\geq 0}} : {}_{\mathcal{Z}}U_q^{\geq 0} \to {}_{\mathcal{Z}}\mathscr{S}^{\geq 0}_{n,1}, \qquad \rho\,|_{\mathcal{Z}U_q^{\leq 0}} : {}_{\mathcal{Z}}U_q^{\leq 0} \to {}_{\mathcal{Z}}\mathscr{S}^{\leq 0}_{n,1}.$$

We can describe precisely the image of the generators of U_q under the homomorphism ρ in Theorem 4.6 as follows.

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Proposition 4.7. (see [S2])

(i) *For* e_i $(1 \le i \le m-1)$, *we have*

$$\rho(e_i) = \sum_{\mu \in \Lambda_{n,1}} q^{-\mu_{i+1}+1} \psi^1_{\mu+\alpha_i,\mu},$$

where if $\mu + \alpha_i \notin \Lambda_{n,1}$, *we define* $\psi^1_{\mu+\alpha_i,\mu} = 0$.

(ii) *For* f_i $(1 \le i \le m-1)$, *we have*

$$\rho(f_i) = \sum_{\mu \in \Lambda_{n,1}} q^{-\mu_i+1} \psi^1_{\mu-\alpha_i,\mu},$$

where if $\mu - \alpha_i \notin \Lambda_1$, *we define* $\psi^1_{\mu-\alpha_i,\mu} = 0$.

(iii) *For* $K_i^{\pm}$ $(1 \le i \le m)$, *we have*

$$\rho(K_i^{\pm}) = \sum_{\mu \in \Lambda_{n,1}} q^{\pm\mu_i} \psi^1_{\mu,\mu}.$$

Clearly, $\psi^1_{\mu,\mu}$ *is an identity map on* M^μ.

Proof. See Appendix A. ∎

4.8.

By Theorem 4.6, the q-Schur algebra $\mathscr{S}_{n,1}$ is a quotient algebra of U_q. Thus, $\mathscr{S}_{n,1}$ is generated by the generators of U_q. Doty and Giaquinto [DG] described the kernel of the surjection $\rho : U_q \to \mathscr{S}_{n,1}$ precisely. Moreover, they also gave a presentation of the q-Schur algebra ${}_{\mathcal{Z}}\mathscr{S}_{n,1}$ over $\mathcal{Z}$.

Theorem 4.9. ([DG, Theorems 3.1 and 3.3])

(i) *The* q*-Schur algebra* $\mathscr{S}_{n,1}$ *is isomorphic to the associative algebra over* $\mathbb{Q}(q)$ *generated by* e_i, f_i $(1 \le i \le m-1)$ *and* $K_i^{\pm}$ $(1 \le i \le m)$ *with the defining relations (1.2.1)–(1.2.6) together with the following two relations:*

$$K_1 K_2 \cdots K_m = q^n, \tag{4.9.1}$$

$$(K_i - 1)(K_i - q)(K_i - q^2) \cdots (K_i - q^n) = 0. \tag{4.9.2}$$

(ii) ${}_{\mathcal{Z}}\mathscr{S}_{n,1}$ *is the* $\mathcal{Z}$*-subalgebra of* $\mathscr{S}_{n,1}$ *generated by all* $e_i^{(k)}, f_i^{(k)}, K_j^{\pm}$, *and* $\left[\begin{smallmatrix} K_j;0 \\ t \end{smallmatrix}\right]$ *for* $1 \le i \le m-1, 1 \le j \le m, k \ge 1, t \ge 1$.

In [DG], they gave an alternative presentation of $\mathscr{S}_{n,1}$ by generators and relations as follows.

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THEOREM 4.10. ([DG, Theorem 3.4])

(i) *The q-Schur algebra $\mathscr{S}_{n,1}$ is isomorphic to an associative algebra over $\mathbb{Q}(q)$ generated by E_i, F_i $(1 \le i \le m-1)$ and 1_λ $(\lambda \in \Lambda_{n,1})$ with the following defining relations:*

$$1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda, \qquad \sum_{\lambda \in \Lambda_{n,1}} 1_\lambda = 1;$$

$$E_i 1_\lambda = \begin{cases} 1_{\lambda+\alpha_i} E_i & \text{if } \lambda + \alpha_i \in \Lambda_{n,1}, \\ 0 & \text{otherwise}; \end{cases}$$

$$F_i 1_\lambda = \begin{cases} 1_{\lambda-\alpha_i} F_i & \text{if } \lambda - \alpha_i \in \Lambda_{n,1}, \\ 0 & \text{otherwise}; \end{cases}$$

$$1_\lambda E_i = \begin{cases} E_i 1_{\lambda-\alpha_i} & \text{if } \lambda - \alpha_i \in \Lambda_{n,1}, \\ 0 & \text{otherwise}; \end{cases}$$

$$1_\lambda F_i = \begin{cases} F_i 1_{\lambda+\alpha_i} & \text{if } \lambda + \alpha_i \in \Lambda_{n,1}, \\ 0 & \text{otherwise}; \end{cases}$$

$$E_i F_j - F_j E_i = \delta_{ij} \sum_{\lambda \in \Lambda_{n,1}} [\lambda_i - \lambda_{i+1}] 1_\lambda;$$

$$E_{i\pm 1} E_i^2 - (q + q^{-1}) E_i E_{i\pm 1} E_i + E_i^2 E_{i\pm 1} = 0;$$

$$E_i E_j = E_j E_i \quad (|i-j| \ge 2);$$

$$F_{i\pm 1} F_i^2 - (q + q^{-1}) F_i F_{i\pm 1} F_i + F_i^2 F_{i\pm 1} = 0;$$

$$F_i F_j = F_j F_i \quad (|i-j| \ge 2).$$

(ii) *${}_{\mathcal{Z}}\mathscr{S}_{n,1}$ is the $\mathcal{Z}$-subalgebra of $\mathscr{S}_{n,1}$ generated by all $E_i^{(k)}, F_i^{(k)}$ $(1 \le i \le m-1, k \ge 1)$, and 1_λ $(\lambda \in \Lambda_{n,1})$.*

REMARK 4.11. For $\lambda \in \Lambda_{n,1}$ and $i = 1, \dots, m-1$, put $\eta_i^\lambda = [\lambda_i - \lambda_{i+1}] 1_\lambda$, and put $\eta_{\Lambda_{n,1}} = \{\eta_i^\lambda \mid 1 \le i \le m-1, \lambda \in \Lambda_{n,1}\}$. It is clear that $\mathscr{S}_{n,1}$ is isomorphic to $\mathcal{S}_q^{\eta_{\Lambda_{n,1}}}$ defined in Section 2.5. Clearly, $\mathcal{S}_q^{\eta_{\Lambda_{n,1}}}$ satisfies condition (C-1). It is known that the q-Schur algebra ${}_{\mathcal{Z}}\mathscr{S}_{n,1}$ over $\mathcal{Z}$ has a triangular decomposition which coincides with the triangular decomposition of ${}_{\mathcal{Z}}\mathcal{S}_q$ in Proposition 3.2, and that ${}_{\mathcal{Z}}\mathscr{S}_{n,1}$ is a cellular algebra. Moreover, ${}_{\mathcal{Z}}\Delta(\lambda)$ for $\lambda \in \Lambda_{n,1}^+$ coincides with a cell module of ${}_{\mathcal{Z}}\mathscr{S}_{n,1}$ thanks to Theorem 3.4. In particular, $\Lambda_{n,1}^+$ coincides with the set of partitions of size n (see [DR2] and [M] for the results on q-Schur algebra ${}_{\mathbb{Z}}\mathscr{S}_{n,1}$). Thus, $\mathscr{S}_{n,1}$ ($\cong \mathcal{S}_q^\eta(\Lambda_{n,1})$) satisfies conditions (A-1), (A-2), and (C-1).

In [DP], a presentation of Borel subalgebras $\mathscr{S}_{n,1}^{\leq 0}$ and $\mathscr{S}_{n,1}^{\geq 0}$ was given as follows.

Theorem 4.12. ([DP, Theorem 8.1]) *The Borel subalgebra* $\mathscr{S}_{n,1}^{\leq 0}$ *(resp.,* $\mathscr{S}_{n,1}^{\geq 0}$*) is isomorphic to the associative algebra generated by* f_i *(resp.,* e_i*)* $(1 \leq i \leq m-1)$ *and* $K_i^{\pm}$ $(1 \leq i \leq m)$ *with defining relations (1.2.1), (1.2.3), (1.2.6), (4.9.1), and (4.9.2) (resp., (1.2.1), (1.2.2), (1.2.5), (4.9.1), and (4.9.2)).*

Remark 4.13. The above presentation of Borel subalgebras is not exactly the same as the one given in [DP, Theorem 8.1]. However, it is equivalent to their presentation (see [DP, Remarks 4.4]).

§5. Review of cyclotomic q-Schur algebras

In this section, we recall the definition of cyclotomic q-Schur algebras $\mathscr{S}_{n,r}$ introduced by [DJM], and we review some results on Borel subalgebras of $\mathscr{S}_{n,r}$ obtained by [DR2] which have an important role in later arguments to obtain presentations of $\mathscr{S}_{n,r}$.

5.1.

Let R be an integral domain, and take parameters $q, Q_1, \dots, Q_r \in R$, where q is invertible in R. The Ariki-Koike algebra ${}_R\mathscr{H}_{n,r}$ associated to $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ is the associative algebra with 1 over R generated by $T_0, T_1, \dots, T_{n-1}$ with the following defining relations:

$$\begin{aligned}
&(T_0 - Q_1)(T_0 - Q_2)\cdots(T_0 - Q_r) = 0, && \\
&(T_i - q)(T_i + q^{-1}) = 0 && (1 \leq i \leq n-1),\\
&T_0T_1T_0T_1 = T_1T_0T_1T_0, && \\
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} && (1 \leq i \leq n-2),\\
&T_iT_j = T_jT_i && (|i-j| \geq 2).
\end{aligned}$$

The subalgebra of ${}_R\mathscr{H}_{n,r}$ generated by $T_1, \dots, T_{n-1}$ is isomorphic to the Iwahori-Hecke algebra ${}_R\mathscr{H}_n$. We define an algebra antiautomorphism $* : {}_R\mathscr{H}_{n,r} \ni x \mapsto x^* \in {}_R\mathscr{H}_{n,r}$ by $T_i^* = T_i$ for $i = 0, \dots, n-1$.

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5.2.
Put

$$\Lambda_{n,r} = \left\{ \mu = (\mu^{(1)}, \dots, \mu^{(r)}) \;\middle|\; \begin{array}{c} \mu^{(k)} = (\mu^{(k)}_1, \dots, \mu^{(k)}_n) \in \mathbb{Z}^n_{\geq 0} \\ \sum_{k=1}^r \sum_{i=1}^n \mu^{(k)}_i = n \end{array} \right\}.$$

Thus, $\Lambda_{n,r}$ is the set of r-tuples of compositions with n parts whose size is equal to n. Put $m = rn$, and put $p_k = (k-1)n$ for $k = 1, \dots, r$. Then, there exists a bijection from $\Lambda_{n,r}$ to $\Lambda_{n,1}$ such that $\mu \mapsto \overline{\mu}$, where $\overline{\mu} = (\overline{\mu}_1, \overline{\mu}_2, \dots, \overline{\mu}_m) \in \Lambda_{n,1}$ obtained by $\overline{\mu}_{p_k+i} = \mu^{(k)}_i$.

5.3.
For $i = 1, \dots, n$, put $L_1 = T_0$ and $L_i = T_{i-1} L_{i-1} T_{i-1}$. For $\mu \in \Lambda_{n,r}$, put

$$u^+_\mu = \prod_{k=1}^{r} \prod_{i=1}^{a_k} (L_i - Q_k), \qquad m_\mu = x_{\overline{\mu}} u^+_\mu, \qquad M^\mu = m_\mu \cdot {}_R\mathscr{H}_{n,r},$$

where $a_k = \sum_{j=1}^{k-1} |\mu^{(j)}|$ with $a_1 = 0$. Note that $(m_\mu)^* = m_\mu$, and we define $(M^\mu)^* = {}_R\mathscr{H}_{n,r} \cdot m_\mu$. The cyclotomic q-Schur algebra ${}_R\mathscr{S}_{n,r}$ associated to ${}_R\mathscr{H}_{n,r}$ is defined by

$${}_R\mathscr{S}_{n,r} = \mathrm{End}_{{}_R\mathscr{H}_{n,r}} \Big(\bigoplus_{\mu \in \Lambda_{n,r}} M^\mu \Big).$$

The following properties are well known, and one can check them by direct calculations by using the defining relations of ${}_R\mathscr{H}_{n,r}$.

LEMMA 5.4. *We have the following.*

(i) *L_i and L_j commute with each other for any $1 \leq i, j \leq n$.*
(ii) *T_i and L_j commute with each other if $j \neq i, i+1$.*
(iii) *T_i commutes with both $L_i L_{i+1}$ and $L_i + L_{i+1}$.*
(iv) *For $a \in R$ and $i = 1, \dots, n-1$, T_i commutes with $\prod_{j=1}^k (L_j - a)$ if $k \neq i$.*
(v) *$L_{i+1} T_i = (q - q^{-1}) L_{i+1} + T_i L_i$, $T_i L_{i+1} = (q - q^{-1}) L_{i+1} + L_i T_i$.*
(vi) *$L_i T_i = (q^{-1} - q) L_{i+1} + T_i L_{i+1}$, $T_i L_i = (q^{-1} - q) L_{i+1} + L_{i+1} T_i$.*

5.5.
For $\lambda, \mu \in \Lambda_{n,r}$ and $d \in \mathscr{D}_{\overline{\lambda}\overline{\mu}}$ such that $\mathbf{i}_{\overline{\lambda}} \cdot d \succeq \mathbf{i}_{\overline{\mu}}$, we define $\varphi^d_{\lambda,\mu} \in {}_R\mathscr{S}_{n,r}$ by

$$\varphi^d_{\lambda,\mu}(m_\nu \cdot h) = \delta_{\mu\nu} \Big(\sum_{w \in \mathfrak{S}_{\overline{\lambda}} d \mathfrak{S}_{\overline{\mu}}} q^{\ell(w)} T_w \Big) u^+_\mu \cdot h \quad (\nu \in \Lambda_{n,r}, h \in {}_R\mathscr{H}_{n,r}).$$

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This is well defined by Lemma 4.3, and we have $\varphi^d_{\lambda,\mu} \in \mathrm{Hom}_{R\mathscr{H}_{n,r}}(M^\mu, M^\lambda)$ by [DR2, Lemma 5.6].

For $\lambda, \mu \in \Lambda_{n,r}$ and $d \in \mathscr{D}_{\overline{\mu}\overline{\lambda}}$ such that $\mathbf{i}_{\overline{\lambda}} \succeq \mathbf{i}_{\overline{\mu}} \cdot d$, we have $\mathbf{i}_{\overline{\lambda}} \cdot d^{-1} \succeq \mathbf{i}_{\overline{\mu}}$ and $d^{-1} \in \mathscr{D}_{\overline{\lambda}\overline{\mu}}$ from definitions immediately. Thus, we can define $\varphi^{d^{-1}}_{\lambda,\mu} \in \mathrm{Hom}_{R\mathscr{H}_{n,r}}(M^\mu, M^\lambda)$ as above. On the other hand, by [DJM, Corollary 5.17], we have $\varphi^{d^{-1}}_{\lambda,\mu}(m_\mu) \in (M^\mu)^* \cap M^\lambda$; hence, $(\varphi^{d^{-1}}_{\lambda,\mu}(m_\mu))^* \in M^\mu \cap (M^\lambda)^*$. Thus, we define $\varphi'^d_{\mu,\lambda} \in {}_R\mathscr{S}_{n,r}$ by

$$\varphi'^d_{\mu,\lambda}(m_\nu \cdot h) = \delta_{\lambda\nu}\big(\varphi^{d^{-1}}_{\lambda,\mu}(m_\mu)\big)^* \cdot h \quad (\nu \in \Lambda_{n,r}, h \in {}_R\mathscr{H}_{n,r}),$$

and we have $\varphi'^d_{\mu\lambda} \in \mathrm{Hom}_{R\mathscr{H}_{n,r}}(M^\lambda, M^\mu)$.

Let ${}_R\mathscr{S}^{\leq 0}_{n,r}$ (resp., ${}_R\mathscr{S}^{\geq 0}_{n,r}$) be the free R-submodule of ${}_R\mathscr{S}_{n,r}$ spanned by $\{\varphi^d_{\lambda,\mu} \mid (\overline{\lambda}, d) \in \Omega^{\succeq}(\overline{\mu}), \mu \in \Lambda_{n,r}\}$ (resp., $\{\varphi'^d_{\mu,\lambda} \mid (\overline{\lambda}, d) \in \Omega^{\preceq}(\overline{\mu}), \mu \in \Lambda_{n,r}\}$). Then ${}_R\mathscr{S}^{\leq 0}_{n,r}$ (resp., ${}_R\mathscr{S}^{\geq 0}_{n,r}$) is a subalgebra of ${}_R\mathscr{S}_{n,r}$, and $\{\varphi^d_{\lambda,\mu} \mid (\overline{\lambda}, d) \in \Omega^{\succeq}(\overline{\mu}), \mu \in \Lambda_{n,r}\}$ (resp., $\{\varphi'^d_{\mu,\lambda} \mid (\overline{\lambda}, d) \in \Omega^{\preceq}(\overline{\mu}), \mu \in \Lambda_{n,r}\}$) gives a free R-basis of ${}_R\mathscr{S}^{\leq 0}_{n,r}$ (resp., ${}_R\mathscr{S}^{\geq 0}_{n,r}$) by [DR2, Lemma 5.12, Theorem 5.13].

Moreover, Du and Rui [DR2] proved the following theorem.

THEOREM 5.6 ([DR2, Theorems 5.13 and 5.16]).

(i) *There exists an algebra isomorphism* $\mathcal{F}^{\leq 0} : {}_R\mathscr{S}^{\leq 0}_{n,r} \to {}_R\mathscr{S}^{\leq 0}_{n,1}$ *such that* $\mathcal{F}^{\leq 0}(\varphi^d_{\lambda,\mu}) = \psi^d_{\overline{\lambda},\overline{\mu}}$ *for* $\varphi^d_{\lambda,\mu} \in \{\varphi^d_{\lambda,\mu} \mid (\overline{\lambda}, d) \in \Omega^{\succeq}(\overline{\mu}), \mu \in \Lambda_{n,r}\}$.

(ii) *There exists an algebra isomorphism* $\mathcal{F}^{\geq 0} : {}_R\mathscr{S}^{\geq 0}_{n,r} \to {}_R\mathscr{S}^{\geq 0}_{n,1}$ *such that* $\mathcal{F}^{\geq 0}(\varphi'^d_{\mu,\lambda}) = \psi^d_{\overline{\mu},\overline{\lambda}}$ *for* $\varphi'^d_{\mu,\lambda} \in \{\varphi'^d_{\mu,\lambda} \mid (\overline{\lambda}, d) \in \Omega^{\preceq}(\overline{\mu}), \mu \in \Lambda_{n,r}\}$.

(iii) ${}_R\mathscr{S}_{n,r}$ *has a decomposition*

$${}_R\mathscr{S}_{n,r} = {}_R\mathscr{S}^{\leq 0}_{n,r} \cdot {}_R\mathscr{S}^{\geq 0}_{n,r} = \sum_{\lambda \in \Lambda_{n,r}} {}_R\mathscr{S}^{\leq 0}_{n,r} \cdot \varphi^1_{\lambda,\lambda} \cdot {}_R\mathscr{S}^{\geq 0}_{n,r}.$$

§6. The cyclotomic q-Schur algebra as a quotient algebra of $\widetilde{U}_q$

In this section, we give a surjective homomorphism from $\widetilde{U}_q$ to $\mathscr{S}_{n,r}$.

6.1.

As in Section 5, let n, r be positive integers, and put $m = nr$. (We need this condition to apply Theorem 5.6 later.) Let $\Gamma = \{1, \dots, n\} \times \{1, \dots, r\}$, and let $\Gamma' = \Gamma \setminus \{(n, r)\}$. As a convention, we set $(n+1, k) = (1, k+1)$ and $(0, k+1) = (n, k)$ for $k = 1, \dots, r-1$. For $(i, k) \in \Gamma$, put $\varepsilon_{(i,k)} = \varepsilon_{p_k+i}$, where $p_k = (k-1)n$. Thus, we can rewrite the weight lattice P by $P =$

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$\bigoplus_{(i,k)\in\varGamma}\mathbb{Z}\varepsilon_{(i,k)}$, and we regard $\varLambda_{n,r}$ as a subset of P by the injective map from $\varLambda_{n,r}$ to P given by $\varLambda_{n,r}\ni\mu\mapsto\sum_{(i,k)\in\varGamma}\mu_i^{(k)}\varepsilon_{(i,k)}\in P$. For $(i,k)\in\varGamma$, put $h_{(i,k)}=h_{p_k+i}$; then the dual weight lattice $P^\vee$ can be rewritten as $P^\vee=\bigoplus_{(i,k)\in\varGamma}\mathbb{Z}h_{(i,k)}$. Moreover, for $(i,k)\in\varGamma'$, put $\alpha_{(i,k)}=\alpha_{p_k+i}=\varepsilon_{(i,k)}-\varepsilon_{(i+1,k)}$. Thus, for $\mu\in\varLambda_{n,r}$, $\mu\pm\alpha_{(i,k)}$ makes sense in P.

6.2.

For $\mu\in\varLambda_{n,r}$ and $(i,k)\in\varGamma'$, if $\mu+\alpha_{(i,k)}\in\varLambda_{n,r}$ then we have $\mathbf{i}_{\overline{\mu}}\succeq\mathbf{i}_{\overline{\mu+\alpha_{(i,k)}}}$ from definitions. On the other hand, if $\mu-\alpha_{(i,k)}\in\varLambda_{n,r}$ then we have $\mathbf{i}_{\overline{\mu-\alpha_{(i,k)}}}\succeq\mathbf{i}_{\overline{\mu}}$. Then, for $(i,k)\in\varGamma'$, we define elements $\varphi^{\pm}_{(i,k)}\in{}_R\mathscr{S}_{n,r}$ by

$$\varphi^{+}_{(i,k)}=\sum_{\mu\in\varLambda_{n,r}}q^{-\mu_{i+1}^{(k)}+1}\varphi'^{1}_{\mu+\alpha_{(i,k)},\mu},$$

$$\varphi^{-}_{(i,k)}=\sum_{\mu\in\varLambda_{n,r}}q^{-\mu_{i}^{(k)}+1}\varphi^{1}_{\mu-\alpha_{(i,k)},\mu},$$

where we set $\varphi'^{1}_{\mu+\alpha_{(i,k)},\mu}=0$ (resp., $\varphi^{1}_{\mu-\alpha_{(i,k)},\mu}=0$) if $\mu+\alpha_{(i,k)}\notin\varLambda_{n,r}$ (resp., $\mu-\alpha_{(i,k)}\notin\varLambda_{n,r}$).

For $(i,k)\in\varGamma$, we define $\kappa^{\pm}_{(i,k)}\in{}_R\mathscr{S}_{n,r}$ by

$$\kappa^{\pm}_{(i,k)}=\sum_{\mu\in\varLambda_{n,r}}q^{\pm\mu_i^{(k)}}\varphi^{1}_{\mu,\mu},$$

and we write $\kappa^{+}_{(i,k)}$ by $\kappa_{(i,k)}$ for simplicity.

For $\mu\in\varLambda_{n,r}$ and $(i,k)\in\varGamma$, put $N=\sum_{l=1}^{k-1}|\mu^{(l)}|+\sum_{j=1}^{i-1}\mu_j^{(k)}$. By Lemma 5.4, one sees that $(L_{N+1}+L_{N+2}+\cdots+L_{N+\mu_i^{(k)}})$ commutes with m_μ. Thus, we can define $\sigma^{\mu}_{(i,k)}\in{}_R\mathscr{S}_{n,r}$ by

$$\sigma^{\mu}_{(i,k)}(m_\nu\cdot h)=\delta_{\mu,\nu}\big(m_\mu(L_{N+1}+\cdots+L_{N+\mu_i^{(k)}})\big)\cdot h\quad(\nu\in\varLambda_{n,r}h\in{}_R\mathscr{H}_{n,r}),$$

where we set $\sigma^{\mu}_{(i,k)}=0$ if $\mu_i^{(k)}=0$. Moreover, we define

$$\sigma_{(i,k)}=\sum_{\mu\in\varLambda_{n,r}}\sigma^{\mu}_{(i,k)}.$$

6.3.

Recall that $\mathcal{A}=\mathcal{Z}[\gamma_1,\ldots,\gamma_r]$ is the polynomial ring over $\mathcal{Z}=\mathbb{Z}[q,q^{-1}]$ with indeterminate elements $\gamma_1,\ldots,\gamma_r$ and that $\mathcal{K}=\mathbb{Q}(q,\gamma_1,\ldots,\gamma_r)$ is the

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quotient field of $\mathcal{A}$. We denote ${}_{\mathcal{K}}\mathscr{S}_{n,r}$ simply by $\mathscr{S}_{n,r}$, where we set $Q_i = \gamma_i$ $(1 \le i \le r)$. Now, we can define a surjective homomorphism of $\mathcal{K}$-algebras from $\widetilde{U}_q$ to $\mathscr{S}_{n,r}$ as in the following proposition.

Proposition 6.4. *There exists a surjective homomorphism $\widetilde{\rho} : \widetilde{U}_q \to \mathscr{S}_{n,r}$ such that, for $(i,k) \in \Gamma'$,*

$$\widetilde{\rho}(e_{p_k+i}) = \varphi^+_{(i,k)}, \tag{6.4.1}$$

$$\widetilde{\rho}(f_{p_k+i}) = \varphi^-_{(i,k)}, \tag{6.4.2}$$

$$\widetilde{\rho}(\tau_{p_k+i}) = \begin{cases} -\gamma_{k+1}\frac{\kappa_{(n,k)}\kappa^-_{(1,k+1)} - \kappa^-_{(n,k)}\kappa_{(1,k+1)}}{q-q^{-1}} \\ \qquad + \kappa_{(n,k)}\kappa^-_{(1,k+1)}(q^{-1}\sigma_{(n,k)} - q\sigma_{(1,k+1)}) & \textit{if } i = n, \\ \frac{\kappa_{(i,k)}\kappa^-_{(i+1,k)} - \kappa^-_{(i,k)}\kappa_{(i+1,k)}}{q-q^{-1}} & \textit{otherwise}, \end{cases} \tag{6.4.3}$$

and that, for $(i,k) \in \Gamma$,

$$\widetilde{\rho}(K^{\pm}_{p_k+i}) = \kappa^{\pm}_{(i,k)}. \tag{6.4.4}$$

Moreover, by restricting $\widetilde{\rho}$ to ${}_{\mathcal{A}}\widetilde{U}_q$, $\widetilde{\rho}\,|_{{}_{\mathcal{A}}\widetilde{U}_q}$ gives a surjective homomorphism from ${}_{\mathcal{A}}\widetilde{U}_q$ to ${}_{\mathcal{A}}\mathscr{S}_{n,r}$.

6.5.

The rest of this section is devoted to the proof of the proposition. The following relations are clear from the definitions:

$$\kappa_{(i,k)}\kappa_{(j,l)} = \kappa_{(j,l)}\kappa_{(i,k)}, \qquad \kappa_{(i,k)}\kappa^-_{(i,k)} = \kappa^-_{(i,k)}\kappa_{(i,k)} = 1. \tag{6.5.1}$$

Since $\varphi^1_{\nu,\nu}$ is the identity map on M^ν and $\sigma^\mu_{(i,k)} \in \mathrm{Hom}_{\mathscr{H}_{n,r}}(M^\mu, M^\mu)$, we have

$$\sigma^\mu_{(i,k)}\varphi^1_{\nu,\nu} = \varphi^1_{\nu,\nu}\sigma^\mu_{(i,k)} = \delta_{\mu,\nu}\sigma^\mu_{(i,k)}.$$

This relation combined with (6.5.1) implies that

$$\begin{aligned} &\kappa_{(j,l)}\big(\kappa_{(n,k)}\kappa^-_{(1,k+1)}(q^{-1}\sigma_{(n,k)} - q\sigma_{(1,k+1)})\big) \\ &\quad = \big(\kappa_{(n,k)}\kappa^-_{(1,k+1)}(q^{-1}\sigma_{(n,k)} - q\sigma_{(1,k+1)})\big)\kappa_{(j,l)}. \end{aligned} \tag{6.5.2}$$

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6.6.

By the definitions of $\varphi^{\pm}_{(i,k)}, \kappa^{\pm}_{(i,k)}$, it is clear that the elements $\varphi^{+}_{(i,k)}$ (resp., $\varphi^{-}_{(i,k)}$) for $(i,k) \in \Gamma'$ are in $\mathscr{S}^{\geq 0}_{n,r}$ (resp., $\mathscr{S}^{\leq 0}_{n,r}$), and that $\kappa^{\pm}_{(i,k)}$ for $(i,k) \in \Gamma$ is included in both of $\mathscr{S}^{\geq 0}_{n,r}$ and $\mathscr{S}^{\leq 0}_{n,r}$. Recall that, in the type A case, there exists a surjective homomorphism $\rho : U_q \to \mathscr{S}_{n,1}$ (Theorem 4.6). Here, we extend this homomorphism to a homomorphism over $\mathcal{K}$. By using the isomorphism $\mathcal{F}^{\geq 0} : \mathscr{S}^{\geq 0}_{n,r} \to {}_{\mathcal{K}}\mathscr{S}^{\geq 0}_{n,1}$ (resp., $\mathcal{F}^{\leq 0} : \mathscr{S}^{\leq 0}_{n,r} \to {}_{\mathcal{K}}\mathscr{S}^{\leq 0}_{n,1}$) in Theorem 5.6, we have the following proposition.

Proposition 6.7. *We have the following.*

(i) $\mathscr{S}^{\geq 0}_{n,r}$ *is generated by* $\varphi^{+}_{(i,k)}$ $((i,k) \in \Gamma')$ *and* $\kappa^{\pm}_{(i,k)}$ $((i,k) \in \Gamma)$.

(ii) $\mathscr{S}^{\leq 0}_{n,r}$ *is generated by* $\varphi^{-}_{(i,k)}$ $((i,k) \in \Gamma')$ *and* $\kappa^{\pm}_{(i,k)}$ $((i,k) \in \Gamma)$.

Proof. We show only (i) since (ii) is shown in a similar way. By the above arguments, $\varphi^{+}_{(i,k)}$ and $\kappa^{\pm}_{(i,k)}$ are elements of $\mathscr{S}^{\geq 0}_{n,r}$. On the other hand, by Proposition 4.7 and Theorem 5.6, we have $((\mathcal{F}^{\geq 0})^{-1} \circ \rho)(e_{p_k+i}) = \varphi^{+}_{(i,k)}$ and $((\mathcal{F}^{\geq 0})^{-1} \circ \rho)(K^{\pm}_{p_k+i}) = \kappa^{\pm}_{(i,k)}$. Moreover, ${}_{\mathcal{K}}\mathscr{S}^{\geq 0}_{n,1}$ is the image of $U^{\geq 0}_q$ under ρ by Theorem 4.6(ii), and $U^{\geq 0}_q$ is generated by e_i $(1 \leq i \leq m-1)$ and $K^{\pm}_i$ $(1 \leq i \leq m)$. This implies (i). ∎

6.8.

In the proof of Proposition 6.7, we have a surjection $(\mathcal{F}^{\geq 0})^{-1} \circ \rho : U^{\geq 0}_q \to \mathscr{S}^{\geq 0}_{n,r}$. Under this surjection, relations (1.2.2) and (1.2.5) imply the following relations (6.8.1) and (6.8.3). Similarly, the following relations (6.8.2) and (6.8.4) follow from relations (1.2.3) and (1.2.6).

$$\kappa_{(i,k)}\varphi^{+}_{(j,l)}\kappa^{-}_{(i,k)} = q^{\langle \alpha_{(j,l)}, h_{(i,k)}\rangle}\varphi^{+}_{(j,l)}, \tag{6.8.1}$$

$$\kappa_{(i,k)}\varphi^{-}_{(j,l)}\kappa^{-}_{(i,k)} = q^{-\langle \alpha_{(j,l)}, h_{(i,k)}\rangle}\varphi^{-}_{(j,l)}, \tag{6.8.2}$$

$$\begin{aligned} &\varphi^{+}_{(i\pm 1,k)}(\varphi^{+}_{(i,k)})^2 - (q+q^{-1})\varphi^{+}_{(i,k)}\varphi^{+}_{(i\pm 1,k)}\varphi^{+}_{(i,k)} + (\varphi^{+}_{(i,k)})^2\varphi^{+}_{(i\pm 1,k)} = 0, \\ &\varphi^{+}_{(i,k)}\varphi^{+}_{(j,l)} = \varphi^{+}_{(j,l)}\varphi^{+}_{(i,k)} \quad \big(|(p_k+i)-(p_l-j)| \geq 2\big), \end{aligned} \tag{6.8.3}$$

$$\begin{aligned} &\varphi^{-}_{(i\pm 1,k)}(\varphi^{-}_{(i,k)})^2 - (q+q^{-1})\varphi^{-}_{(i,k)}\varphi^{-}_{(i\pm 1,k)}\varphi^{-}_{(i,k)} + (\varphi^{-}_{(i,k)})^2\varphi^{-}_{(i\pm 1,k)} = 0, \\ &\varphi^{-}_{(i,k)}\varphi^{-}_{(j,l)} = \varphi^{-}_{(j,l)}\varphi^{-}_{(i,k)} \quad \big(|(p_k+i)-(p_l-j)| \geq 2\big). \end{aligned} \tag{6.8.4}$$

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6.9.

For $i=1,\dots,n-1$, let $s_i=(i,i+1)\in\mathfrak{S}_n$ be the adjacent transposition. For $\mu,\nu\in\Lambda_{n,r}$, put $X_\mu^\nu=\{x\in\mathscr{D}_{\overline{\mu}}\mid\overline{\nu}(\mathfrak{t}^{\overline{\mu}}\cdot x)=\overline{\nu}(\mathfrak{t}^{\overline{\mu}})\}$. One can check that

(6.9.1) $\quad X_\mu^{\mu-\alpha_{(i,k)}}=\{1,s_N,(s_Ns_{N+1}),\dots,(s_Ns_{N+1}\cdots s_{N+\mu_{i+1}^{(k)}-1})\},$

(6.9.2) $\quad X_{\mu-\alpha_{(i,k)}}^{\mu}=\{1,s_{N-1},(s_{N-1}s_{N-2}),\dots,(s_{N-1}s_{N-2}\cdots s_{N-\mu_i^{(k)}+1})\},$

(6.9.3) $\quad X_\mu^{\mu+\alpha_{(i,k)}}=\{1,s_N,(s_Ns_{N-1}),\dots,(s_Ns_{N-1}\cdots s_{N-\mu_i^{(k)}+1})\},$

(6.9.4) $\quad X_{\mu+\alpha_{(i,k)}}^{\mu}=\{1,s_{N+1},(s_{N+1}s_{N+2}),\dots,(s_{N+1}s_{N+2}\cdots s_{N+\mu_{i+1}^{(k)}-1})\},$

where $N=\sum_{l=1}^{k-1}|\mu^{(l)}|+\sum_{j=1}^{i}\mu_j^{(k)}$, and put $\mu_{n+1}^{(k)}=\mu_1^{(k+1)}$ if $i=n$. Then, we have the following lemma.

Lemma 6.10. *For $\mu\in\Lambda_{n,r}$ and $(i,k)\in\Gamma'$, we have the following.*

(i)

$$\varphi^+_{(i,k)}(m_\mu)=q^{-\mu_{i+1}^{(k)}+1}m_{\mu+\alpha_{(i,k)}}\Big(\sum_{y\in X^\mu_{\mu+\alpha_{(i,k)}}}q^{\ell(y)}T_y\Big)$$
$$=q^{-\mu_{i+1}^{(k)}+1}\Big(\sum_{x\in X_\mu^{\mu+\alpha_{(i,k)}}}q^{\ell(x)}T_x^*\Big)h^\mu_{+(i,k)}m_\mu,$$

where $h^\mu_{+(i,k)}=\begin{cases}1 & (i\neq n)\\ L_{N+1}-Q_{k+1} & (i=n)\quad (N=|\mu^{(1)}|+\cdots+|\mu^{(k)}|).\end{cases}$

(ii)

$$\varphi^-_{(i,k)}(m_\mu)=q^{-\mu_i^{(k)}+1}\Big(\sum_{y\in X_\mu^{\mu-\alpha_{(i,k)}}}q^{\ell(y)}T_y^*\Big)m_\mu$$
$$=q^{-\mu_i^{(k)}+1}m_{\mu-\alpha_{(i,k)}}h^\mu_{-(i,k)}\Big(\sum_{x\in X^\mu_{\mu-\alpha_{(i,k)}}}q^{\ell(x)}T_x\Big),$$

where $h^\mu_{-(i,k)}=\begin{cases}1 & (i\neq n),\\ L_N-Q_{k+1} & (i=n)\quad (N=|\mu^{(1)}|+\cdots+|\mu^{(k)}|).\end{cases}$

Proof. This is a direct consequence of the definitions and Lemma 4.3. ∎

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This lemma implies the following proposition.

PROPOSITION 6.11. *For $(i,k),(j,l)\in\Gamma'$, we have the following relations.*

(i) *If $(i,k)\neq(j,l)$ then we have*

$$\varphi^+_{(i,k)}\varphi^-_{(j,l)}-\varphi^-_{(j,l)}\varphi^+_{(i,k)}=0.$$

(ii) *If $(i,k)=(j,l)$ and $i\neq n$, then we have*

$$\varphi^+_{(i,k)}\varphi^-_{(i,k)}-\varphi^-_{(i,k)}\varphi^+_{(i,k)}=\frac{\kappa_{(i,k)}\kappa^-_{(i+1,k)}-\kappa^-_{(i,k)}\kappa_{(i+1,k)}}{q-q^{-1}}.$$

(iii) *If $(i,k)=(j,l)=(n,k)$, then we have*

$$\begin{aligned}&\varphi^+_{(n,k)}\varphi^-_{(n,k)}-\varphi^-_{(n,k)}\varphi^+_{(n,k)}\\&\quad=-\gamma_{k+1}\frac{\kappa_{(n,k)}\kappa^-_{(1,k+1)}-\kappa^-_{(n,k)}\kappa_{(1,k+1)}}{q-q^{-1}}\\&\qquad+\kappa_{(n,k)}\kappa^-_{(1,k+1)}(q^{-1}\sigma_{(n,k)}-q\sigma_{(1,k+1)}).\end{aligned}$$

Proof. By Lemma 6.10, for $\mu\in\Lambda_{n,r}$ and $(i,k),(j,l)\in\Gamma'$, we have

$$\begin{aligned}&\varphi^+_{(i,k)}\varphi^-_{(j,l)}(m_\mu)\\&\quad=\varphi^+_{(i,k)}\Big(q^{-\mu^{(l)}_j+1}m_{\mu-\alpha_{(j,l)}}h^\mu_{-(j,l)}\Big(\sum_{x\in X^\mu_{\mu-\alpha_{(j,l)}}}q^{\ell(x)}T_x\Big)\Big)\\&\quad=q^{-\mu^{(l)}_j+1}q^{-(\mu-\alpha_{(j,l)})^{(k)}_{i+1}+1}m_\mu\Big(\sum_{y\in X^{(\mu-\alpha_{(j,l)})}_{(\mu-\alpha_{(j,l)})+\alpha_{(i,k)}}}q^{\ell(y)}T_y\Big)h^\mu_{-(j,l)}\\&\qquad\times\Big(\sum_{x\in X^\mu_{\mu-\alpha_{(j,l)}}}q^{\ell(x)}T_x\Big).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\varphi^-_{(j,l)}\varphi^+_{(i,k)}(m_\mu)\\&\quad=\varphi^-_{(i,k)}\Big(q^{-\mu^{(k)}_{i+1}+1}m_{\mu+\alpha_{(i,k)}}\Big(\sum_{x\in X^\mu_{\mu+\alpha_{(i,k)}}}q^{\ell(x)}T_x\Big)\Big)\end{aligned}$$

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$$= q^{-\mu_{i+1}^{(k)}+1} q^{-(\mu+\alpha_{(i,k)})_j^{(l)}+1} m_\mu h_{-(j,l)}^{\mu+\alpha_{(i,k)}} \Big(\sum_{y \in X_{(\mu+\alpha_{(i,k)})-\alpha_{(j,l)}}^{(\mu+\alpha_{(i,k)})}} q^{\ell(y)} T_y \Big)$$
$$\times \Big(\sum_{x \in X_{\mu+\alpha_{(i,k)}}^{\mu}} q^{\ell(x)} T_x \Big).$$

One sees that $q^{-\mu_j^{(l)}+1} q^{-(\mu-\alpha_{(j,l)})_{i+1}^{(k)}+1} = q^{-\mu_{i+1}^{(k)}+1} q^{-(\mu+\alpha_{(i,k)})_j^{(l)}+1}$ for any case. Put

$$A = \Big(\sum_{y \in X_{(\mu-\alpha_{(j,l)})+\alpha_{(i,k)}}^{(\mu-\alpha_{(j,l)})}} q^{\ell(y)} T_y \Big), \qquad B = \Big(\sum_{x \in X_{\mu-\alpha_{(j,l)}}^{\mu}} q^{\ell(x)} T_x \Big),$$
$$C = \Big(\sum_{y \in X_{(\mu+\alpha_{(i,k)})-\alpha_{(j,l)}}^{(\mu+\alpha_{(i,k)})}} q^{\ell(y)} T_y \Big), \qquad D = \Big(\sum_{x \in X_{\mu+\alpha_{(i,k)}}^{\mu}} q^{\ell(x)} T_x \Big).$$

Using this, one can prove the three assertions of the proposition as follows.

(i) First, we assume that $(i,k) \neq (j,l)$. Then we have $h_{-(j,l)}^{\mu} = h_{-(j,l)}^{\mu+\alpha_{(i,k)}}$, and $h_{-(j,l)}^{\mu}$ commutes with A. If $(p_j + l) - (p_k + i) \neq 1$, then we have $X_{(\mu-\alpha_{(j,l)})+\alpha_{(i,k)}}^{(\mu-\alpha_{(j,l)})} = X_{\mu+\alpha_{(i,k)}}^{\mu}$ and $X_{(\mu+\alpha_{(i,k)})-\alpha_{(j,l)}}^{(\mu+\alpha_{(i,k)})} = X_{\mu-\alpha_{(j,l)}}^{\mu}$. Thus, we have $A = D$ and $B = C$. Moreover, one sees that A commutes with B. If $(p_j+1)-(p_k+i) = 1$, then we have $X_{(\mu-\alpha_{(j,l)})+\alpha_{(i,k)}}^{(\mu-\alpha_{(j,l)})} = X_{(\mu+\alpha_{(i,k)})-\alpha_{(j,l)}}^{(\mu+\alpha_{(i,k)})}$ and $X_{\mu-\alpha_{(j,l)}}^{\mu} = X_{\mu+\alpha_{(i,k)}}^{\mu}$. Hence, we have $A = C$ and $B = D$. This implies (i).

(ii) Next, we assume that $(i,k) = (j,l)$ and $i \neq n$. Then we have $h_{-(j,l)}^{\mu} = h_{-(j,l)}^{\mu+\alpha_{(i,k)}} = 1$. Put $N = \sum_{l=1}^{k-1} |\mu^{(l)}| + \sum_{j=1}^{i} \mu_j^{(k)}$. Then by (6.9.4) and (6.9.2), we have that

$$(6.11.1) \quad X_{(\mu-\alpha_{(i,k)})+\alpha_{(i,k)}}^{(\mu-\alpha_{(i,k)})} = \{1, s_N, (s_N s_{N+1}), \dots, (s_N s_{N+1} \cdots s_{N+\mu_{i+1}^{(k)}-1})\},$$

$$(6.11.2) \quad X_{(\mu+\alpha_{(i,k)})-\alpha_{(i,k)}}^{(\mu+\alpha_{(i,k)})} = \{1, s_N, (s_N s_{N-1}), \dots, (s_N s_{N-1} \cdots s_{N-\mu_i^{(k)}+1})\}.$$

Combined with (6.9.2) and (6.9.4), we have $AB - CD = B - D$. Note that $m_\mu T_w = q^{\ell(w)} m_\mu$ for $w \in \mathfrak{S}_\mu$; then we have

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$$
\begin{aligned}
&(\varphi^+_{(i,k)}\varphi^-_{(i,k)} - \varphi^-_{(i,k)}\varphi^+_{(i,k)})(m_\mu) \\
&= q^{-\mu_i^{(k)}-\mu_{i+1}^{(k)}+1}\Big(\Big(\sum_{a=0}^{\mu_i^{(k)}-1}(q^a)^2\Big) - \Big(\sum_{b=0}^{\mu_{i+1}^{(k)}-1}(q^b)^2\Big)\Big)m_\mu \\
&= \frac{\kappa_{(i,k)}\kappa^-_{(i+1,k)} - \kappa^-_{(i,k)}\kappa_{(i+1,k)}}{q-q^{-1}}(m_\mu).
\end{aligned}
$$

This implies (ii).

(iii) Finally, we assume that $(i,k)=(j,l)=(n,k)$. Put $N=\sum_{l=1}^k |\mu^{(l)}|$; then, we have $h^\mu_{-(n,k)} = L_N - Q_{k+1}$ and $h^{\mu+\alpha_{(n,k)}}_{-(n,k)} = L_{N+1} - Q_{k+1}$. Hence, we have

$$
\begin{aligned}
&(\varphi^+_{(n,k)}\varphi^-_{(n,k)} - \varphi^-_{(n,k)}\varphi^+_{(n,k)})(m_\mu) \\
&= q^{-\mu_n^{(k)}-\mu_1^{(k)}+1}m_\mu(A\cdot L_N\cdot B - L_{N+1}\cdot C\cdot D) \\
&\quad - Q_{k+1}q^{-\mu_n^{(k)}-\mu_1^{(k)}+1}m_\mu(AB-CD).
\end{aligned}
\tag{6.11.3}
$$

In a way similar to the case of (ii), we have

$$
q^{-\mu_n^{(k)}-\mu_1^{(k)}+1}m_\mu(AB-CD) = \frac{\kappa_{(n,k)}\kappa^-_{(1,k+1)} - \kappa^-_{(n,k)}\kappa_{(1,k+1)}}{q-q^{-1}}(m_\mu).
\tag{6.11.4}
$$

By Lemma 5.4, we can prove the following formula by induction on c:

$$
\begin{aligned}
&L_N(T_{N-1}T_{N-2}\cdots T_{N-c}) \\
&= (q-q^{-1})\Big(\sum_{\xi=1}^{c} T_{N-1}T_{N-2}\cdots \check{T}_{N-\xi}\cdots T_{N-c}L_{N-\xi+1}\Big) \\
&\quad + T_{N-1}T_{N-2}\cdots T_{N-c}L_{N-c},
\end{aligned}
\tag{6.11.5}
$$

where $\check{T}_{N-\xi}$ means removing $T_{N-\xi}$ from the product $T_{N-1}T_{N-2}\cdots T_{N-c}$. Combining this with (6.9.2), we have

$$
\begin{aligned}
L_N\cdot B &= L_N + \sum_{c=1}^{\mu_n^{(k)}-1}\big(q^c L_N(T_N T_{N-1}\cdots T_{N-c})\big) \\
&= L_N + \sum_{c=1}^{\mu_n^{(k)}-1}\Big\{q^c(q-q^{-1})
\end{aligned}
$$

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$$
\begin{aligned}
&\times \Big(\sum_{\xi=1}^{c} T_{N-1}T_{N-2}\cdots \check{T}_{n-\xi}\cdots T_{N-c}L_{N-\xi+1}\Big)\\
(6.11.6)\qquad &+ q^{c}T_{N-1}T_{N-2}\cdots T_{N-c}L_{N-c}\Big\}\\
= L_N + &\sum_{\xi=1}^{\mu_n^{(k)}-1}\Big(\sum_{c=\xi}^{\mu_n^{(k)}-1} q^{c}(q-q^{-1})T_{N-1}T_{N-2}\cdots \check{T}_{n-\xi}\cdots T_{N-c}\Big)\\
&\times L_{N-\xi+1}\\
&+ \sum_{c=1}^{\mu_n^{(k)}-1} q^{c}T_{N-1}\cdots T_{N-c}L_{N-c}
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
L_{N+1}\cdot C = L_{N+1}&\\
&+ \sum_{\xi=0}^{\mu_n^{(k)}-1}\Big(\sum_{c=\xi}^{\mu_n^{(k)}-1} q^{c+1}(q-q^{-1})T_NT_{N-1}\cdots \check{T}_{n-\xi}\cdots T_{N-c}\Big)\\
(6.11.7)\qquad &\times L_{N-\xi+1}\\
&+ \sum_{c=0}^{\mu_n^{(k)}-1} q^{c+1}T_NT_{N-1}\cdots T_{N-c}L_{N-c}
\end{aligned}
$$

by using the formula. We also have

$$
\begin{aligned}
&L_{N+1}(T_{N+1}T_{N+2}\cdots T_{N+c})\\
(6.11.8)\qquad &= \Big((q^{-1}-q)^{c} + \sum_{\xi=1}^{c}(q^{-1}-q)^{c-\xi}\\
&\quad\times \Big(\sum_{\substack{(i_1,\dots,i_\xi)\text{ s.t.}\\ 1\le i_1<i_2<\cdots<i_\xi\le c}} T_{N+i_1}T_{N+i_2}\cdots T_{N+i_\xi}\Big)\Big)\cdot L_{N+c+1},
\end{aligned}
$$

which is proved by induction on c thanks to Lemma 5.4. Relations (6.11.6) and (6.11.7), by making use of (6.11.1) and (6.11.2), combined with Lemma 5.4, imply that

$$\begin{aligned} &A\cdot L_N\cdot B - L_{N+1}\cdot C\cdot D\\ &= L_N\cdot B - \Big(1+q(q-q^{-1})\\ &\quad + \sum_{c=1}^{\mu_n^{(k)}-1}(q^{c+1}(q-q^{-1})T_{N-1}T_{N-2}\cdots T_{N-c}\Big)\cdot L_{N+1}\cdot D. \end{aligned} \tag{6.11.9}$$

Note that $m_\mu T_w = q^{\ell(w)}m_\mu$ for $w \in \mathfrak{S}_\mu$, and so (6.11.6) implies that

$$m_\mu\cdot(L_N\cdot B) = m_\mu q^{2(\mu_n^{(k)}-1)}(L_N + L_{N-1} + \cdots + L_{N-\mu_n^{(k)}+1}). \tag{6.11.10}$$

Similarly, (6.9.4) and (6.11.8) imply that

$$\begin{aligned} &m_\mu\cdot\Big(1+q(q-q^{-1}) + \sum_{c=1}^{\mu_n^{(k)}-1}(q^{c+1}(q-q^{-1})T_{N-1}T_{N-2}\cdots T_{N-c}\Big)\\ &\quad\cdot L_{N+1}\cdot D\\ &= m_\mu q^{2(\mu_n^{(k)})}(L_{N+1}+L_{N+2}+\cdots+L_{N+\mu_1^{(k+1)}}). \end{aligned} \tag{6.11.11}$$

By (6.11.9)–(6.11.11), we have

$$\begin{aligned} &q^{-\mu_n^{(k)}-\mu_1^{(k)}+1}m_\mu(A\cdot L_N\cdot B - L_{N+1}\cdot C\cdot D)\\ &= m_\mu q^{\mu_n^{(k)}-\mu_1^{(k+1)}}\big(q^{-1}(L_N+L_{N-1}+\cdots+L_{N-\mu_n^{(k)}+1})\\ &\quad - q(L_{N+1}+L_{N+2}+\cdots+L_{N+\mu_1^{(k+1)}})\big)\\ &= \kappa_{n,k}\kappa^-_{(1,k+1)}(q^{-1}\sigma_{(n,k)} - q\sigma_{(1,k+1)})(m_\mu). \end{aligned} \tag{6.11.12}$$

Now (6.11.3), (6.11.4), and (6.11.12) imply (iii). ∎

We can now prove Proposition 6.4.

Proof of Proposition 6.4. By relations (6.5.1), (6.5.2), and (6.8.1)–(6.8.4) together with Proposition 6.11, one sees that the homomorphism $\widetilde{\rho}$ in Proposition 6.4 is well defined. On the other hand, by Proposition 6.7, we have $\widetilde{\rho}(\widetilde{U}_q^{\geq 0}) = \mathscr{S}_{n,r}^{\geq 0}$ and $\widetilde{\rho}(\widetilde{U}_q^{\leq 0}) = \mathscr{S}_{n,r}^{\leq 0}$. Moreover, we know that $\mathscr{S}_{n,r} = \mathscr{S}_{n,r}^{\leq 0}\times\mathscr{S}_{n,r}^{\geq 0}$ by Theorem 5.6. Thus, we see that $\widetilde{\rho}$ is surjective.

By Theorem 4.6(iii),(iv) combined with Theorem 5.6, $\widetilde{\rho}\,|_{{}_{\mathcal{A}}\widetilde{U}_q}$ gives a surjection from ${}_{\mathcal{A}}\widetilde{U}_q$ to ${}_{\mathcal{A}}\mathscr{S}_{n,r}$. The proposition is now proved. ∎

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§7. Presentations of cyclotomic q-Schur algebras

In this section, we give two presentations of cyclotomic q-Schur algebras by generators and defining relations (see Theorem 7.16(ii),(iii)).

Recall that $\mathscr{S}_{n,r}$ is the cyclotomic q-Schur algebra over $\mathcal{K} = \mathbb{Q}(q, \gamma_1, \ldots, \gamma_r)$ with parameters $q, \gamma_1, \ldots, \gamma_r$.

7.1.

For presenting cyclotomic q-Schur algebras by generators and relations, we prepare some notations. Let $\mathcal{K}\langle x_1, \ldots, x_{m-1}\rangle$ be the noncommutative polynomial ring over $\mathcal{K}$ with indeterminate elements $x_1, \ldots, x_{m-1}$. Note that $\mathcal{K}\langle x_1, \ldots, x_{m-1}\rangle$ is isomorphic to the free $\mathcal{K}$-algebra generated by $x_1, \ldots, x_{m-1}$. Put $\mathbf{x} = \{x_1, \ldots, x_{m-1}\}$. For $(i,k) \in \Gamma'$, set $x_{(i,k)} = x_{p_k+i}$, where $p_k = (k-1)n$. Thus, we have $\mathbf{x} = \{x_{(i,k)} \mid (i,k) \in \Gamma'\}$ and $\mathcal{K}\langle x_1, \ldots, x_{m-1}\rangle = \mathcal{K}\langle \mathbf{x}\rangle = \mathcal{K}\langle x_{(i,k)} \mid (i,k) \in \Gamma'\rangle$.

For $g(\mathbf{x}) \in \mathcal{K}\langle \mathbf{x}\rangle$, let $g(\varphi^+)$ (resp., $g(\varphi^-)$) be the element of $\mathscr{S}_{n,r}$ obtained by replacing $x_{(i,k)}$ with $\varphi^+_{(i,k)}$ (resp., $\varphi^-_{(i,k)}$) in $g(\mathbf{x})$. Then, we have the following lemma.

Lemma 7.2. *For $\lambda \in \Lambda_{n,r}$ and $(i,k) \in \Gamma$, there exists an element*

$$g^\lambda_{(i,k)} = \sum_j r_j g^-_j(\mathbf{x}) \otimes g^+_j(\mathbf{x}) \in \mathcal{K}\langle \mathbf{x}\rangle \otimes_{\mathcal{K}} \mathcal{K}\langle \mathbf{x}\rangle \quad (r_j \in \mathcal{K}, g^-_j(\mathbf{x}), g^+_j(\mathbf{x}) \in \mathcal{K}\langle \mathbf{x}\rangle)$$

such that $\sigma^\lambda_{(i,k)} = \sum_j r_j g^-_j(\varphi^-) g^+_j(\varphi^+) \varphi^1_{\lambda,\lambda}$.

Proof. By Theorem 5.6(iii), we have $\mathscr{S}_{n,r} = \mathscr{S}^{\leq 0}_{n,r} \cdot \mathscr{S}^{\geq 0}_{n,r}$. On the other hand, by Proposition 6.7, $\mathscr{S}^{\leq 0}_{n,r}$ (resp., $\mathscr{S}^{\geq 0}_{n,r}$) is generated by $\varphi^-_{(i,k)}$ (resp., $\varphi^+_{(i,k)}$) for $(i,k) \in \Gamma'$ and by $\kappa^\pm_{(i,k)}$ for $(i,k) \in \Gamma$. Recall that $\kappa^\pm_{(i,k)} = \sum_{\mu \in \Lambda_{n,r}} q^{\pm \mu^{(k)}_i} \varphi^1_{\mu,\mu}$ and that $\varphi^1_{\mu,\mu}$ is the identity map on M^μ and the zero map on M^τ $(\tau \neq \mu)$. Moreover, $\{\varphi^1_{\mu,\mu} \mid \mu \in \Lambda_{n,r}\}$ is a set of pairwise orthogonal idempotents. Combined with relations (6.8.1) and (6.8.2), we obtain the lemma. □

7.3.

In general, the element $g^\lambda_{(i,k)} \in \mathcal{K}\langle \mathbf{x}\rangle \otimes_{\mathcal{K}} \mathcal{K}\langle \mathbf{x}\rangle$ satisfying the condition in Lemma 7.2 is not unique. Throughout the rest of this article, for $(i,k) \in \Gamma'$ and $\lambda \in \Lambda_{n,r}$, we fix $g^\lambda_{(i,k)}$ once and for all.

Let $\mathcal{K}\langle F_1, \ldots, F_{m-1}, E_1, \ldots, E_{m-1}\rangle$ be the noncommutative polynomial ring over $\mathcal{K}$ with indeterminate elements $F_1, \ldots, F_{m-1}, E_1, \ldots, E_{m-1}$. Put

$F = \{F_i \mid 1 \le i \le m-1\}$, and put $E = \{E_i \mid 1 \le i \le m-1\}$. For $(i,k) \in \varGamma'$, set $F_{(i,k)} = F_{p_k+i}$, and set $E_{(i,k)} = E_{p_k+i}$. For $g(\mathbf{x}) \in \mathcal{K}\langle \mathbf{x}\rangle$, let $g(F)$ (resp., $g(E)$) be the element of $\mathcal{K}\langle F\rangle$ (resp., $\mathcal{K}\langle E\rangle$) obtained by replacing $x_{(i,k)}$ with $F_{(i,k)}$ (resp., $E_{(i,k)}$) in $g(\mathbf{x})$. For $g^{\lambda}_{(i,k)} = \sum_j r_j g_j^-(\mathbf{x}) \otimes g_j^+(\mathbf{x}) \in \mathcal{K}\langle \mathbf{x}\rangle \otimes_{\mathcal{K}} \mathcal{K}\langle \mathbf{x}\rangle$ ($(i,k) \in \varGamma, \mu \in \varLambda_{n,r}$) in Lemma 7.2, put

$$g^{\lambda}_{(i,k)}(F,E) = \sum_j r_j g_j^-(F) \cdot g_j^+(E) \in \mathcal{K}\langle F, E\rangle. \tag{7.3.1}$$

7.4.

Let $\mathcal{S}_{n,r}$ be the associative algebra over $\mathbb{Q}(q,\gamma_1,\dots,\gamma_r)$ with 1 generated by $E_{(i,k)}, F_{(i,k)}$ ($(i,k) \in \varGamma'$) and 1_λ ($\lambda \in \varLambda_{n,r}$) with the following defining relations:

$$1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda, \qquad \sum_{\lambda \in \varLambda_{n,r}} 1_\lambda = 1; \tag{7.4.1}$$

$$E_{(i,k)} 1_\lambda = \begin{cases} 1_{\lambda+\alpha_{(i,k)}} E_{(i,k)} & \text{if } \lambda + \alpha_{(i,k)} \in \varLambda_{n,r}, \\ 0 & \text{otherwise;} \end{cases} \tag{7.4.2}$$

$$F_{(i,k)} 1_\lambda = \begin{cases} 1_{\lambda-\alpha_{(i,k)}} F_{(i,k)} & \text{if } \lambda - \alpha_{(i,k)} \in \varLambda_{n,r}, \\ 0 & \text{otherwise;} \end{cases} \tag{7.4.3}$$

$$1_\lambda E_{(i,k)} = \begin{cases} E_{(i,k)} 1_{\lambda-\alpha_{(i,k)}} & \text{if } \lambda - \alpha_{(i,k)} \in \varLambda_{n,r}, \\ 0 & \text{otherwise;} \end{cases} \tag{7.4.4}$$

$$1_\lambda F_{(i,k)} = \begin{cases} F_{(i,k)} 1_{\lambda+\alpha_{(i,k)}} & \text{if } \lambda + \alpha_{(i,k)} \in \varLambda_{n,r}, \\ 0 & \text{otherwise;} \end{cases} \tag{7.4.5}$$

$$E_{(i,k)} F_{(j,l)} - F_{(j,l)} E_{(i,k)} = \delta_{(i,k),(j,l)} \sum_{\lambda \in \varLambda_{n,r}} \eta^{\lambda}_{(i,k)}; \tag{7.4.6}$$

$$\begin{aligned} &E_{(i\pm1,k)}(E_{(i,k)})^2 - (q+q^{-1}) E_{(i,k)} E_{(i\pm1,k)} E_{(i,k)} \\ &\quad + (E_{(i,k)})^2 E_{(i\pm1,k)} = 0; \\ &E_{(i,k)} E_{(j,l)} = E_{(j,l)} E_{(i,k)} \qquad \big(|(p_k+i) - (p_l+j)| \ge 2\big); \end{aligned} \tag{7.4.7}$$

$$\begin{aligned} &F_{(i\pm1,k)}(F_{(i,k)})^2 - (q+q^{-1}) F_{(i,k)} F_{(i\pm1,k)} F_{(i,k)} \\ &\quad + (F_{(i,k)})^2 F_{(i\pm1,k)} = 0; \\ &F_{(i,k)} F_{(j,l)} = F_{(j,l)} F_{(i,k)} \qquad \big(|(p_k+i) - (p_l+j)| \ge 2\big), \end{aligned} \tag{7.4.8}$$

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where

$$\eta^\lambda_{(i,k)} = \begin{cases} \big(-\gamma_{k+1}[\lambda^{(k)}_n - \lambda^{(k+1)}_1] \\ \quad + q^{\lambda^{(k)}_n - \lambda^{(k+1)}_1}(q^{-1}g^\lambda_{(n,k)}(F,E) - qg^\lambda_{(1,k+1)}(F,E))\big)1_\lambda & \text{if } i = n, \\ [\lambda^{(k)}_i - \lambda^{(k)}_{i+1}]1_\lambda & \text{otherwise.} \end{cases}$$

7.5.

It is clear that $\mathcal{S}_{n,r}$ is a homomorphic image of $\widetilde{\mathcal{S}}_q(\varLambda_{n,r})$ defined in Section 2. Thus, $\mathcal{S}_{n,r}$ is a homomorphic image of $\widetilde{U}_q$. In fact, as the following lemma shows, $\mathcal{S}_{n,r}$ is isomorphic to $\mathcal{S}_q^{\eta_{\varLambda_{n,r}}}$, where $\eta_{\varLambda_{n,r}} = \{\eta^\lambda_{(i,k)} \mid (i,k) \in \varGamma', \lambda \in \varLambda_{n,r}\}$.

LEMMA 7.6. *For $(i,k) \in \varGamma'$ and $\lambda \in \varLambda_{n,r}$, we have $\eta^\lambda_{(i,k)} \in \widetilde{\mathcal{S}}_q^- \widetilde{\mathcal{S}}_q^+ 1_\lambda$ and $\deg(\eta^\lambda_{(i,k)}) = 0$. Thus, $\mathcal{S}_{n,r}$ is isomorphic to $\mathcal{S}_q^{\eta_{\varLambda_{n,r}}}$.*

Proof. From the definitions of $g^\lambda_{(n,k)}(F,E)$ and $g^\lambda_{(1,k+1)}(F,E)$, it is clear that $\eta^\lambda_{(i,k)} \in \widetilde{\mathcal{S}}_q^- \widetilde{\mathcal{S}}_q^+ 1_\lambda$. Note that $\sigma^\lambda_{(i,k)} \in \mathrm{Hom}_{\mathscr{H}_{n,r}}(M^\lambda, M^\lambda)$. Lemma 7.2, together with the definitions of $\varphi^\pm_{(j,l)}$, implies that $\deg(g^\lambda_{(i,k)}(F,E)) = 0$. Thus, we have $\deg(\eta^\lambda_{(i,k)}) = 0$. ∎

From now on, under the isomorphism $\mathcal{S}_{n,r} \cong \mathcal{S}_q^{\eta_{\varLambda_{n,r}}}$, we apply to $\mathcal{S}_{n,r}$ the results in Sections 2 and 3 for $\mathcal{S}_q^{\eta_{\varLambda_{n,r}}}$. Recall that $\widetilde{\rho} : \widetilde{U}_q \to \mathscr{S}_{n,r}$ and $\Psi : \widetilde{U}_q \to \mathcal{S}_{n,r}$ are surjective homomorphisms of algebras given in Proposition 6.4 and Section 2.5, respectively. We have the following proposition.

PROPOSITION 7.7. *There exists a surjective homomorphism of algebras $\Phi : \mathcal{S}_{n,r} \to \mathscr{S}_{n,r}$ such that*

$$(7.7.1) \quad \Phi(E_{(i,k)}) = \varphi^+_{(i,k)}, \qquad \Phi(F_{(i,k)}) = \varphi^-_{(i,k)}, \qquad \Phi(1_\lambda) = \varphi^1_{\lambda,\lambda}.$$

In particular, the surjection $\widetilde{\rho} : \widetilde{U}_q \to \mathscr{S}_{n,r}$ factors through the algebra $\mathcal{S}_{n,r}$; namely, we have $\widetilde{\rho} = \Phi \circ \Psi$. Moreover, by restricting Φ to ${}_{\mathcal{A}}\mathcal{S}_{n,r}$, we have a surjective homomorphism $\Phi\,|_{{}_{\mathcal{A}}\mathcal{S}_{n,r}} : {}_{\mathcal{A}}\mathcal{S}_{n,r} \to {}_{\mathcal{A}}\mathscr{S}_{n,r}$.

Proof. First, we prove that Φ gives a well-defined algebra homomorphism from $\mathcal{S}_{n,r}$ to $\mathscr{S}_{n,r}$. One can easily check that relations (7.4.1)–(7.4.5) hold in the images of Φ for the corresponding generators. By (6.8.3) and (6.8.4), relations (7.4.7) and (7.4.8) hold in the image of Φ. Proposition 6.11 together with the definition of $\eta^\lambda_{(i,k)}$ implies that (7.4.6) holds in the image of Φ. Thus,

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Φ is well defined. By investigating the images of generators under each map, we have $\widetilde{\rho} = \Phi \circ \Psi$, and Φ is surjective. The last assertion follows from the restriction of $\widetilde{\rho} = \Phi \circ \Psi$ to ${}_{\mathcal{A}}\widetilde{U}_q$ together with Proposition 6.4. ∎

Since $\varphi^1_{\lambda,\lambda} \neq 0$ in $\mathscr{S}_{n,r}$ for $\lambda \in \Lambda_{n,r}$, and since $\Phi(1_\lambda) = \varphi^1_{\lambda,\lambda}$, we have the following corollary.

Corollary 7.8. *For $\lambda \in \Lambda_{n,r}$, $1_\lambda \neq 0$ in $\mathcal{S}_{n,r}$.*

7.9.

For $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \Lambda_{n,r}$, we say that λ is an r-partition of size n if all $\lambda^{(k)}$ $(1 \leq k \leq r)$ are partitions; namely, all $\lambda^{(k)}$ are weakly decreasing sequences. On the other hand, we have $\Lambda^+_{n,r} = \{\lambda \in \Lambda_{n,r} \mid 1_\lambda \notin \mathcal{S}_{n,r}(> \lambda)\}$ by (2.10.1). Then, we obtain the parameterization of the isomorphism classes of simple $\mathcal{S}_{n,r}$-modules as follows.

Lemma 7.10. *For $\mathcal{S}_{n,r} (\cong \mathcal{S}_q^{\eta_{\Lambda_{n,r}}})$, $\Lambda^+_{n,r} = \{\lambda \in \Lambda_{n,r} \mid 1_\lambda \notin \mathcal{S}_{n,r}(> \lambda)\}$ coincides with the set of r-partitions of size n. In particular, the isomorphism classes of simple $\mathcal{S}_{n,r}$-modules are parameterized by $\Lambda^+_{n,r}$.*

Proof. Let $(i,k) \in \Gamma'$ be such that $i \neq n$. For $a \in \mathbb{Z}_{>0}$ and $\lambda \in \Lambda_{n,r}$, we can prove, by induction on $a \in \mathbb{Z}_{>0}$ together with (7.4.6), that

$$E^a_{(i,k)} F^a_{(i,k)} 1_\lambda \equiv [a]! \Big(\prod_{j=1}^{a} [\lambda^{(k)}_i - \lambda^{(k)}_{i+1} - a + j] \Big) 1_\lambda \mod \mathcal{S}_{n,r}(> \lambda). \tag{7.10.1}$$

Assume that $\lambda \in \Lambda_{n,r}$ is not an r-partition. Then, there exists i, k such that $\lambda^{(k)}_i < \lambda^{(k)}_{i+1}$, where $1 \leq i \leq n-1$ and $1 \leq k \leq r$. Thus, by (7.10.1), we have

$$\begin{aligned} &E^{\lambda^{(k)}_i + 1}_{(i,k)} F^{\lambda^{(k)}_i + 1}_{(i,k)} 1_\lambda \\ &\quad \equiv [\lambda^{(k)}_i + 1]! \Big(\prod_{j=1}^{\lambda^{(k)}_i + 1} [j - \lambda^{(k)}_{i+1} - 1] \Big) 1_\lambda \mod \mathcal{S}_{n,r}(> \lambda). \end{aligned} \tag{7.10.2}$$

Since $\lambda - (\lambda^{(k)}_i + 1)\alpha_{(i,k)} \notin \Lambda_{n,r}$, the left-hand side of (7.10.2) is equal to 0 by (7.4.3). On the other hand, since $\lambda^{(k)}_i < \lambda^{(k)}_{i+1}$, we have $[\lambda^{(k)}_i + 1]! (\prod_{j=1}^{\lambda^{(k)}_i + 1} [j - \lambda^{(k)}_{i+1} - 1]) \neq 0$. Thus, (7.10.2) implies that $1_\lambda \in \mathcal{S}_{n,r}(> \lambda)$ if λ is not an

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r-partition. Hence, we have $\{\lambda \in \Lambda_{n,r} \mid \lambda : r\text{-partition}\} \supset \Lambda^+_{n,r}$. By Theorem 2.16(iii), the isomorphism classes of simple $\mathcal{S}_{n,r}$-modules are parameterized by $\Lambda^+_{n,r}$. On the other hand, through the surjection $\Phi : \mathcal{S}_{n,r} \to \mathscr{S}_{n,r}$ in Proposition 7.7, one can regard a simple $\mathscr{S}_{n,r}$-module as a simple $\mathcal{S}_{n,r}$-module. Moreover, it is known that the isomorphism classes of simple $\mathscr{S}_{n,r}$-modules are parameterized by the set of r-partitions of size n by [DJM]. Thus, we obtain the lemma. □

7.11.

Since $\mathcal{S}_{n,r}$ is a quotient algebra of $\widetilde{U}_q$, one can describe $\mathcal{S}_{n,r}$ by generators and relations of $\widetilde{U}_q$ together with some additional relations. Here, we give such additional relations precisely. For $(i,k) \in \Gamma'$ and $\lambda \in \Lambda_{n,r}$, we define $g^{\lambda}_{(i,k)}(f,e) \in \widetilde{U}_q$ in a way similar to that in (7.3.1). Recall the bijection from $\Lambda_{n,r}$ to $\Lambda_{n,1}$ such that $\mu \mapsto \overline{\mu}$ in Section 5.2. For $\lambda \in \Lambda_{n,r}$, put $K_\lambda = K_{\overline{\lambda}} \in \widetilde{U}_q$, where $K_{\overline{\lambda}}$ is defined in (2.2.1). For $(i,k) \in \Gamma'$, put

$$g_{(i,k)}(f,e) = \sum_{\lambda \in \Lambda_{n,r}} \big(g^{\lambda}_{(i,k)}(f,e) K_\lambda\big),$$

and put

$$\eta_{(i,k)} = \begin{cases} \big(-\gamma_{k+1} \frac{K_{(n,k)}K^-_{(1,k+1)} - K^-_{(n,k)}K_{(1,k+1)}}{q-q^{-1}} \\ \quad + K_{(n,k)}K^{-1}_{(1,k+1)}(q^{-1}g_{(n,k)}(f,e) - q g_{(1,k+1)}(f,e))\big) & \text{if } i = n, \\ \frac{K_{(i,k)}K^-_{(i+1,k)} - K^-_{(i,k)}K_{(i+1,k)}}{q-q^{-1}} & \text{otherwise.} \end{cases}$$

Let $\widetilde{I}_{n,r}$ be the two-sided ideal of $\widetilde{U}_q$ generated by $\tau_{p_k+i} - \eta_{(i,k)}$ ($(i,k) \in \Gamma'$), $K_1K_2\cdots K_m - q^n$, and $(K_i - 1)(K_i - q)(K_i - q^2)\cdots(K_i - q^n)$ ($1 \le i \le m$). Let $U_{n,r} = \widetilde{U}_q/\widetilde{I}_{n,r}$ be a quotient algebra of $\widetilde{U}_q$. One sees that $U_{n,r}$ is isomorphic to the algebra generated by E_i, F_i ($1 \le i \le m-1$) and $K_i^{\pm}$ ($1 \le i \le m$) with defining relations (1.5.1)–(1.5.3), (1.5.6), and (1.5.7) together with the following relations:

(7.11.1) $$e_{(i,k)}f_{(j,l)} - f_{(j,l)}e_{(i,k)} = \delta_{(i,k),(j,l)}\eta_{(i,k)},$$

(7.11.2) $$K_1K_2\cdots K_m = q^n,$$

(7.11.3) $$(K_i - 1)(K_i - q)(K_i - q^2)\cdots(K_i - q^n) = 0,$$

where we identify $e_{(i,k)} \leftrightarrow e_{p_k+i}$, $f_{(i,k)} \leftrightarrow f_{p_k+i}$ and $K^{\pm}_{(i,k)} \leftrightarrow K^{\pm}_{p_k+i}$.

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Proposition 7.12. *$\widetilde{I}_{n,r}$ contains the kernel of the surjection $\Psi : \widetilde{U}_q \to \mathcal{S}_{n,r}$. Thus, Ψ induces the surjection $\Psi' : U_{n,r} \to \mathcal{S}_{n,r}$. Moreover, Ψ' gives an isomorphism of algebras.*

Proof. From the definition, we have $\Psi(\eta_{(i,k)}) = \sum_{\lambda\in\Lambda_{n,r}} \eta^{\lambda}_{(i,k)}$; thus, we have $\Psi(\tau_{p_k+i} - \eta_{(i,k)}) = 0$. Note that $\Psi(K_i) = \sum_{\lambda\in\Lambda_{n,r}} q^{\overline{\lambda}_i} 1_\lambda$; we see easily that $\Psi(K_1 \cdots K_m) = q^n$ and $\Psi((K_i - 1)(K_i - q)\cdots(K_i - q^n)) = 0$. Thus, we have $\widetilde{I}_{n,r} \subset \operatorname{Ker}\Psi$, and Ψ induces the surjection $\Psi' : U_{n,r} \to \mathcal{S}_{n,r}$.

Let $U^0_{n,r}$ be the subalgebra of $U_{n,r}$ generated by K_i $(1 \le i \le m)$. In a way similar to that in the proof of [DDPW, Lemma 13.39], the restriction of Ψ' to $U^0_{n,r}$ gives an isomorphism $U^0_{n,r} \cong \mathcal{S}^0_{n,r}$. (Note that, in the proof of [DDPW, Lemma 13.39], the authors use only the relations of K_i which coincide with the relations in $U^0_{n,r}$.) Through the isomorphism $U^0_{n,r} \cong \mathcal{S}^0_{n,r}$, we have

$$K_\lambda K_\mu = \delta_{\lambda,\mu} K_\lambda, \qquad \sum_{\lambda\in\Lambda_{n,r}} K_\lambda = 1 \tag{7.12.1}$$

in $U_{n,r}$. Moreover, for $1 \le i \le m$ and $\lambda \in \Lambda_{n,r}$, we have $K_i K_\lambda = q^{\overline{\lambda}_i} K_\lambda$; thus, we have

$$K_i = K_i\Big(\sum_{\lambda\in\Lambda_{n,r}} K_\lambda\Big) = \sum_{\lambda\in\Lambda_{n,r}} q^{\overline{\lambda}_i} K_\lambda. \tag{7.12.2}$$

Let $\Psi^\dagger : \mathcal{S}_{n,r} \to U_{n,r}$ be the homomorphism of algebras given by $\Psi^\dagger(E_{(i,k)}) = e_{(i,k)}$, $\Psi^\dagger(F_{(i,k)}) = f_{(i,k)}$, and $\Psi^\dagger(1_\lambda) = K_\lambda$. In order to see that $\Psi^\dagger$ is well defined, we may check relations (7.4.1)–(7.4.8) in the image of $\Psi^\dagger$ for the corresponding generators. Relation (7.4.1) follows from (7.12.1). We can check relations (7.4.2)–(7.4.5) in a way similar to that in the proof of [DDPW, Lemma 13.40]. Relation (7.4.6) follows from the definition of $\eta_{(i,k)}$. Relations (7.4.7) and (7.4.8) are just (1.5.6) and (1.5.7), respectively. Thus, $\Psi^\dagger$ is well defined. Moreover, by (7.12.2), we see that $\Psi^\dagger$ is surjective and gives the inverse map of Ψ'; thus, we have $U_{n,r} \cong \mathcal{S}_{n,r}$. ∎

7.13.

Our goal is to show that the surjection $\Phi : \mathcal{S}_{n,r} \to \mathscr{S}_{n,r}$ in Proposition 7.7 is actually an isomorphism. Let

$$\{\varphi_{ST} \mid S, T \in \mathcal{T}(\lambda) \text{ for some } \lambda \in \Lambda^+_{n,r}\}$$

be the cellular basis of $\mathscr{S}_{n,r}$ constructed in [DJM], where $\mathcal{T}(\lambda)$ is the set of semistandard tableaux of shape λ (see [DJM] for the definition). For

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$\lambda \in \Lambda^+_{n,r}$, let $\mathscr{S}_{n,r}(\geq \lambda)$ (resp., $\mathscr{S}_{n,r}(> \lambda)$) be a subspace of $\mathscr{S}_{n,r}$ spanned by $\{\varphi_{ST} \mid S, T \in \mathcal{T}(\mu)$ for some $\mu \in \Lambda^+_{n,r}$ such that $\mu \geq \lambda\}$ (resp., $\{\varphi_{ST} \mid S, T \in \mathcal{T}(\mu)$ for some $\mu \in \Lambda^+_{n,r}$ such that $\mu > \lambda\}$); then both of $\mathscr{S}_{n,r}(\geq \lambda)$ and $\mathscr{S}_{n,r}(> \lambda)$ are two-sided ideals of $\mathscr{S}_{n,r}$.

It is known that $\varphi^1_{\lambda,\lambda} \in \mathscr{S}_{n,r}(\geq \lambda) \setminus \mathscr{S}_{n,r}(> \lambda)$ for $\lambda \in \Lambda^+_{n,r}$. ($\varphi^1_{\lambda,\lambda}$ is denoted by $\varphi_{T^\lambda T^\lambda}$ in [DJM].) For $\lambda \in \Lambda^+_{n,r}$, the left $\mathscr{S}_{n,r}$-module $W(\lambda)$ of $\mathscr{S}_{n,r}$ (the Weyl module) is defined by

$$W(\lambda) = \big(\mathscr{S}_{n,r} \cdot \varphi^1_{\lambda,\lambda} + \mathscr{S}_{n,r}(> \lambda)\big)/\mathscr{S}_{n,r}(> \lambda).$$

Note that $W(\lambda)$ is an $\mathscr{S}_{n,r}$-submodule of $\mathscr{S}_{n,r}(\geq \lambda)/\mathscr{S}_{n,r}(> \lambda)$. By [DR2, Theorem 5.15] (and its proof), for $S, T \in \mathcal{T}(\mu)$, we have

(7.13.1) $\quad \varphi_{ST} = \varphi_{ST^\mu}\varphi^1_{\mu,\mu}\varphi_{T^\mu T}, \quad$ where $\varphi_{ST^\mu} \in \mathscr{S}^{\leq 0}_{n,r}$ and $\varphi_{T^\mu T} \in \mathscr{S}^{\geq 0}_{n,r}$.

One sees from this that

$$W(\lambda) \cong \mathscr{S}^{\leq 0} \cdot \varphi^1_{\lambda,\lambda}/\big(\mathscr{S}^{\leq 0} \cdot \varphi^1_{\lambda,\lambda} \cap \mathscr{S}_{n,r}(> \lambda)\big) \text{ as } \mathcal{K}\text{-vector spaces.}$$

It is known that $\{W(\lambda) \mid \lambda \in \Lambda^+_{n,r}\}$ gives a complete set of isomorphism classes of (left) simple $\mathscr{S}_{n,r}$-modules. Similarly, we have a complete set of isomorphism classes of (right) simple $\mathscr{S}_{n,r}$-modules $\{W^\sharp(\lambda) \mid \lambda \in \Lambda^+_{n,r}\}$ such that

$$W^\sharp(\lambda) = \varphi^1_{\lambda,\lambda} \cdot \mathscr{S}^{\geq 0}/\big(\varphi^1_{\lambda,\lambda} \cdot \mathscr{S}^{\geq 0}_{n,r} \cap \mathscr{S}_{n,r}(> \lambda)\big) \quad \text{as } \mathcal{K}\text{-vector spaces.}$$

Recall that $\mathcal{S}^{\leq 0}_{n,r}$ (resp., $\mathcal{S}^{\geq 0}_{n,r}$) is a subalgebra of $\mathcal{S}_{n,r}$ defined in Section 2.17. Then we have the following lemma.

LEMMA 7.14. *The restriction of the surjection Φ (in Proposition 7.7) to $\mathcal{S}^{\leq 0}_{n,r}$ (resp., $\mathcal{S}^{\geq 0}_{n,r}$) gives an isomorphism $\Phi\,|_{\mathcal{S}^{\leq 0}_{n,r}} \colon \mathcal{S}^{\leq 0}_{n,r} \to \mathscr{S}^{\leq 0}_{n,r}$ (resp., $\Phi\,|_{\mathcal{S}^{\geq 0}_{n,r}} \colon \mathcal{S}^{\geq 0}_{n,r} \to \mathscr{S}^{\geq 0}_{n,r}$) of algebras.*

Proof. By Proposition 6.7, the restriction of $\widetilde{\rho}$ (in Proposition 6.4) to $\widetilde{U}^{\leq 0}_q$ gives a surjective homomorphism $\widetilde{\rho}\,|_{\widetilde{U}^{\leq 0}_q} \colon \widetilde{U}^{\leq 0}_q \to \mathscr{S}^{\leq 0}_{n,r}$. Since $\Phi \circ \Psi = \widetilde{\rho}$ (see Proposition 7.7) and $\Psi(\widetilde{U}^{\leq 0}_q) = \mathscr{S}^{\leq 0}_{n,r}$, we have a surjective homomorphism $\Phi\,|_{\mathcal{S}^{\leq 0}_{n,r}} \colon \mathcal{S}^{\leq 0}_{n,r} \to \mathscr{S}^{\leq 0}_{n,r}$.

On the other hand, thanks to Theorem 4.12, we can define a homomorphism $\Phi'^{\leq 0}$ of algebras from $\mathscr{S}^{\leq 0}_{n,1}$ to $U_{n,r}$ by sending the elements f_i $(1 \leq i \leq m-1)$ and $K^\pm_i$ $(1 \leq i \leq m)$ of $\mathscr{S}^{\leq 0}_{n,1}$ to the corresponding elements of $U_{n,r}$. Combining this with isomorphisms $\mathscr{S}_{n,1}^{\leq 0} \cong \mathscr{S}_{n,r}^{\leq 0}$ and $U_{n,r} \cong \mathcal{S}_{n,r}$, $\Phi'^{\leq 0}$ induces a surjective homomorphism from $\mathscr{S}_{n,r}^{\leq 0}$ to $\mathcal{S}_{n,r}^{\leq 0}$. Thus, $\Phi\,|_{\mathcal{S}_{n,r}^{\leq 0}}$ is an isomorphism. The case of $\mathcal{S}_{n,r}^{\geq 0}$ is similar. ∎

LEMMA 7.15. *For $\lambda \in \Lambda_{n,r}^{+}$, the restriction of Φ to $\mathcal{S}_{n,r}(\geq \lambda)$ (resp., $\mathcal{S}_{n,r}(> \lambda)$) gives a surjective homomorphism of $(\mathcal{S}_{n,r}, \mathcal{S}_{n,r})$-bimodules $\Phi\,|_{\mathcal{S}_{n,r}(\geq\lambda)}: \mathcal{S}_{n,r}(\geq \lambda) \to \mathscr{S}_{n,r}(\geq \lambda)$ (resp., $\Phi\,|_{\mathcal{S}_{n,r}(>\lambda)}: \mathcal{S}_{n,r}(> \lambda) \to \mathscr{S}_{n,r}(> \lambda)$).*

Proof. Note that $\Phi(1_\mu) = \varphi_{\mu,\mu}^{1}$, and note that $\varphi_{\mu,\mu}^{1} \in \mathscr{S}_{n,r}(\geq \lambda)$ if $\mu \geq \lambda$. We have $\Phi(\mathcal{S}_{n,r}(\geq \lambda)) \subset \mathscr{S}_{n,r}(\geq \lambda)$ since $\mathscr{S}_{n,r}(\geq \lambda)$ is a two-sided ideal of $\mathscr{S}_{n,r}$.

On the other hand, one sees easily that

$$\mathcal{S}_{n,r}(\geq \lambda) = \sum_{\substack{\mu \in \Lambda_{n,r}^{+} \\ \mu \geq \lambda}} \mathcal{S}_{n,r}^{\leq 0} 1_\mu \mathcal{S}_{n,r}^{\geq 0}.$$

Combining this with (7.13.1) and Lemma 7.14, we have $\varphi_{ST} \in \Phi(\mathcal{S}_{n,r}(\geq \lambda))$ for any $S, T \in \mathcal{T}(\mu)$ ($\mu \in \Lambda_{n,r}^{+}$ such that $\mu \geq \lambda$). Thus, $\Phi\,|_{\mathcal{S}_{n,r}(\geq\lambda)}$ is a surjection from $\mathcal{S}_{n,r}(\geq \lambda)$ to $\mathscr{S}_{n,r}(\geq \lambda)$. The case of $\mathcal{S}_{n,r}(> \lambda)$ is similar. ∎

The following theorem is the main result of our article.

THEOREM 7.16. *We have the following.*

(i) *$\Phi : \mathcal{S}_{n,r} \to \mathscr{S}_{n,r}$ gives an isomorphism of algebras. Moreover, by restricting Φ to ${}_{\mathcal{A}}\mathcal{S}_{n,r}$, $\Phi\,|_{{}_{\mathcal{A}}\mathcal{S}_{n,r}}$ gives an isomorphism from ${}_{\mathcal{A}}\mathcal{S}_{n,r}$ to ${}_{\mathcal{A}}\mathscr{S}_{n,r}$.*

(ii) *$\mathscr{S}_{n,r}$ is presented by generators $E_{(i,k)}, F_{(i,k)}$ ($(i,k) \in \Gamma'$) and 1_λ ($\lambda \in \Lambda_{n,r}$) with the defining relations (7.4.1)–(7.4.8).*

(iii) *$\mathscr{S}_{n,r}$ is also presented by generators e_i, f_i ($1 \leq i \leq m-1$) and $K_i^{\pm}$ ($1 \leq i \leq m$) with the defining relations (1.5.1)–(1.5.3), (1.5.6), (1.5.7), and (7.11.1)–(7.11.3).*

Proof. Through the surjection $\Phi : \mathcal{S}_{n,r} \to \mathscr{S}_{n,r}$, we can regard the simple $\mathscr{S}_{n,r}$-module $W(\lambda)$ ($\lambda \in \Lambda_{n,r}^{+}$) as a simple $\mathcal{S}_{n,r}$-module, and $\{W(\lambda) \mid \lambda \in \Lambda_{n,r}^{+}\}$ gives a complete set of isomorphism classes of simple $\mathcal{S}_{n,r}$-modules by Lemma 7.10. As $\widetilde{U}_q$-modules, both $\Delta(\lambda)$ and $W(\lambda)$ are highest-weight modules with a highest weight λ. Thus, by investigating the action on highest-weight vectors of $\Delta(\lambda)$ and $W(\lambda)$, we have a surjective homomorphism

$$\Delta(\lambda) \to W(\lambda) \quad \text{as } \mathcal{S}_{n,r}\text{-modules.} \tag{7.16.1}$$

We claim the following.

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Claim. *For any $\lambda \in \Lambda^+_{n,r}$, we have*

$$\Delta(\lambda) \cong W(\lambda) \text{ as left } \mathcal{S}_{n,r}\text{-modules},$$
$$\Delta^\sharp(\lambda) \cong W^\sharp(\lambda) \text{ as right } \mathcal{S}_{n,r}\text{-modules},$$
$$\Delta(\lambda) \otimes_{\mathcal{K}} \Delta^\sharp(\lambda) \cong \mathcal{S}_{n,r}(\geq \lambda)/\mathcal{S}_{n,r}(> \lambda) \text{ as } (\mathcal{S}_{n,r}, \mathcal{S}_{n,r})\text{-bimodules.}$$

If we assume the claim, then we have

$$\begin{aligned} \dim_{\mathcal{K}} \mathcal{S}_{n,r} &= \sum_{\lambda \in \Lambda^+_{n,r}} \big(\dim_{\mathcal{K}} \Delta(\lambda)\big)^2 \\ &= \sum_{\lambda \in \Lambda^+_{n,r}} \big(\dim_{\mathcal{K}} W(\lambda)\big)^2 \\ &= \dim_{\mathcal{K}} \mathscr{S}_{n,r}. \end{aligned}$$

This implies that Φ gives an isomorphism from $\mathcal{S}_{n,r}$ to $\mathscr{S}_{n,r}$. Thus, it is enough to show the claim.

We recall that

$$\Delta(\lambda) \cong \mathcal{S}^{\leq 0}_{n,r} \cdot 1_\lambda / \big(\mathcal{S}^{\leq 0}_{n,r} \cdot 1_\lambda \cap \mathcal{S}_{n,r}(> \lambda)\big), \tag{7.16.2}$$

$$W(\lambda) \cong \mathscr{S}^{\leq 0} \cdot \varphi^1_{\lambda,\lambda} / \big(\mathscr{S}^{\leq 0} \cdot \varphi^1_{\lambda,\lambda} \cap \mathscr{S}_{n,r}(> \lambda)\big) \tag{7.16.3}$$

as $\mathcal{K}$-vector spaces. Lemma 7.14 implies the following isomorphism:

$$\Phi\,|_{\mathcal{S}^{\leq 0}_{n,r} 1_\lambda} : \mathcal{S}^{\leq 0}_{n,r} 1_\lambda \cong \mathscr{S}^{\leq 0}_{n,r} \varphi^1_{\lambda,\lambda} \quad \text{as } \mathcal{K}\text{-vector spaces.} \tag{7.16.4}$$

We prove the claim by backward induction on the partial order of $\Lambda^+_{n,r}$.

First, we suppose that λ is maximal in $\Lambda^+_{n,r}$. In this case, we have $\mathcal{S}_{n,r}(> \lambda) = \{0\}$ and $\mathscr{S}_{n,r}(> \lambda) = \{0\}$. Thus, (7.16.1)–(7.16.4) imply that $\Delta(\lambda) \cong W(\lambda)$ as left $\mathcal{S}_{n,r}$-modules. Similarly, we have $\Delta^\sharp(\lambda) \cong W^\sharp(\lambda)$ as right $\mathcal{S}_{n,r}$-modules. Since $\Delta(\lambda)$ (resp., $\Delta^\sharp(\lambda)$) is a simple left (resp., right) $\mathcal{S}_{n,r}$-module, the surjective homomorphism of $\mathcal{S}_{n,r}$-bimodules $\Delta(\lambda) \otimes_{\mathcal{K}} \Delta^\sharp(\lambda) \to \mathcal{S}_{n,r}(\geq \lambda)/\mathcal{S}_{n,r}(> \lambda)$ is an isomorphism.

Next, we suppose that λ is not maximal in $\Lambda^+_{n,r}$. The induction hypothesis implies that the surjection $\Phi\,|_{\mathcal{S}_{n,r}(>\lambda)} : \mathcal{S}_{n,r}(> \lambda) \to \mathscr{S}_{n,r}(> \lambda)$ in Lemma 7.15 is an isomorphism by comparing dimensions. Combined with (7.16.1)–(7.16.4), this implies that $\Delta(\lambda) \cong W(\lambda)$ as left $\mathcal{S}_{n,r}$-modules. Similarly, we have $\Delta^\sharp(\lambda) \cong W^\sharp(\lambda)$ as right $\mathcal{S}_{n,r}$-modules. This implies that $\Delta(\lambda) \otimes_{\mathcal{K}}$

$\Delta^\sharp(\lambda) \cong \mathcal{S}_{n,r}(\geq \lambda)/\mathcal{S}_{n,r}(> \lambda)$. Thus, we have the claim and (i) follows. The remaining assertions (ii) and (iii) follow from Section 7.4 and Proposition 7.12. ∎

REMARKS 7.17. (i) In the case where $r = 1$, generators and defining relations of $\mathcal{S}_{n,r}$ (resp., $U_{n,r}$) in Section 7.4 (resp., 7.11) coincide with generators and defining relations of q-Schur algebras of type A in Theorem 4.10 (resp., Theorem 4.9).

(ii) By an argument similar to the case where $r = 1$ (see Remark 4.11), $\mathcal{S}_{n,r}$ ($\cong \mathscr{S}_{n,r}$) satisfies conditions (A-1), (A-2), and (C-1).

§8. An algorithm for computing decomposition numbers

In this section, we give an algorithm for computing the decomposition numbers of ${}_F\mathcal{S}_{n,r} \cong {}_F\mathscr{S}_{n,r}$ on an arbitrary field F and parameters $q, Q_1, \dots, Q_r \in F$. Throughout this section, we consider the objects over a fixed field F, so we will omit the subscript F (e.g., ${}_F\mathcal{S}_{n,r}$, ${}_F\Delta(\lambda), \dots$) unless it causes some confusion.

8.1.

Since $\mathcal{S}_{n,r}$ satisfies condition (C-1), we can define a bilinear form $\langle\, ,\, \rangle_\iota : \Delta(\lambda) \times \Delta(\lambda) \to F$ by

$$\langle \overline{y1_\lambda}, \overline{x1_\lambda} \rangle_\iota 1_\lambda \equiv \iota(y1_\lambda)x1_\lambda \mod \mathcal{S}_{n,r}(> \lambda) \qquad \text{for } x, y \in \mathcal{S}^-_{n,r}.$$

Note that $\langle\, ,\, \rangle_\iota$ is symmetric. Put $\operatorname{rad}_\iota \Delta(\lambda) = \{\overline{x} \in \Delta(\lambda) \mid \langle \overline{y}, \overline{x} \rangle_\iota = 0$ for any $\overline{y} \in \Delta(\lambda)\}$. One sees easily that $\langle \overline{y}, \overline{x} \rangle_\iota = \langle \overline{\iota(y)}, \overline{x} \rangle$ for $\overline{x}, \overline{y} \in \Delta(\lambda)$; thus, we have $\operatorname{rad}_\iota \Delta(\lambda) = \operatorname{rad} \Delta(\lambda)$. Hence, from now on we denote $\langle\, ,\, \rangle_\iota$ (resp., $\operatorname{rad}_\iota \Delta(\lambda)$) simply by $\langle\, ,\, \rangle$ (resp., $\operatorname{rad} \Delta(\lambda)$).

8.2.

For an $\mathcal{S}_{n,r}$-module M, we have the weight-space decomposition

$$M = \bigoplus_{\mu \in \Lambda_{n,r}} M_\mu,$$

where $M_\mu = 1_\mu \cdot M$. Since $\Delta(\lambda) = \mathcal{S}^-_{n,r} \cdot \overline{1_\lambda}$, we see that $\lambda \geq \mu$ if $\Delta(\lambda)_\mu \neq 0$. It is clear that $\Delta(\lambda)_\mu$ is spanned by

$$\Xi(\lambda - \mu) = \big\{ F^{(c_1)}_{(i_1,k_1)} F^{(c_2)}_{(i_2,k_2)} \cdots F^{(c_l)}_{(i_l,k_l)} \cdot \overline{1_\lambda} \mid c_1\alpha_{(i_1,k_1)} + c_2\alpha_{(i_2,k_2)} + \cdots + c_l\alpha_{(i_l,k_l)} = \lambda - \mu \big\}.$$

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Note that $\Xi(\lambda-\mu)$ is a finite set. Then we can pick up a homogeneous basis of $\Delta(\lambda)_\mu$ from $\Xi(\lambda-\mu)$. We take a homogeneous basis $\mathcal{B}(\lambda)_\mu$ of $\Delta(\lambda)_\mu$, and fix it.

For $\lambda \in \Lambda^+_{n,r}, \mu \in \Lambda_{n,r}$, let

$$M(\lambda)_\mu = (\langle \overline{b'}, \overline{b} \rangle)_{\overline{b}, \overline{b'} \in \mathcal{B}(\lambda)_\mu}$$

be the Gram matrix of the weight space $\Delta(\lambda)_\mu$. Put $\operatorname{rad}\Delta(\lambda)_\mu = \operatorname{rad}\Delta(\lambda) \cap \Delta(\lambda)_\mu$; then we have the following lemma.

LEMMA 8.3. *We have*

$$\dim_F \operatorname{rad}\Delta(\lambda)_\mu = \operatorname{corank} M(\lambda)_\mu.$$

Proof. For $\overline{x} \in \Delta(\lambda)_\mu$, $\overline{y} \in \Delta(\lambda)_\nu$, we have $\langle \overline{y}, \overline{x} \rangle = 0$ unless $\mu = \nu$ by (2.13.3). Thus, $\overline{x} \in \operatorname{rad}\Delta(\lambda)_\mu$ if and only if $\langle \overline{b'}, \overline{x} \rangle = 0$ for any $\overline{b'} \in \mathcal{B}(\lambda)_\mu$. This implies the lemma. □

Algorithm for computing decomposition numbers of $\mathcal{S}_{n,r}$

STEP 1. Compute the value of $\langle \overline{b'}, \overline{b} \rangle$ for all $\overline{b}, \overline{b'} \in \mathcal{B}(\lambda)_\mu$ $(\lambda \in \Lambda^+_{n,r}, \mu \in \Lambda_{n,r})$.

Note that by (2.13.1) and the definition of the bilinear form, we can compute $\langle \overline{b'}, \overline{b} \rangle$ by using the commutative relation (7.4.6) repeatedly.

STEP 2. Compute the corank of $M(\lambda)_\mu$ for all $\lambda \in \Lambda^+_{n,r}, \mu \in \Lambda_{n,r}$.

This is an elementary calculation of linear algebra.

STEP 3. Compute $\dim_F(L(\lambda)_\mu)$ for all $\lambda \in \Lambda^+_{n,r}$, $\mu \in \Lambda_{n,r}$.

Since $L(\lambda) = \Delta(\lambda)/\operatorname{rad}\Delta(\lambda)$, we have

$$\dim_F\big(L(\lambda)_\mu\big) = \dim_F\big(\Delta(\lambda)_\mu\big) - \dim_F\big(\operatorname{rad}\Delta(\lambda)_\mu\big).$$

Thus, we can compute $\dim_F(L(\lambda)_\mu)$ by Lemma 8.3 and Step 2.

STEP 4. Compute the decomposition numbers $d_{\lambda\mu} = [\Delta(\lambda) : L(\mu)]$ for $\lambda, \mu \in \Lambda^+_{n,r}$ by the following inductive process.

By Theorem 3.6, we have $d_{\lambda\lambda} = 1$ for $\lambda \in \Lambda^+_{n,r}$. By induction, we may assume that $d_{\lambda\mu}$ is known for $\mu \in \Lambda^+_{n,r}$ such that $\lambda \geq \mu > \nu$, and we compute the decomposition number $d_{\lambda\nu}$.

Note the following facts:

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- $\operatorname{rad}\Delta(\lambda)$ is the unique maximal $\mathcal{S}_{n,r}$-submodule of $\Delta(\lambda)$.
- $d_{\lambda\mu}\neq 0$ $(\lambda\neq\mu)$ only if $\lambda>\mu$.
- $L(\mu)_\nu\neq 0$ only if $\mu\geq\nu$.
- $\dim_F L(\nu)_\nu=1$.

These facts imply that

$$\text{(8.3.1)}\qquad \begin{aligned}\dim_F\bigl(\operatorname{rad}\Delta(\lambda)_\nu\bigr)&=\sum_{\mu\in\Lambda^+_{n,r}\setminus\{\lambda\}} d_{\lambda\mu}\cdot\bigl(\dim_F L(\mu)_\nu\bigr)\\&=\sum_{\substack{\mu\in\Lambda^+_{n,r}\\ \lambda>\mu>\nu}} d_{\lambda\mu}\cdot\bigl(\dim_F L(\mu)_\nu\bigr)+d_{\lambda\nu}.\end{aligned}$$

By Lemma 8.3 and Step 2, we know that $\dim_F(\operatorname{rad}\Delta(\lambda)_\nu)$. By the assumption of the induction together with Step 3, we know that $\sum_{\substack{\mu\in\Lambda^+_{n,r}\\ \lambda>\mu>\nu}} d_{\lambda\mu}\cdot(\dim_F L(\mu)_\nu)$. Thus, we can compute the decomposition number $d_{\lambda\nu}$ from equation (8.3.1).

REMARKS 8.4. (i) In fact, in order to compute the decomposition numbers, it is enough to consider the Gram matrix $M(\lambda)_\mu$ only for $\lambda,\mu\in\Lambda^+_{n,r}$ since we have

$$\dim_F L(\mu)_\nu=\dim_F\Delta(\mu)_\nu-\sum_{\tau\in\Lambda^+_{n,r}} d_{\mu\tau}\dim_F L(\tau)_\nu.$$

In this case, we should skip Step 3 and add the following process of another induction on $\Lambda^+_{n,r}$ in Step 4:

$$\begin{aligned}&d_{\mu\tau}\text{ is known for }\mu,\tau\in\Lambda^+_{n,r}\text{ such that }\lambda>\mu.\\&\quad\Leftrightarrow\ \dim_F L(\mu)_\nu\text{ is known for }\mu\in\Lambda^+_{n,r},\ \nu\in\Lambda_{n,r}\text{ such that }\lambda>\mu.\end{aligned}$$

(ii) Thanks to Theorem 3.4 and [DR2, Theorem 5.16(f)] (or directly by comparing the highest weights as $\widetilde{U}_q$-modules), we have ${}_F\Delta(\lambda)\cong{}_FW(\lambda)$ for $\lambda\in\Lambda^+_{n,r}$. In particular, we have ${}_F\Delta(\lambda)=F\otimes_{\mathcal{A}}{}_{\mathcal{A}}\Delta(\lambda)$ since it is known that ${}_FW(\lambda)=F\otimes_{\mathcal{A}}{}_{\mathcal{A}}W(\lambda)$.

(iii) Our algorithm can be applied to an arbitrary field which is not necessarily of characteristic 0.

(iv) In order to implement this algorithm, we need the following two data. One is a homogeneous basis $\mathcal{B}(\lambda)_\mu$ of $\Delta(\lambda)_\mu$. Though we can pick up a homogeneous basis $\mathcal{B}(\lambda)_\mu$ of $\Delta(\lambda)_\mu$ from the finite set $\Xi(\lambda-\mu)$, we do not

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know an algorithm to obtain such a basis. Another is a precise description of $\eta^{\lambda}_{(i,k)}$ ($(i,k)\in\Gamma'$, $\lambda\in\Lambda_{n,r}$) in relation (7.4.6) by using the generators of $\mathcal{S}_{n,r}$. Actually, we can compute $\eta^{\lambda}_{(i,k)}$ in $\mathscr{S}_{n,r}$ (see Lemma 7.2). However, the calculations take too long, and there is no algorithm to give a precise description of $\eta^{\lambda}_{(i,k)}$ at this time.

(v) There exists a surjective homomorphism ${}_{\mathcal{A}}\widetilde{U}_q^- \to {}_{\mathcal{A}}\mathcal{S}_q^-$ as algebras, and we have ${}_{\mathcal{A}}\widetilde{U}_q^- \cong {}_{\mathcal{A}}U_q^-$. Thus, we have a surjective homomorphism of ${}_{\mathcal{A}}U_q^-$-modules

$$ {}_{\mathcal{A}}U_q^- \to {}_{\mathcal{A}}\Delta(\lambda)\ (= {}_{\mathcal{A}}\mathcal{S}_q^-\cdot\overline{1_\lambda}) \quad \text{such that } 1\mapsto \overline{1_\lambda}. $$

It may be useful that we take a homogeneous basis of ${}_{\mathcal{A}}\Delta(\lambda)$ from the image of a certain homogeneous basis of ${}_{\mathcal{A}}U_q^-$ (e.g., monomial basis, Poincaré-Birkhoff-Witt (PBW) basis, canonical basis).

(vi) We can apply this algorithm to compute the decomposition numbers of ${}_F\mathcal{S}_q$ under the general setting in Section 3. Moreover, we can also apply it to compute the decomposition numbers of ${}_F\mathcal{S}_q$ associated to any Cartan matrix of finite type, which includes the generalized q-Schur algebra constructed in [Do].

Appendix A. A proof of Proposition 4.7

In this section, we give a proof of Proposition 4.7.

A.1.

Let V be a vector space over $\mathbb{Q}(q)$ with a basis $\{v_1,\dots,v_m\}$. Then, $U_q = U_q(\mathfrak{gl}_m)$ acts on V from the left by

$$ e_i\cdot v_j = \begin{cases} v_{j-1} & \text{if } j=i+1,\\ 0 & \text{otherwise;}\end{cases} $$

$$ f_i\cdot v_j = \begin{cases} v_{j+1} & \text{if } j=i,\\ 0 & \text{otherwise;}\end{cases} $$

$$ K_i^{\pm}\cdot v_j = \begin{cases} q^{\pm1}v_j & \text{if } j=i,\\ v_j & \text{otherwise.}\end{cases} $$

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This action is called the *vector representation* of U_q. We extend this action to a tensor space $V^{\otimes n}$ by using the comultiplication Δ of U_q defined by

$$\begin{aligned}\Delta(e_i) &= e_i \otimes K_i K_{i+1}^- + 1 \otimes e_i,\\ \Delta(f_i) &= f_i \otimes 1 + K_i^- K_{i+1} \otimes f_i,\\ \Delta(K_i^{\pm}) &= K_i^{\pm} \otimes K_i^{\pm}.\end{aligned}$$

We denote this action by $\rho' : U_q(\mathfrak{gl}_m) \to \mathrm{End}(V^{\otimes n})$.

On the other hand, $\mathscr{H}_n$ acts on $V^{\otimes n}$ from the right as follows. We define $\widetilde{T} \in \mathrm{End}(V \otimes V)^{\mathrm{op}}$ by

$$(v_i \otimes v_j) \cdot \widetilde{T} = \begin{cases} q v_i \otimes v_j & \text{if } i = j,\\ v_j \otimes v_i & \text{if } i < j,\\ v_j \otimes v_i + (q - q^{-1}) v_i \otimes v_j & \text{if } i > j,\end{cases}$$

where $\mathrm{End}(V \otimes V)^{\mathrm{op}}$ means the opposite algebra of $\mathrm{End}(V \otimes V)$. For $i = 1, \ldots, n-1$, we define $\widetilde{T}_i \in \mathrm{End}(V^{\otimes n})^{\mathrm{op}}$ by

$$\widetilde{T}_i = \mathrm{id}_V^{\otimes(i-1)} \otimes \widetilde{T} \otimes \mathrm{id}_V^{\otimes(n-1-i)}.$$

Then, we define an algebra homomorphism $\theta : \mathscr{H}_n \to \mathrm{End}(V^{\otimes n})^{\mathrm{op}}$ by $\theta(T_i) = \widetilde{T}_i$. By [J], it is known that the action of U_q and the action of $\mathscr{H}_n$ on $V^{\otimes n}$ commute. Moreover, we have

$$\rho'(U_q) = \mathrm{End}_{\mathscr{H}_n}(V^{\otimes n}).$$

A.2.

For $\mu = (\mu_1, \ldots, \mu_m) \in \Lambda_{n,1}$, let $V_\mu^{\otimes n}$ be the subspace of $V^{\otimes n}$ spanned by $\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \mid \mu_j = \sharp\{k \mid i_k = j\} \text{ for } j = 1, \ldots, m\}$. One sees easily that $V_\mu^{\otimes n}$ is the weight space of $V^{\otimes n}$ with weight μ as a U_q-module, and we have the weight-space decomposition

$$V^{\otimes n} = \bigoplus_{\mu \in \Lambda_{n,1}} V_\mu^{\otimes n}.$$

Since the action of $\mathscr{H}_n$ commutes with the action of U_q, $V_\mu^{\otimes n}$ is invariant under the action of $\mathscr{H}_n$. For $\mu \in \Lambda_{n,1}$, put

$$v_\mu = \underbrace{v_1 \otimes \cdots \otimes v_1}_{\mu_1 \text{ terms}} \otimes \underbrace{v_2 \otimes \cdots \otimes v_2}_{\mu_2 \text{ terms}} \otimes \cdots \otimes \underbrace{v_m \otimes \cdots \otimes v_m}_{\mu_m \text{ terms}}.$$

Then, we have $V_\mu^{\otimes n} = v_\mu \cdot \mathscr{H}_n$. Moreover, one can check that there exists an isomorphism $V_\mu^{\otimes n} \to M^\mu$ of $\mathscr{H}_n$-modules such that $v_\mu \mapsto x_\mu$. Thus, we have

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the following isomorphism of algebras:

$$\begin{aligned}\rho'(U_q) &= \mathrm{End}_{\mathscr{H}_n}(V^{\otimes n}) \\ &= \mathrm{End}_{\mathscr{H}_n}\Big(\bigoplus_{\mu\in\Lambda_{n,1}} V_\mu^{\otimes n}\Big) \\ &\cong \mathrm{End}_{\mathscr{H}_n}\Big(\bigoplus_{\mu\in\Lambda_{n,1}} M^\mu\Big).\end{aligned}$$

This isomorphism gives the surjection $\rho : U_q \to \mathscr{S}_{n,1}$ in Theorem 4.6.

A.3.
For $\mu \in \Lambda_{n,1}$, put

$$\begin{aligned}A &= \underbrace{v_1\otimes\cdots\otimes v_1}_{\mu_1\text{ terms}}\otimes\underbrace{v_2\otimes\cdots\otimes v_2}_{\mu_2\text{ terms}}\otimes\cdots\otimes\underbrace{v_i\otimes\cdots\otimes v_i}_{\mu_i\text{ terms}},\\ B &= \underbrace{v_{i+2}\otimes\cdots\otimes v_{i+2}}_{\mu_{i+2}\text{ terms}}\otimes\underbrace{v_{i+3}\otimes\cdots\otimes v_{i+3}}_{\mu_{i+3}\text{ terms}}\otimes\cdots\otimes\underbrace{v_m\otimes\cdots\otimes v_m}_{\mu_m\text{ terms}}.\end{aligned}$$

Then, we have

$$\begin{aligned}v_\mu &= A\otimes\underbrace{v_{i+1}\otimes\cdots\otimes v_{i+1}}_{\mu_{i+1}\text{ terms}}\otimes B,\\ v_{\mu+\alpha_i} &= A\otimes v_i\otimes\underbrace{v_{i+1}\otimes\cdots\otimes v_{i+1}}_{\mu_{i+1}-1\text{ terms}}\otimes B.\end{aligned}$$

By the definitions, one can compute that

$$\begin{aligned}\rho'(e_i)(v_\mu) &= \sum_{j=1}^{\mu_{i+1}} q^{-(\mu_{i+1}-j)} A\otimes\underbrace{v_{i+1}\otimes\cdots\otimes v_{i+1}}_{j-1\text{ terms}}\otimes v_i\otimes\underbrace{v_{i+1}\otimes\cdots\otimes v_{i+1}}_{\mu_{i+1}-j\text{ terms}}\otimes B\\ &= q^{-\mu_{i+1}+1}\sum_{x\in X^\mu_{\mu+\alpha_i}} q^{\ell(x)} v_{(\mu+\alpha_i)}\cdot T_x.\end{aligned}$$

Under the isomorphism $V_\mu^{\otimes n}\cong M^\mu$, this implies that $\rho(e_i)(m_\mu) = q^{-\mu_{i+1}+1}\times \psi^1_{\mu+\alpha_i,\mu}(m_\mu)$. Thus, we have Proposition 4.7(i). We can prove Proposition 4.7(ii),(iii) in a similar way.

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Appendix B. Example: Cyclotomic q-Schur algebra of type $G(2,1,2)$

In this appendix, we consider a cyclotomic q-Schur algebra $\mathscr{S}_{2,2}$ of type $G(2,1,2)$, namely, that associated to the complex reflection group $\mathfrak{S}_2 \ltimes (\mathbb{Z}/2\mathbb{Z})^2$. In this case, we will describe elements $\eta^{\lambda}_{(i,k)}$ explicitly and compute the Gram matrices $M(\lambda)_{\mu}$ and decomposition numbers of ${}_{\mathbb{C}}\mathscr{S}_{2,2}$. Throughout this appendix, we replace γ_i with Q_i $(i=1,2)$; thus, $\mathscr{S}_{2,2}$ is an algebra over $\mathcal{K}=(q,Q_1,Q_2)$, where q,Q_1,Q_2 are indeterminate elements.

B.1.

The cyclotomic q-Schur algebra $\mathscr{S}_{2,2}$ of type $G(2,1,2)$ is generated by the generators $E_{(1,1)},E_{(2,1)},E_{(1,2)},F_{(1,1)},F_{(2,1)},F_{(1,2)},1_{\lambda}(\lambda\in\Lambda)$, where

$$\Lambda=\left\{\begin{matrix}\lambda_{\langle 0\rangle}=((2,0),(0,0)), & \lambda_{\langle 1\rangle}=((1,1),(0,0)), & \lambda_{\langle 2\rangle}=((1,0),(1,0)),\\ \lambda_{\langle 3\rangle}=((1,0),(0,1)), & \lambda_{\langle 4\rangle}=((0,2),(0,0)), & \lambda_{\langle 5\rangle}=((0,1),(1,0)),\\ \lambda_{\langle 6\rangle}=((0,1),(0,1)), & \lambda_{\langle 7\rangle}=((0,0),(2,0)), & \lambda_{\langle 8\rangle}=((0,0),(1,1)),\\ \lambda_{\langle 9\rangle}=((0,0),(0,2)) & & \end{matrix}\right\},$$

with the defining relations (7.4.1)–(7.4.8). By Lemma 7.10, we have

$$\Lambda^{+}=\{\lambda_{\langle 0\rangle},\lambda_{\langle 1\rangle},\lambda_{\langle 2\rangle},\lambda_{\langle 7\rangle},\lambda_{\langle 8\rangle}\}.$$

By Lemma 7.2 and (7.3.1), we have

$$\begin{aligned}
g^{\lambda_{\langle 1\rangle}}_{(2,1)}(F,E)&=Q_1\big((q-q^{-1})F_{(1,1)}E_{(1,1)}+q^{-2}\big),\\
g^{\lambda_{\langle 4\rangle}}_{(2,1)}(F,E)&=Q_1(q^2+1),\\
g^{\lambda_{\langle 5\rangle}}_{(2,1)}(F,E)&=Q_1,\\
g^{\lambda_{\langle 6\rangle}}_{(2,1)}(F,E)&=Q_1,\\
g^{\lambda_{\langle 2\rangle}}_{(1,2)}(F,E)&=F_{(2,1)}E_{(2,1)}+Q_2,\\
g^{\lambda_{\langle 5\rangle}}_{(1,2)}(F,E)&=F_{(1,1)}F_{(2,1)}E_{(2,1)}E_{(1,1)}+Q_2,\\
g^{\lambda_{\langle 7\rangle}}_{(1,2)}(F,E)&=qF_{(2,1)}E_{(2,1)}+Q_2(1+q^2),\\
g^{\lambda_{\langle 8\rangle}}_{(1,2)}(F,E)&=F_{(2,1)}E_{(2,1)}+Q_2;
\end{aligned}$$

$g^{\lambda}_{(2,1)}(F,E)$ (resp., $g^{\lambda}_{(1,2)}(F,E)$), which does not appear in the above list, is equal to 0.

As an example, we compute only $g^{\lambda_{\langle 1\rangle}}_{(2,1)}(F,E)$. By the definitions, we have

$$
\begin{aligned}
&\sigma^{\lambda_{\langle 1\rangle}}_{(2,1)}(m_{\lambda_{\langle 1\rangle}})\\
&\quad = m_{\lambda_{\langle 1\rangle}} L_2\\
&\quad = (L_1 - Q_2)(L_2 - Q_2)T_1 L_1 T_1\\
&\quad = T_1(L_1 - Q_2)L_1(L_2 - Q_2)T_1 \quad (\because \text{ Lemma 5.4(i),(iv)})\\
&\quad = Q_1 T_1 (L_1 - Q_2)(L_2 - Q_2)T_1\\
&\quad = Q_1(L_1 - Q_2)(L_2 - Q_2)\big((q - q^{-1})T_1 + 1\big) \quad (\because T_1^2 = (q - q^{-1})T_1 + 1)\\
&\quad = Q_1\big((q - q^{-1})m_{\lambda_{\langle 1\rangle}} T_1 + m_{\lambda_{\langle 1\rangle}}\big),
\end{aligned}
$$

where the fourth equality follows from $L_1 = T_0$ and $T_0^2 = (Q_1 + Q_2)T_0 - Q_1Q_2$. On the other hand, we have

$$
\begin{aligned}
\varphi^-_{(1,1)}\varphi^+_{(1,1)}(m_{\lambda_{\langle 1\rangle}}) &= q^{-1} m_{\lambda_{\langle 1\rangle}}(1 + qT_1)\\
&= m_{\lambda_{\langle 1\rangle}} T_1 + q^{-1} m_{\lambda_{\langle 1\rangle}}.
\end{aligned}
$$

Thus, we have $\sigma^{\lambda_{\langle 1\rangle}}_{(2,1)} = Q_1((q - q^{-1})\varphi^-_{(1,1)}\varphi^+_{(1,1)} + q^{-2})\varphi^1_{\lambda_{\langle 1\rangle},\lambda_{\langle 1\rangle}}$. This implies that

$$
g^{\lambda_{\langle 1\rangle}}_{(2,1)}(F,E) = Q_1\big((q - q^{-1})F_{(1,1)}E_{(1,1)} + q^{-2}\big).
$$

Since $\eta^{\lambda}_{(2,1)} = \big(-Q_2[\lambda^{(1)}_2 - \lambda^{(2)}_1] + q^{\lambda^{(1)}_2 - \lambda^{(2)}_1}(q^{-1}g^{\lambda}_{(2,1)}(F,E) - qg^{\lambda}_{(1,2)}(F,E))\big)1_{\lambda}$, we have

$$
\begin{aligned}
\eta^{\lambda_{\langle 1\rangle}}_{(2,1)} &= \big(Q_1(q - q^{-1})F_{(1,1)}E_{(1,1)} + (Q_1 q^{-2} - Q_2)\big)1_{\lambda_{\langle 1\rangle}},\\
\eta^{\lambda_{\langle 2\rangle}}_{(2,1)} &= -F_{(2,1)}E_{(2,1)}1_{\lambda_{\langle 2\rangle}},\\
\eta^{\lambda_{\langle 4\rangle}}_{(2,1)} &= \big(Q_1(q^3 + q) - Q_2(q + q^{-1})\big)1_{\lambda_{\langle 4\rangle}},\\
\eta^{\lambda_{\langle 5\rangle}}_{(2,1)} &= \big(-qF_{(1,1)}F_{(2,1)}E_{(2,1)}E_{(1,1)} + (Q_1 q^{-1} - Q_2 q)\big)1_{\lambda_{\langle 5\rangle}},\\
\eta^{\lambda_{\langle 6\rangle}}_{(2,1)} &= (Q_1 - Q_2)1_{\lambda_{\langle 6\rangle}},\\
\eta^{\lambda_{\langle 7\rangle}}_{(2,1)} &= -F_{(2,1)}E_{(2,1)}1_{\lambda_{\langle 7\rangle}},
\end{aligned}
$$

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$$\eta_{(2,1)}^{\lambda_{\langle 8\rangle}} = -F_{(2,1)}E_{(2,1)}1_{\lambda_{\langle 8\rangle}},$$
$$\eta_{(2,1)}^{\lambda_{\langle 0\rangle}} = \eta_{(2,1)}^{\lambda_{\langle 3\rangle}} = \eta_{(2,1)}^{\lambda_{\langle 9\rangle}} = 0.$$

B.2.

We can take a homogeneous basis of ${}_{\mathcal{A}}\Delta(\lambda)$ for $\lambda \in \Lambda^{+}$ as follows.

Basis of ${}_{\mathcal{A}}\Delta(\lambda_{\langle 0\rangle})$	
Weight	Basis
$\lambda_{\langle 0\rangle}$	$\overline{1_{\lambda_{\langle 0\rangle}}}$
$\lambda_{\langle 1\rangle}$	$\overline{F_{(1,1)}1_{\lambda_{\langle 0\rangle}}}$
$\lambda_{\langle 2\rangle}$	$\overline{F_{(2,1)}F_{(1,1)}1_{\lambda_{\langle 0\rangle}}}$
$\lambda_{\langle 3\rangle}$	$\overline{F_{(1,2)}F_{(2,1)}F_{(1,1)}1_{\lambda_{\langle 0\rangle}}}$
$\lambda_{\langle 4\rangle}$	$\overline{F_{(1,1)}^{(2)}1_{\lambda_{\langle 0\rangle}}}$
$\lambda_{\langle 5\rangle}$	$\overline{F_{(2,1)}F_{(1,1)}^{(2)}1_{\lambda_{\langle 0\rangle}}}$
$\lambda_{\langle 6\rangle}$	$\overline{F_{(1,2)}F_{(2,1)}F_{(1,1)}^{(2)}1_{\lambda_{\langle 0\rangle}}}$
$\lambda_{\langle 7\rangle}$	$\overline{F_{(2,1)}^{(2)}F_{(1,1)}^{(2)}1_{\lambda_{\langle 0\rangle}}}$
$\lambda_{\langle 8\rangle}$	$\overline{F_{(1,2)}F_{(2,1)}^{(2)}F_{(1,1)}^{(2)}1_{\lambda_{\langle 0\rangle}}}$
$\lambda_{\langle 9\rangle}$	$\overline{F_{(1,2)}^{(2)}F_{(2,1)}^{(2)}F_{(1,1)}^{(2)}1_{\lambda_{\langle 0\rangle}}}$

Basis of ${}_{\mathcal{A}}\Delta(\lambda_{\langle 1\rangle})$	
Weight	Basis
$\lambda_{\langle 1\rangle}$	$\overline{1_{\lambda_{\langle 1\rangle}}}$
$\lambda_{\langle 2\rangle}$	$\overline{F_{(2,1)}1_{\lambda_{\langle 1\rangle}}}$
$\lambda_{\langle 3\rangle}$	$\overline{F_{(1,2)}F_{(2,1)}1_{\lambda_{\langle 1\rangle}}}$
$\lambda_{\langle 5\rangle}$	$\overline{F_{(1,1)}F_{(2,1)}1_{\lambda_{\langle 1\rangle}}}$
$\lambda_{\langle 6\rangle}$	$\overline{F_{(1,2)}F_{(1,1)}F_{(2,1)}1_{\lambda_{\langle 1\rangle}}}$
$\lambda_{\langle 8\rangle}$	$\overline{F_{(2,1)}F_{(1,2)}F_{(1,1)}F_{(2,1)}1_{\lambda_{\langle 1\rangle}}}$

Basis of ${}_{\mathcal{A}}\Delta(\lambda_{\langle 2\rangle})$	
Weight	Basis
$\lambda_{\langle 2\rangle}$	$\overline{1_{\lambda_{\langle 2\rangle}}}$
$\lambda_{\langle 3\rangle}$	$\overline{F_{(1,2)}1_{\lambda_{\langle 2\rangle}}}$
$\lambda_{\langle 5\rangle}$	$\overline{F_{(1,1)}1_{\lambda_{\langle 2\rangle}}}$
$\lambda_{\langle 6\rangle}$	$\overline{F_{(1,2)}F_{(1,1)}1_{\lambda_{\langle 2\rangle}}}$
$\lambda_{\langle 7\rangle}$	$\overline{F_{(2,1)}F_{(1,1)}1_{\lambda_{\langle 2\rangle}}}$
$\lambda_{\langle 8\rangle}$	$\overline{F_{(2,1)}F_{(1,2)}F_{(1,1)}1_{\lambda_{\langle 2\rangle}}}$, $\overline{F_{(1,2)}F_{(2,1)}F_{(1,1)}1_{\lambda_{\langle 2\rangle}}}$
$\lambda_{\langle 9\rangle}$	$\overline{F_{(1,2)}F_{(2,1)}F_{(1,2)}F_{(1,1)}1_{\lambda_{\langle 2\rangle}}}$

Basis of ${}_{\mathcal{A}}\Delta(\lambda_{\langle 7\rangle})$	
Weight	Basis
$\lambda_{\langle 7\rangle}$	$\overline{1_{\lambda_{\langle 7\rangle}}}$
$\lambda_{\langle 8\rangle}$	$\overline{F_{(1,2)}1_{\lambda_{\langle 7\rangle}}}$,
$\lambda_{\langle 9\rangle}$	$\overline{F_{(1,2)}^{(2)}1_{\lambda_{\langle 7\rangle}}}$

Basis of ${}_{\mathcal{A}}\Delta(\lambda_{\langle 8\rangle})$	
Weight	Basis
$\lambda_{\langle 8\rangle}$	$\overline{1_{\lambda_{\langle 8\rangle}}}$

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B.3.

We can compute the Gram matrix of ${}_{\mathcal{A}}\Delta(\lambda)_\mu$ $\lambda,\mu\in\Lambda^+$ with respect to the above basis. Here, as an example, we compute $M(\lambda_{\langle 0\rangle})_{\lambda_{\langle 2\rangle}}$. Note that ${}_{\mathcal{A}}\Delta(\lambda_{\langle 0\rangle})_{\langle 2\rangle}$ has a basis $\{\overline{F_{(2,1)}F_{(1,1)}1_{\lambda_{\langle 0\rangle}}}\}$. We have

$$\begin{aligned}
&1_{\lambda_{\langle 0\rangle}}E_{(1,1)}E_{(2,1)}F_{(2,1)}F_{(1,1)}1_{\lambda_{\langle 0\rangle}}\\
&\quad= E_{(1,1)}\big(Q_1(q-q^{-1})F_{(1,1)}E_{(1,1)} + (Q_1q^{-2}-Q_2)\big)F_{(1,1)}1_{\lambda_{\langle 0\rangle}}\\
&\quad= \big(Q_1(q-q^{-1})[2][2] + (Q_1q^{-2}-Q_2)[2]\big)1_{\lambda_{\langle 0\rangle}}\\
&\qquad(\because E_{(1,1)}F_{(1,1)}1_{\lambda_{\langle 0\rangle}} = [2]1_{\lambda_{\langle 0\rangle}})\\
&\quad= [2](Q_1q^2-Q_2)1_{\lambda_{\langle 0\rangle}}.
\end{aligned}$$

This implies that $\langle\overline{F_{(2,1)}F_{(1,1)}1_{\lambda_{\langle 0\rangle}}},\overline{F_{(2,1)}F_{(1,1)}1_{\lambda_{\langle 0\rangle}}}\rangle = [2](Q_1q^2-Q_2)$. Thus, we have $M(\lambda_{\langle 0\rangle})_{\lambda_{\langle 2\rangle}} = ([2](Q_1q^2-Q_2))$.

In a similar way, we can compute the Gram matrix $M(\lambda)_\mu$ for $\lambda,\mu\in\Lambda^+_{n,r}$, and we have

$\Delta(\lambda_{\langle 0\rangle})$:

$$\begin{aligned}
M(\lambda_{\langle 0\rangle})_{\lambda_{\langle 1\rangle}} &= \big([2]\big)\\
M(\lambda_{\langle 0\rangle})_{\lambda_{\langle 2\rangle}} &= \big([2](q^2Q_1-Q_2)\big)\\
M(\lambda_{\langle 0\rangle})_{\lambda_{\langle 7\rangle}} &= \big((Q_1-Q_2)(q^2Q_1-Q_2)\big)\\
M(\lambda_{\langle 0\rangle})_{\lambda_{\langle 8\rangle}} &= \big([2](Q_1-Q_2)(q^2Q_1-Q_2)\big)
\end{aligned}$$

$\Delta(\lambda_{\langle 1\rangle})$:

$$\begin{aligned}
M(\lambda_{\langle 1\rangle})_{\lambda_{\langle 2\rangle}} &= \big((q^{-2}Q_1-Q_2)\big)\\
M(\lambda_{\langle 1\rangle})_{\lambda_{\langle 8\rangle}} &= \big((Q_1-Q_2)(q^{-2}Q_1-Q_2)\big)
\end{aligned}$$

$\Delta(\lambda_{\langle 2\rangle})$:

$$\begin{aligned}
M(\lambda_{\langle 2\rangle})_{\lambda_{\langle 7\rangle}} &= \big(q(q^{-2}Q_1-Q_2)\big)\\
M(\lambda_{\langle 2\rangle})_{\lambda_{\langle 8\rangle}} &= \begin{pmatrix}(Q_1-Q_2) & q(q^{-2}Q_1-Q_2)\\ q(q^{-2}Q_1-Q_2) & [2]q(q^{-2}Q_1-Q_2)\end{pmatrix}\\
&\big(\det M(\lambda_{\langle 2\rangle})_{\lambda_{\langle 8\rangle}} = (q^{-2}Q_1-Q_2)(q^2Q_1-Q_2)\big)
\end{aligned}$$

$\Delta(\lambda_{\langle 7\rangle})$:

$$M(\lambda_{\langle 7\rangle})_{\lambda_{\langle 8\rangle}} = \big([2]\big)$$

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B.4.

Let $\mathcal{A} \to \mathbb{C}$ be a ring homomorphism, and we express the image of q, Q_1, Q_2 in $\mathbb{C}$ by the same symbol. We can compute the decomposition numbers of ${}_{\mathbb{C}}\mathscr{S}_{2,2} = \mathbb{C} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathscr{S}_{2,2}$ by using the algorithm in Section 8, and we have the following decomposition matrix of ${}_{\mathbb{C}}\mathscr{S}_{2,2}$.

($q^2 \neq \pm 1, 0,\ Q_1 = Q_2 \neq 0$)

$\Delta(\lambda) \backslash L(\mu)$	λ_8	λ_7	λ_2	λ_1	λ_0
λ_8	1				
λ_7	0	1			
λ_2	0	0	1		
λ_1	1	0	0	1	
λ_0	0	1	0	0	1

($q^2 \neq \pm 1, 0,\ q^{-2}Q_1 = Q_2 \neq 0$)

$\Delta(\lambda) \backslash L(\mu)$	λ_8	λ_7	λ_2	λ_1	λ_0
λ_8	1				
λ_7	0	1			
λ_2	0	1	1		
λ_1	0	0	1	1	
λ_0	0	0	0	0	1

($q^2 \neq \pm 1, 0,\ q^2Q_1 = Q_2 \neq 0$)

$\Delta(\lambda) \backslash L(\mu)$	λ_8	λ_7	λ_2	λ_1	λ_0
λ_8	1				
λ_7	0	1			
λ_2	1	0	1		
λ_1	0	0	0	1	
λ_0	0	0	1	0	1

($q^2 = -1,\ \pm Q_1 \neq Q_2$)

$\Delta(\lambda) \backslash L(\mu)$	λ_8	λ_7	λ_2	λ_1	λ_0
λ_8	1				
λ_7	1	1			
λ_2	0	0	1		
λ_1	0	0	0	1	
λ_0	0	0	0	1	1

($q^2 = -1,\ Q_1 = Q_2 \neq 0$)

$\Delta(\lambda) \backslash L(\mu)$	λ_8	λ_7	λ_2	λ_1	λ_0
λ_8	1				
λ_7	1	1			
λ_2	0	0	1		
λ_1	1	0	0	1	
λ_0	1	1	0	1	1

($q^2 = -1,\ -Q_1 = Q_2 \neq 0$)

$\Delta(\lambda) \backslash L(\mu)$	λ_8	λ_7	λ_2	λ_1	λ_0
λ_8	1				
λ_7	1	1			
λ_2	1	1	1		
λ_1	0	0	1	1	
λ_0	0	1	1	1	1

($q^2 = 1,\ Q_1 = Q_2 = 0$)

$\Delta(\lambda) \backslash L(\mu)$	λ_8	λ_7	λ_2	λ_1	λ_0
λ_8	1				
λ_7	0	1			
λ_2	1	1	1		
λ_1	1	0	1	1	
λ_0	0	1	1	0	1

($q^2 \neq -1, 0,\ Q_1 = Q_2 = 0$)

$\Delta(\lambda) \backslash L(\mu)$	λ_8	λ_7	λ_2	λ_1	λ_0
λ_8	1				
λ_7	0	1			
λ_2	1	1	1		
λ_1	1	0	1	1	
λ_0	0	1	1	0	1

($q^2 = -1,\ Q_1 = Q_2 = 0$)

$\Delta(\lambda) \backslash L(\mu)$	λ_8	λ_7	λ_2	λ_1	λ_0
λ_8	1				
λ_7	1	1			
λ_2	2	1	1		
λ_1	1	0	1	1	
λ_0	1	1	1	1	1

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Appendix C. Example: The case of $\eta_i^\lambda = 0$

In this appendix, we give an extreme example of an $\mathcal{S}_q$ which is not a cyclotomic q-Schur algebra. In this example, we see that the isomorphism classes of nonisomorphic simple $\mathcal{S}_q$-modules are indexed by $\varLambda$ (i.e., $\varLambda^+ = \varLambda$), and all simple $\mathcal{S}_q$-modules are 1-dimensional. Moreover, this is an example such that condition (C-2) does not hold. We also see that $\mathcal{S}_q$ is not semisimple over any field and parameters.

We take $\mathcal{K} = \mathbb{Q}(q)$. Put $\varLambda = \{\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}_{\geq 0}^m \mid \lambda_1 + \cdots + \lambda_m = n\}$, and $\eta_i^\lambda = 0$ for any $i = 1, \dots, m-1$ and $\lambda \in \varLambda$. Then, $\mathcal{S}_q = \mathcal{S}_q^{\eta_\varLambda}$ is the algebra generated by E_i, F_i $(1 \leq i \leq m-1)$ and 1_λ $(\lambda \in \varLambda)$ with the defining relations (2.1.1)–(2.1.6), (2.1.8), and (2.1.9) together with the relation

$$E_i F_j - F_j E_i = 0. \tag{2.1.7$'$}$$

In this case, one sees easily that $\varLambda = \varLambda^+$. We denote a monomial of F_i (resp., E_i) for $i = 1, \dots, m-1$ by $X(F)$ (resp., $Y(E)$). Then, one sees that

$$X(F) 1_\lambda \notin \mathcal{S}_q(> \lambda) \qquad \big(\text{resp., } 1_\lambda Y(E) \notin \mathcal{S}_q(> \lambda)\big),$$

if $\lambda + \deg(X(F)) \in \varLambda$ (resp., $\lambda - \deg(Y(E)) \in \varLambda$). On the other hand, we have

$$\begin{aligned} X(F) 1_\lambda Y(E) &= X(F) Y(E) 1_{\lambda - \deg(Y(E))} \\ &= Y(E) X(F) 1_{\lambda - \deg(Y(E))} \\ &= Y(E) 1_{\lambda - \deg(Y(E)) + \deg(X(F))} X(F). \end{aligned}$$

Thus, we have $X(F) 1_\lambda Y(E) = 0$ if $\lambda - \deg(Y(E)) + \deg(X(F)) \notin \varLambda$. It happens that $\lambda + \deg(X(F)) \in \varLambda$, $\lambda - \deg(Y(E)) \in \varLambda$, and $\lambda - \deg(Y(E)) + \deg(X(F)) \notin \varLambda$. This shows that the natural surjection $\Delta(\lambda) \otimes_{\mathcal{K}} \Delta^\sharp(\lambda) \to \mathcal{S}_q(\geq \lambda)/\mathcal{S}_q(> \lambda)$ is not an isomorphism in general. (Note that (C-2) $\Leftrightarrow$ (C$'$-2).)

For $\lambda, \mu \in \varLambda^+ (= \varLambda)$, one sees that

$$M(\lambda)_\mu = 0 \qquad \text{unless } \lambda = \mu,$$

where 0 means the zero matrix. This implies that $\dim_{\mathcal{K}} L(\lambda)_\mu = 0$ unless $\lambda = \mu$, and that

$$[\Delta(\lambda) : L(\mu)] = \dim_{\mathcal{K}} \Delta(\lambda)_\mu.$$

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