PROPERTIES (V) AND (u) ARE NOT THREE-SPACE PROPERTIES

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In his fundamental papers [**7, 8**], Pelczynski introduced properties (u), (V), and (V^*) as tools as study the structure of Banach spaces. Let X be a Banach space. It is said that X has property (u) if, for every weak Cauchy sequence (x_n) in X, there exists a weakly unconditionally Cauchy (wuC) series $\sum_n z_n$ in X such that the sequence $(x_n - \sum_{j=1}^{j=n} z_j)$ is weakly null. It is said that X has property (V) if, for every Banach space Z, every unconditionally converging operator from X into Z is weakly compact; equivalently, whenever K is a bounded subset of X^* such that $\lim_{n\to\infty} \sup\{|f(x_n)| : f \in K\} = 0$ for every wuC series $\sum_n x_n$ in X, then K is relatively weakly compact. A Banach space X is said to have property (V^*) if whenever K is a bounded subset of X such that $\lim_{n\to\infty} \sup\{|f_n(x)| : x \in K\} = 0$ for every wuC series $\sum_n f_n$ in X^*, then K is relatively weakly compact. Some well-known results which shall be needed later are contained in the following.

PROPOSITION 1 [**7**, Proposition 2, Proposition 6 and Corollary 5]. *If X has property (u) and does not contain isomorphic copies of l_1, then X has property (V). If X has property (V), then X^* is weakly sequentially complete. If X has property (V^*), then X is weakly sequentially complete.*

A property P is said to be a *three-space property* if, whenever a closed subspace Y of a Banach space X, and the corresponding quotient space X/Y have P, then also X has P. In [**5**], it is shown that properties (V) (resp. (V^*)) satisfy a restricted version of the three-space property, namely, when X/Y (resp. Y) is reflexive.

In this note we show that the properties (u) and (V) are not three-space properties, which solves a problem of G. Godefroy and P. Saab [**5**]. Moreover, we shall show that a Banach lattice, or the space of Bourgain and Delbaen, or the dual spaces of the spaces X_p of Figiel, Ghoussoub and Johnson, cannot provide a counterexample to the three-space problem for property (V^*).

Let us describe the space X_p, $1 \le p < \infty$, of [**3, 4**]. Denoting by $\mathbb{N}$ the set of integers, by $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$, and c the space of converging sequences, we set $X = l_1(c)$, i.e. the space of doubly-indexed sequences $a = (a_{ij})$, $i \in \mathbb{N}$, $j \in \mathbb{N}^*$, such that $\lim_{j\to\infty} a_{ij} = a_{i\omega}$ and $\|a\|_X = \sum_{i=1}^{\infty} (\sup_j |a_{ij}|) < \infty$. Let $f^n \in X$ be given by

$$f^n_{ij} = \begin{cases} 1, & \text{if } i \le n \le j \\ 0, & \text{otherwise.} \end{cases}$$

The gauge $\|\ \ \|$ of the closed absolutely convex solid hull of the unit ball of X and the

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sequence (f^n) is a lattice norm on X. For $1 \leq p < +\infty$, $\|x\|_p = \| |x|^p \|^{1/p}$ defines a new lattice norm on X, whose completion shall be denoted X_p. Let $T : X \to c_0$ be the operator defined by $T(a_{ij}) = (a_{i\omega})$, and let T_p be its continuous extension to X_p. The following result is in [**3**, Ex. 3.1].

PROPOSITION 2. *For $1 < p < \infty$, X_p does not contain copies of l_1 and T_p is surjective.*

LEMMA 3. *$l_1(c_0)$ is a dense subspace of* Ker T_p.

Proof. It is enough to note that given a sequence $u^n \in l_1(c)$ converging to a point $x \in \operatorname{Ker} T_p$, the point $u^n \wedge x$ belongs to $\operatorname{Ker} T_p \cap l_1(c_0)$. Since the spaces X_p are Banach lattices, $\|u^n \wedge x - x\|_p \leq \|u^n - x\|_p$. □

PROPOSITION 4. *For $1 \leq p < +\infty$,* Ker T_p *has property* (u).

Proof. It is clear that the norm $\| \ \|$ is an order continuous norm on $l_1(c_0)$. Hence if (x_n) is a downward directed sequence in $l_1(c_0)$ with $\inf(x_n) = 0$, then the sequence (x_n^p) is also directed downward with $\inf(x_n^p) = 0$, therefore $\lim_n \|x_n^p\| = 0$, and thus $\lim_n \|x_n\|_p = 0$. This shows that $\| \ \|_p$ is an order continuous norm on $l_1(c_0)$. It follows from a result of Luxembourg (see [**1**, Theorem 12.10, p. 179]), that Ker T_p is then an order continuous Banach lattice and hence it has property (u). □

REMARK. If (x^n) denotes a weakly Cauchy sequence of Ker T_p, the vectors $(y^k) \in$ Ker T_p defined by

$$y_{ij}^n = \begin{cases} x_{ij}, & \text{for } k = i, \\ 0, & \text{otherwise,} \end{cases}$$

where x_{ij} is the pointwise limit of the (i, j) coordinate of the x^n, form a w.u.C. sequence such that $(x^n - \sum_{k=1}^{n} y^k)$ is weakly null.

THEOREM 5. *Properties (u) and (V) are not three-space properties.*

Proof. It is clear that $X_p/\operatorname{Ker} T_p = c_0$ has properties (u) and (V). From Propositions 1, 2, and 4 it follows that, for $1 < p < +\infty$, Ker T_p has properties (u) and (V). It is also clear that X_p fails property (V) since T_p is unconditionally converging [**4**], but not weakly compact. Moreover, since X_p contains no subspace isomorphic to l_1, it follows from Proposition 1 that X_p fails property (u). □

Concerning the three-space problem for property (V^*), we shall give another partial answer.

PROPOSITION 6. *If X is a Banach lattice containing a closed subspace M such that both M and X/M have property (V^*), then X has property (V^*).*

Proof. If Y and X/Y have property (V^*) then Y and X/Y are weakly sequentially complete. Since this is a three-space property, X is weakly sequentially complete. By [**9**, Theorem 4], X has property (V^*). □

PROPOSITION 7. *Assume that X is a non-reflexive Banach space such that a subspace Y and the corresponding quotient X/Y have property (V^*). Then X^* contains a subspace isomorphic to c_0.*

Proof. If X^* contains no subspace isomorphic to c_0, then $(X/Y)^* = Y^\perp$ will contain no subspace isomorphic to c_0 and thus X/Y will be reflexive since it has property (V^*). It follows that $Y^\perp$ is reflexive and thus $Y^* = X^*/Y^\perp$ contains no subspace isomorphic to c_0. This in turn shows that Y is reflexive since Y is assumed to have property (V^*). It follows that X itself is reflexive. This contradiction finishes the proof. □

Observation. It follows from Propositions 6 and 7 that the spaces X_p^* and the space BD of Bourgain and Delbaen [**2**], which are natural candidates to verify the failure of the three-space property for (V^*), cannot provide such a counterexample.

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