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Abstract

A recent paper of Totaro developed a theory of q-ample bundles in characteristic 0. Specifically, a line bundle L on X is q-ample if for every coherent sheaf $\mathcal{F}$ on X, there exists an integer m_0 such that $m \geqslant m_0$ implies $H^i(X, \mathcal{F} \otimes \mathcal{O}(mL)) = 0$ for $i > q$. We show that a line bundle L on a complex projective scheme X is q-ample if and only if the restriction of L to its augmented base locus is q-ample. In particular, when X is a variety and L is big but fails to be q-ample, then there exists a codimension-one subscheme D of X such that the restriction of L to D is not q-ample.

1. Introduction

A recent paper of Totaro [Tot10] generalized the notion of an ample line bundle, with the object of relating cohomological, numerical, and geometric properties of these line bundles. Let q be a natural number. Totaro called a line bundle L on X q-ample if for every coherent sheaf $\mathcal{F}$ on X, there exists an integer m_0 such that $m \geqslant m_0$ implies $H^i(X, \mathcal{F} \otimes \mathcal{O}(mL)) = 0$ for $i > q$.

Totaro [Tot10] has shown that in characteristic 0, this notion of q-amplitude is equivalent to others previously studied by Demailly, Peternell, and Schneider in [DPS96]. As a result, the q-amplitude of a line bundle depends only on its numerical class, and the cone of such bundles is open. This means that there is some hope of recovering geometric and numerical information about X and its subvarieties from knowing when a line bundle is q-ample, though at present such results are known only in limited cases. In general, much is known about the 0-ample cone (which is the ample cone) and the $(n-1)$-ample cone of an n-dimensional variety X is known to be the negative of the complement of the pseudoeffective cone of X. For values of q between 1 and $n-2$, the relation between numerical and cohomological data remains mysterious. The Kleiman criterion tells us that 0-amplitude is determined by the restriction of L to the irreducible curves on X, and likewise one gets at least some information about the q-ample cone by looking at restrictions to $(q+1)$-dimensional subvarieties.

However, Totaro [Tot10] has given an example of a smooth toric 3-fold with a line bundle L which is not in the closure of the 1-ample cone, but the restriction of L to every two-dimensional subvariety is in the closure of the 1-ample cone of each subvariety. For completeness, we include this example in § 5. The example shows that the most direct generalization of Kleiman's criterion does not hold for even the first open case: the 1-ample cone of a 3-fold.

The goal of this note is to show that one can in fact test q-amplitude on proper subschemes in the case where L is a big line bundle on a projective variety X. In particular, we show that if L is a big line bundle which is not q-ample, and D is the locus of vanishing of a negative twist of L, then the restriction of L to D is not q-ample either. In a recent paper [Kür10], Küronya proved

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a sort of Fujita vanishing theorem for line bundles whose augmented base locus has dimension at most q. As a consequence, he showed that if the augmented base locus of L has dimension q, then L is q-ample. We prove the following result.

Theorem 1.1. *Let X be a complex projective scheme and let L be a line bundle on X. Let Y be the scheme given by the augmented base locus of L with the unique scheme structure as a reduced closed subscheme of X. Then L is q-ample on X if and only if the restriction of L to Y is q-ample.*

Matsumura has shown in [Mat11] that a line bundle admits a Hermitian metric whose curvature form has all but q eigenvalues positive at every point if and only if it admits such a metric when restricted to the augmented base locus. A line bundle with such a metric is q-ample, but it is unknown in general whether every q-ample line bundle admits such a metric.

We also prove a Kleiman-type criterion for $(n-2)$-amplitude for big divisors when X is smooth.

Corollary 1.2. *Let X be a non-singular projective variety. A big line bundle L on X is $(n-2)$-ample if and only if the restriction of $-L$ to every irreducible codimension-one subvariety is not pseudoeffective.*

When X is a 3-fold, a big line bundle L is 1-ample if and only if its dual is not in the pseudoeffective cone when restricted to any surface contained in X. Since a big line bundle on a 3-fold is always 2-ample, our results give a complete description of the intersection of the q-ample cones with the big cone of a 3-fold in terms of restriction to subvarieties.

In the final section, we examine possible geometric criteria for an effective line bundle to be q-ample. In particular, on an n-dimensional Cohen–Macaulay variety, any line bundle which admits a disconnected section must fail to be $(n-2)$-ample. This fact in particular helps to explain some features of Totaro's example, and may lead to more general criteria for q-amplitude.

2. The restriction theorem

In this section, we prove that a line bundle L which fails to be q-ample is still not q-ample when restricted to any section of $L-H$, where H is any ample line bundle.

Theorem 2.1. *Let X be a reduced projective scheme over $\mathbb{C}$. Suppose L is a line bundle on X which is not q-ample on X, and let L' be a line bundle with a non-zero section such that $\mathcal{O}(\alpha L - \beta L')$ is ample for some positive integers α, β. Let D be the subscheme of X given by the vanishing of some non-zero section of L'. Then $L|_D$ is not q-ample on D.*

Before proving Theorem 2.1, we will need a lemma.

Lemma 2.2. *Let X be a projective scheme over $\mathbb{C}$. Fix an ample line bundle H on X. Suppose L is a q-ample line bundle on X for some $q \geqslant 0$. Then for every coherent sheaf $\mathcal{F}$ on X there exist integers a_0 and b_0 such that given $a, b \geqslant 0$, $H^i(X, \mathcal{F} \otimes \mathcal{O}(aL + bH)) = 0$ for $i > q$ whenever $a \geqslant a_0$ or $b \geqslant b_0$.*

Proof. Every coherent sheaf has a possibly infinite resolution by bundles of the form $\bigoplus \mathcal{O}(-dH)$. By [Laz04a, Appendix B], it thus suffices to check for finitely many sheaves of the form $\mathcal{O}(-dH)$. The proof follows by induction on the dimension of X. In the base case, dimension zero, the lemma follows because for every coherent sheaf the groups H^i vanish for $i > 0$.

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Since every ample line bundle has some multiple which is very ample, it suffices to prove the lemma when H is very ample. It is also enough to find the constants a_0 and b_0 such that the cohomology vanishes for a fixed $i > q$. Assume H is very ample, and fix an $i > q$. Now suppose X has dimension n and the lemma is true for projective schemes of dimension $n-1$.

Because L is q-ample, we know there exists a_1 such that $H^i(X, \mathcal{O}(aL - dH)) = 0$ whenever $a \geqslant a_1$. Let D be a hyperplane section under the embedding given by H. By the inductive hypothesis, there exists a_2 such that $H^i(D, \mathcal{O}(aL + (b-d)H)) = 0$ whenever $a \geqslant a_2$ and $b \geqslant 0$. By abuse of notation, we use L to refer to both the line bundle on X and its pull back to D. The projection formula [Har77, II, Example 5.1] along with the preservation of cohomology under push forward by a closed immersion shows that this will not change the cohomology. Thus, we have an exact sequence in cohomology:

$$\cdots \to H^i(X, \mathcal{O}(aL + (b-d)H)) \to H^i(X, \mathcal{O}(aL + (b+1-d)H)) \\ \to H^i(D, \mathcal{O}(aL + (b+1-d)H)_{|D}) \to \cdots .$$

Set $a_0 = \max\{a_1, a_2\}$. Then for $a \geqslant a_0$, we know that $H^i(D, \mathcal{O}((aL + (b+1-d)H))_{|D}) = 0$, so by induction on b we know that $H^i(X, \mathcal{O}(aL + (b-d)H))$ vanishes for all $b > 0$. To find b_0, we know that for each $a < a_0$, there exists b' such that the cohomology vanishes for $b > b'$ since H is ample. Take b_0 as the maximum of all the b'. □

Proof of Theorem 2.1. L is q-ample if and only if αL is, so we may assume $\alpha = 1$. Likewise, Totaro [Tot10, Corollary 7.2] showed that L is q-ample on a scheme X if and only if its restriction to the reduced scheme is q-ample, so we may assume $\beta = 1$. At this point we are assuming that $H = L - L'$ is ample.

We recall another result of Totaro [Tot10, Theorem 7.1]: given H ample there exists a global constant C such that L is q-ample if and only if there exists N such that $H^i(X, \mathcal{O}(NL - jH)) = 0$ for all $i > q$, $1 \leqslant j \leqslant C$. Let us assume L is $(q+1)$-ample but not q-ample. Since L is not q-ample, for all N one of the above groups is non-zero. Since L is $(q+1)$-ample, that group must have $i = q+1$ for large enough N. Now H is ample, so, for sufficiently large e, $H^i(X, \mathcal{O}((e-j)H)) = 0$ for $i > q$, $1 \leqslant j \leqslant C$.

Likewise, for all sufficiently large $e \geqslant 1$, we know that $H^{q+1}(X, \mathcal{O}((e-j)H)) = 0$, and that for some $1 \leqslant j \leqslant C$, $H^{q+1}(X, \mathcal{O}(eL - jH)) \neq 0$. Since $\mathcal{O}(L') = \mathcal{O}(L - H)$, there exist j and k such that $1 \leqslant j \leqslant C$, and $1 \leqslant k \leqslant e$ such that $H^{q+1}(X, \mathcal{O}((e-j)H + (k-1)L')) = 0$ and $H^{q+1}(X, \mathcal{O}((e-j)H + kL')) \neq 0$. To simplify notation, we set $l = e - j$.

Consider the exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X(-L') \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

The section defining D may be given by a section which is not regular when X is reducible and so the sheaf $\mathcal{F}$ may be non-zero. Now write $\mathcal{G} = \operatorname{coker}(\mathcal{F} \to \mathcal{O}_X(-L')) = \ker(\mathcal{O}_X \to \mathcal{O}_D)$. After twisting by $\mathcal{O}(lH + kL')$, we have two resulting long exact sequences in cohomology. The first is

$$\cdots \to H^{q+1}(X, \mathcal{O}(lH + (k-1)L')) \to H^{q+1}(X, \mathcal{G} \otimes \mathcal{O}(lH + kL')) \\ \to H^{q+2}(X, \mathcal{F} \otimes \mathcal{O}(lH + kL')) \cdots .$$

Since $k \leqslant l + j$ and $\mathcal{O}(H + L') = \mathcal{O}(L)$, for sufficiently large e, $H^{q+2}(X, \mathcal{F} \otimes \mathcal{O}(lH + kL')) = H^{q+2}(X, \mathcal{F} \otimes \mathcal{O}((l-k)H + kL)) = 0$, by Lemma 2.2. Thus, $H^{q+1}(X, \mathcal{O}(lH + (k-1)L')) = 0$ implies $H^{q+1}(X, \mathcal{G} \otimes \mathcal{O}(lH + kL')) = 0$.

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The second long exact sequence is given by

$$\cdots \to H^i(X, \mathcal{G} \otimes \mathcal{O}(lH + kL')) \to H^i(X, \mathcal{O}(lH + kL')) \to H^i(D, \mathcal{O}(lH + kL')_{|D}) \to \cdots .$$

The group $H^{q+1}(X, \mathcal{G} \otimes \mathcal{O}(lH + kL')) = 0$, and $H^{q+1}(X, \mathcal{O}(lH + kL')) \neq 0$, so we see that $H^{q+1}(D, \mathcal{O}(lH + kL')_{|D}) \neq 0$. $\mathcal{O}(lH + kD) = \mathcal{O}((l-k)H + kL)$, which has the form $\mathcal{O}(aL + (b-d)H)$, where $d = C$, $a, b \geqslant 0$, and $a + b \geqslant e$. Since we could choose e arbitrarily large, by Lemma 2.2 L is not q-ample when restricted to D. □

In the case where X is irreducible, every non-zero section of a line bundle is regular, and we get the following corollary.

Corollary 2.3. *If X is a complex projective variety (irreducible and reduced) and L is a big line bundle which is not q-ample, there exists a codimension-one subscheme of X on which L is not q-ample.*

Proof. The cone of big line bundles on a projective variety is open, so we may pick L' also big, so some large multiple of L' has a non-zero section whose vanishing is an effective Cartier divisor. □

One subtlety of the Kleiman criterion for ample divisors is that it is possible to have a divisor class which is positive on every irreducible curve but is not ample. One such example is due to Mumford and can be found in [Laz04a, Example 1.5.2]. In particular, this shows that in Corollary 2.3 the hypothesis 'big' cannot be replaced by 'pseudoeffective', already when $q = 0$.

3. Augmented base loci

Let L be a Cartier divisor on a variety X. Write $\mathrm{Bs}(|L|)$ for the base locus of the full linear series of L. It is also helpful to have a notion of the base locus for large multiples of L, as well as for small perturbations by the inverse of an ample line bundle.

Definition 3.1 [Laz04a, Definition 2.1.20]. The stable base locus of L is the algebraic set

$$\mathbf{B}(L) = \bigcap_{m \geqslant 1} \mathrm{Bs}(|mL|).$$

There exists an integer m_0 such that $\mathbf{B}(L) = \mathrm{Bs}(|km_0L|)$ for $k \gg 0$ [Laz04a, Proposition 2.1.20].

Definition 3.2 [Laz04b, Definition 10.3.2]. The augmented base locus of L, denoted by $\mathbf{B}_+(L)$, is the closed algebraic set given by $\mathbf{B}(L - \epsilon\mathcal{H})$ for any ample $\mathcal{H}$ and sufficiently small $\epsilon > 0$.

It is a theorem of Nakamaye [Nak00] that the augmented base locus is well defined. Note that stable and augmented base loci are defined as algebraic sets, not as schemes.

Geometric properties of $\mathbf{B}_+(L)$ reveal information about how much L fails to be ample. For example, $\mathbf{B}_+(L)$ is empty if and only if L is ample. More generally, Küronya has proved in [Kür10] a Fujita-vanishing type result for the cohomology groups H^i, where $i > \dim \mathbf{B}_+(L)$.

Theorem 3.3 [Kür10, Theorem C]. *Let X be a projective scheme, L a Cartier divisor, and $\mathcal{F}$ a coherent sheaf on X. Then there exists m_0 such that $m \geqslant m_0$ implies $H^i(X, \mathcal{F} \otimes \mathcal{O}(mL + D)) = 0$ for all $i > \dim \mathbf{B}_+(L)$ and any nef divisor D.*

In particular, Küronya's theorem implies that L is q-ample for all q at least as big as the dimension of $\mathbf{B}_+(L)$. We show that in fact L is q-ample if and only if the restriction of L to $\mathbf{B}_+(L)$ is q-ample.

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Proof of Theorem 1.1. Certainly if L is q-ample on X it must be q-ample on Y. For the converse, we apply Theorem 2.1 inductively. Suppose L is not q-ample. We may assume all schemes are reduced by [Tot10, Corollary 7.2]. Choose an ample divisor H, and choose a and b such that $L' = aL - bH$ satisfies $\mathrm{Bs}(|L'|) = \mathbf{B}_+(L)$.

Suppose there is a point $x \in X$ which is not contained in Y. Then since Y is the base locus of L', there is a section of L' which does not vanish at x, and let X' be the vanishing of this section. Then, by Theorem 2.1, L is not q-ample on X'. The process only terminates when $X' = Y$, and it must terminate because X was a noetherian topological space. □

4. Towards a numerical criterion for *q*-ample line bundles

The cone of ample line bundles in $N^1(X)$ has a nice description in terms of the geometry of curves in X due to a theorem of Kleiman. (See for example [Laz04a, 1.4.23].)

Theorem 4.1 (Kleiman's criterion). *Let* $\mathrm{Nef}(X)$ *be the cone of nef divisors.* $\mathrm{Nef}(X)$ *is a closed cone, and the cone of ample divisors is the interior of* $\mathrm{Nef}(X)$.

One would like similar criteria to test the q-amplitude of L. A duality argument gives a criterion for the $(n-1)$-ample cone.

Theorem 4.2 [Tot10, Theorem 9.1]. *On a variety X, the $(n-1)$-ample cone is the negative of the complement of the pseudoeffective cone.*

The Kleiman criterion says that L is in the closure of the ample cone if and only if $-L$ is not big on any curve. Theorem 4.2 says that L is in the closure of the $(n-1)$-ample cone if and only if $-L$ is not big on X, which is the only subvariety of X having dimension n. Thus, in some sense, both criteria say that to test if a divisor is in the closure of the q-ample cone it suffices to show that its dual is not in the big cone of any subvarieties of dimension $q+1$. While one would hope that such a criterion holds for all q, we will see in § 5 an example of Totaro which shows this fails for even the case of 3-folds. However, if we also require the divisor to be big, we may combine Corollary 2.3 with a modification of the duality argument to yield Corollary 1.2.

Proof of Corollary 1.2. Certainly if L is $(n-2)$-ample on X it is $(n-2)$-ample on every subvariety. For the other direction, using Corollary 2.3, if L fails to be $(n-2)$-ample we have an effective Cartier divisor D on which L is not $(n-2)$-ample. By [Tot10, Corollary 7.2], we may assume D is reduced. Since X is non-singular, D is a still a Cartier divisor, and the dualizing sheaf $\mathcal{K}_D$ is a line bundle given by $\mathcal{K}_D = (\mathcal{K}_X \otimes \mathcal{O}(D))|_D$.

Let D_i be the components of D and let $f : \coprod D_i \to D$ be the canonical map. Then the map $\mathcal{O}_D \to f_* \bigoplus \mathcal{O}_{D_i}$ is injective, and so yields an injective map $H^0(D, J) \to \bigoplus H^0(D_i, J|_{D_i})$ for any line bundle J on D. Suppose $-L$ is not pseudoeffective on any of the D_i. Then for any line bundle J and sufficiently large m depending on J, $H^0(D_i, \mathcal{O}(J - mL)|_{D_i}) = 0$, so $H^0(D, \mathcal{O}(J - mL)) = 0$.

It follows by duality that $H^{n-1}(D, \mathcal{K}_D \otimes \mathcal{O}(mL - J)) = 0$ for any line bundle J and sufficiently large m. But by [Tot10, Theorem 7.1] this means L is $(n-2)$-ample on D, a contradiction. □

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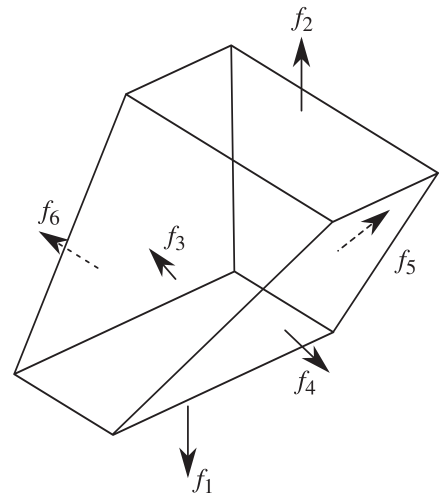

FIGURE 1. The dual polytope to Σ.

5. Totaro's example

In this section, we reproduce Totaro's example from [Tot10] of a line bundle L on a smooth toric Fano 3-fold X such that L is not in the closure of the 1-ample cone of X, but L is in the closure of the 1-ample cone of every proper subvariety of X. Our goal is investigate what sort of additional obstacles beyond the numerical criterion must be considered to say when an effective bundle is q-ample.

DEFINITION 5.1. A line bundle L on X is called q-nef if for every dimension-$(q+1)$ subvariety $V \subset X$, the restriction of $-L$ to V is not big.

The q-nef cone is a closed cone in $N^1(X)$. By Theorem 4.2, a q-ample bundle must be q-nef. Also, when $q=0$ or $q=n-1$, the q-ample cone is the interior of the q-nef cone. Let X be the projectivization of the rank-two vector bundle $\mathcal{O} \oplus \mathcal{O}(1,-1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Then X is a smooth toric Fano 3-fold. One can show that the corresponding fan Σ in $\mathbb{Z}^3 \otimes \mathbb{R}$ has rays

$$f_1 = (0,0,-1), \quad f_2 = (0,0,1), \quad f_3 = (1,0,1),$$
$$f_4 = (0,1,-1), \quad f_5 = (-1,0,0), \quad f_6 = (0,-1,0).$$

The two-dimensional cones are given by

$$(13), (14), (15), (16), (23), (24), (25), (26), (34), (36), (45), (46).$$

The maximal cones are

$$(134), (136), (145), (146), (234), (236), (245), (246).$$

Figure 1 shows the dual polytope to the fan Σ.

Line bundles on X are given by piecewise linear functions on Σ which are integral linear functions on each cone. Let $\langle \Sigma(1) \rangle$ be the $\mathbb{R}$ vector space spanned by the rays of Σ. Since X is simplicial, we have an identification

$$\mathrm{Pic} \otimes \mathbb{R} \cong \langle \Sigma(1) \rangle^* / (\mathbb{Z}^3 \otimes \mathbb{R})^*.$$

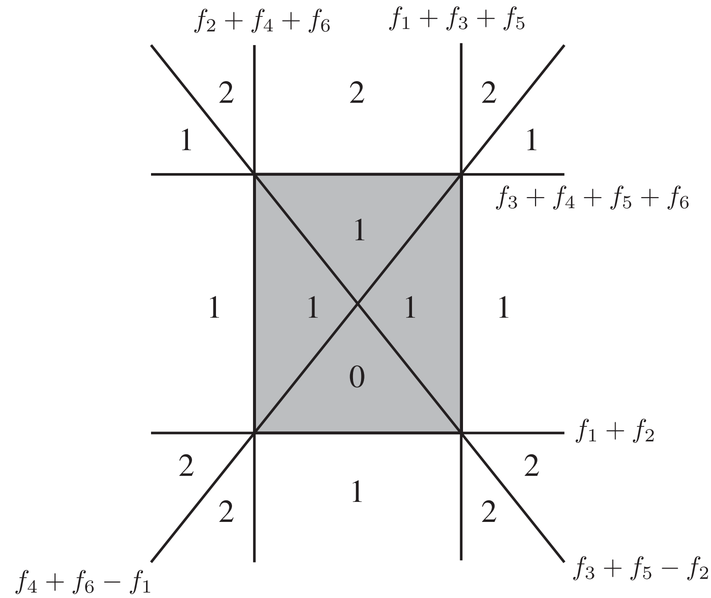

Figure 2. Chambers in $N^1(X)$. The effective cone is shaded, and each chamber is marked with the smallest q such that a line bundle in the interior of the chamber is q-ample. The planes are labeled by the corresponding linear dependence among rays in $\Sigma(1)$.

Write F_i for the function which sends f_i to 1 and $f_{j,j \neq i}$ to 0. Then we can identify F_i with the divisor which is the closure of the torus orbit corresponding to the ray f_i. Let $L = 3F_1 + 3F_2 - F_3 - F_4 - F_5 - F_6$. Then L is not in the closure of the 1-ample cone, but L is 1-nef.

To see that L is not in the closure of the 1-ample cone, it suffices to show that a positive twist of L is not 1-ample. For example, take $H = F_1 + F_2 + F_3 + F_4 + F_5 + F_6$. Then for any sufficiently small rational $\lambda > 0$, a large integral multiple of $L + \lambda H$ has a non-vanishing H^2. This follows from the formula for cohomology of line bundles given in [Ful93, p. 74], along with the fact that the rays with negative coefficients form a non-trivial 1-cycle in $|\Sigma| \setminus \{0\}$.

The 1-nef cone of a toric variety consists of divisors whose restriction to each torus invariant surface is not the negative of a big divisor. It can be shown that L is 1-nef by restricting to each F_i. As an example, we explicitly work out the restriction of L to F_1.

The divisor F_1 is a toric variety and its fan is given by $\Sigma_{F_1} = \mathrm{Star}(f_1)/\langle f_1 \rangle$. Denote the image of the ray f_i in Σ_{F_1} by f_i'. This fan is isomorphic to the fan of $\mathbb{P}^1 \times \mathbb{P}^1$. The most straightforward way of restricting L to F_1 is to choose a linearly equivalent representative in $\langle \Sigma(1) \rangle^*$ which vanishes on f_1. Take $L' = 6F_2 - 4F_3 + 2F_4 - F_5 - F_6$. Then the resulting piecewise linear function ψ on Σ_{F_1} has

$$\psi(f_3') = -4, \quad \psi(f_4') = 2, \quad \psi(f_5') = -1, \quad \psi(f_6') = -1.$$

This corresponds to the divisor $\mathcal{O}(1, -3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, which is not the negative of a big divisor. A similar calculation for the other F_i shows that L is actually 1-nef.

Figure 2 shows a slice of $N^1(X)$, where the effective cone is shaded. The numbers in each region are the largest q such that a line bundle in the interior of that region is q-ample.

6. Further questions

Let X be a variety and L a line bundle on X. When L is not big, $\mathbf{B}_+(L)$ is all of X, and so yields no new information about whether L is q-ample. However, when L is effective, we may hope to see other geometric consequences of q-amplitude reflected in the geometry of a section. In the example in § 5, the divisor $F_1 + F_2$ is not 1-ample, and this cannot be seen via any sort of restriction to proper subvarieties of X. However, $F_1 + F_2$ cannot be 1-ample because it admits a section with disconnected zero set.

PROPOSITION 6.1. *Let X be a normal irreducible Cohen–Macaulay variety of dimension n. If L is a line bundle on X which admits a global section with disconnected zero set, then L is not $(n-2)$-ample.*

Proof. Let D be the vanishing of a section of L, which is disconnected. Then we can take the infinitesimal thickening mD as the vanishing of a section of $\mathcal{O}(mL)$. Consider the restriction exact sequence

$$0 \to \mathcal{O}(-mL) \to \mathcal{O}_X \to \mathcal{O}_{mD} \to 0.$$

Since X is connected $H^0(X, \mathcal{O}_X)$ is one dimensional, but mD is not connected so $H^0(mD, \mathcal{O}_{mD})$ is at least two dimensional. Thus, the associated map $H^0(X, \mathcal{O}_X) \to H^0(mD, \mathcal{O}_{mD})$ is not surjective and so taking the associated long exact sequence we see that $H^1(X, \mathcal{O}(-mL))$ is non-zero. Let $\mathcal{K}_X$ be the dualizing sheaf on X. By Serre duality, $H^{n-1}(X, \mathcal{K}_X \otimes \mathcal{O}(mL))$ is non-vanishing for all m and so L is not $(n-2)$-ample. □

Question 6.2. Given a smooth variety X with an effective line bundle L which is $(n-2)$-nef and such that there is a neighborhood U in $N^1(X)$ so that no line bundle in U admits a section with disconnected vanishing set, must L be $(n-2)$-ample?

One possible way to interpret Proposition 6.1 is as a sort of Lefschetz hyperplane theorem for $(n-2)$-ample divisors. Bott has proved the following generalization of the Lefschetz hyperplane theorem.

THEOREM 6.3 [Bot59, Theorem III]. *Let X be a smooth variety of dimension n, and L a line bundle which admits a Hermitian metric whose curvature form has at least $n-q$ positive eigenvalues (counted with multiplicity) at every point. Suppose also that Y is the vanishing set of a section of L. Then X is obtained from Y as a topological space by attaching cells of dimension at least $n-q$.*

A line bundle is called q-positive if it admits such a Hermitian metric. If Y has 'too much' homology in dimension $n-q-2$, it cannot be a section of a q-positive line bundle. It is a well-known theorem of Andreotti and Grauert [AG62] that a q-positive line bundle is q-ample. The problem of determining when the converse holds was posed by [DPS96], but little progress had been made until recently. Ottem [Ott11] has given examples of line bundles which are q-ample but not q-positive when $\frac{1}{2}\dim X - 1 < q < \dim X - 2$. These examples are effective, and the analogue of the Lefschetz hyperplane theorem holds over $\mathbb{Q}$ but not $\mathbb{Z}$. Matsumura has shown in [Mat11] that if X is a compact n-dimensional complex manifold with a Kähler form ω, and L is a line bundle such that the intersection $\omega^{n-1} \cdot L > 0$, then L is 1-positive.

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