Elementary Notes.

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1. On the Factorisation of a function of n variables.

The following examples illustrate a somewhat obvious extension of the method of factoring given in a former paper (*Proceedings*, Vol. XII., p. 32, q.v.). By means of it, if we are given any function of n variables, no one of which is of higher degree than the second, we can either find the factors of it, *or* prove that it has no factors with rational coefficients.

Example (1)

$$8x^2 + 115xy - 79xz + 52x + 42y^2 - 125yz - 35y + 63z^2 - 13z - 28.$$

This we may write as a trinomial in x, viz.,

$$8x^2 + (115y - 79z + 52)x + (42y^2 - 125yz - 35y + 63z^2 - 13z - 28).$$

Applying the method described in the former paper we have now to find two factors of the product of the coefficient of x^2 and the third term, such that their sum is the coefficient of x.

This requires us to find the factors of the third term, which we may write as a trinomial in y (say) viz.,

$$42y^2 - (125z + 35)y + (63z^2 - 13z - 28).$$

Applying the same method to this function of y, it is now necessary to find the factors of the third term, which may be written as a function of z. It is obvious that the process is quite general however many variables there may be. The work may be arranged as follows. The given function =

$$\begin{aligned} &8x^2 + (115y - 79z + 52)x + \{42y^2 - (125z + 35)y + (63z^2 - 13z - 28)\} \\ &= 8x^2 + (115y - 79z + 52)x + \{42y^2 - (125z + 35)y + (7z + 4)(9z - 7)\} \\ &= 8x^2 + (115y - 79z + 52)x + (3y - 7z - 4)(14y - 9z + 7) \\ &= (8x + 3y - 7z - 4)(x + 14y - 9z + 7). \end{aligned}$$

This example, however, is capable of a shorter treatment, for it is obvious that its factors are of the form

$$(ax + by + cz + d)(a'x + b'y + c'z + d').$$

https://doi.org/10.1017/S0013091500032429 Published online by Cambridge University Press

Consider first, therefore, the factors of

$$8x^2+115xy+42y^2$$

and we get $\quad (8x+3y)(x+14y).$

These give the first terms of the required factors.

Next consider

$$42y^2-125yz+63z^2 \equiv \{\pm(3y-7z)\}\{\pm(14y-9z)\}$$

The upper signs are obviously the suitable ones and we now have

$$(8x+3y-7z\ \ldots)(x+14y-9z\ \ldots)$$

Next $\quad 63z^2-13z-28 \equiv \{\pm(9z-7)\}\{\pm(7z+4)\}$

Whence, taking lower signs and second factor first, we get

$$(8x+3y-7z-4)(x+14y-9z+7).$$

It remains now to test whether these factors suit the terms which have not been used in their determination, viz., $-79xz+52x-35y$.

The following example shows that the method is not limited to functions, the form of whose factors may be determined by inspection.

Example (2)

$$3a^2b^2c-9a^2bc^2+a^2b-3a^2c-3abc^2+2ab^2c+ac+2bc^2 \equiv$$
$$(3a^2c+2ac)b^2-(9a^2c^2-a^2+3ac^2-2c^2)b-(3a^2c-ac) \equiv$$
$$ac(3a+2)b^2-(9a^2c^2-a^2+3ac^2-2c^2)b-ac(3a-1) \equiv \quad \text{(A)}$$
$$\{a\,.\,b-(3a-1)c\}\{c(3a+2)\,.\,b+a\} \equiv$$
$$(\ ab\ -3ac-c)(3abc+2bc+a).$$

Note: This has been arranged as a function of b; it might also have been arranged as a function of a or of c. At (A) we have to find two factors of $\quad ac(3a+2)\,.-ac(3a-1)$

whose sum is $\quad -(9a^2c^2-a^2+3ac^2-2c^2).$

By considering the coefficient of a^2 in this, we see that the factors are $\quad a^2$ and $-c^2(3a+2)(3a-1).$

So long as no variable is of higher degree than the second, the above methods are applicable, and they may be applied even when *some* of the variables are of higher degree than the second, provided we can arrange the function as a trinomial in one of the variables, and the last term and the coefficient of first term be each capable of being factored by known methods.

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Example (3)

$$a^3c^2 + 2a^3c + a^3 + 4ab^2c + b^3 - b^3c^2$$

may be written $(a^3 - b^3)c^2 + (2a^3 + 4ab^2)c + (a^3 + b^3) =$

$$(a-b)(a^2+ab+b^2) \,.\, c^2 + (2a^3+4ab^2) \,.\, c + (a+b)(a^2-ab+b^2) =$$

$$\{(a^2+ab+b^2) \,.\, c + (a^2-ab+b^2)\}\{(a-b) \,.\, c + (a+b)\} =$$

$$(a^2c + a^2 + abc - ab + b^2c + b^2)(ac + a - bc + b).$$

2. On the use of the term "Produced."

One of the features distinguishiug Modern Geometry from Euclidean Geometry is that, by means of suitable conventions, its statements are made perfectly general, *e.g.*, *two straight lines meet in a point.* To one acquainted only with geometry as given in most editions of Euclid, there are two difficulties in this statement: first, parallel straight lines do not meet "even when produced ever so far both ways"; and, secondly, other straight lines may not meet unless *produced.* It would quite change the character of elementary geometry to adopt the convention whereby parallel lines are included in the above proposition, but the convention that straight lines are of unlimited length and do not need to be produced might with advantage be adopted, in teaching the "elements." It is obvious that by such a convention we may both make statements more comprehensive so as to include cases not formerly considered; and also group together cases *apparently* distinct, as has, indeed, been done in propositions in Book II. in many recent editions of Euclid.

But the convention has further advantages, as the following, out of a large number of examples occurring in recent examination papers, may show. They may easily be classified as those in which the term "produced" is (a) useless, or (b) misleading, or (c) wrong.

(a) "ABC is an isosceles triangle in which AB = AC. Through C, CD is drawn perpendicular to BC, meeting BA *produced* in D. Shew that A is the mid-point of BD."

Here it is useless to be given that CD meets BA produced, as this is capable of proof. There is, in fact, a redundancy in the data.

(b) "ABCK is a quadrilateral with AB = AC and angle K a right angle. E is the middle point of BC. From E perpendiculars are drawn to AK and KC *produced* meeting them in H and M respectively. Prove that AE : EC = AH : CM."

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Here, any one beginning by drawing ABCK of Fig. 19 will have no difficulty, but what of the poor examinee who begins with ABCK of Fig 20? From E he can draw a perpendicular neither to AK nor to KC *produced.* The proposition, however, is equally true for both figures. By adopting the suggested convention we get rid both of the difficulty of drawing the figure and of the implication that the truth of the proposition is limited to the case where the perpendiculars lie on opposite sides of BC.

(*c*) "AB, BC are equal arcs of a circle and P is a point on the arc BC, show that BP bisects the angle contained by AP and CP *produced.*"

This proposition is not true unless AB and BC be minor arcs, as may be seen from Figs. 21 and 22, where AB is the arc ACB and BC is the arc BAC. The statement, however, that BP bisects one of the angles between AP and CP is always true, and, moreover, the limitation that P is a point on the arc BC may be removed, for evidently P may be any point on the circumference.

In connection with examples such as this it might be useful to adopt the convention of naming arcs and angles connected with a circle in the clockwise direction; so that AB and BA would denote conjugate arcs, and if O be the centre, AOB would stand on AB while BOA would stand on BA. It would then be unnecessary to distinguish reflex angles, and generality of proof would be gained.