COMPOSITIO MATHEMATICA

Two generalizations of the PRV conjecture

P. L. Montagard, B. Pasquier and N. Ressayre

Compositio Math. **147** (2011), 1321–1336.

doi:10.1112/S0010437X10005233

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

Compositio Math. 147 (2011) 1321–1336
doi:10.1112/S0010437X10005233

Two generalizations of the PRV conjecture

P. L. Montagard, B. Pasquier and N. Ressayre

Abstract

Let G be a complex connected reductive group. The Parthasarathy–Ranga Rao–Varadarajan (PRV) conjecture, which was proved independently by S. Kumar and O. Mathieu in 1989, gives explicit irreducible submodules of the tensor product of two irreducible G-modules. This paper has three aims. First, we simplify the proof of the PRV conjecture, then we generalize it to other branching problems. Finally, we find other irreducible components of the tensor product of two irreducible G-modules that appear for 'the same reason' as the PRV ones.

1. Introduction

1.1 The original PRV conjecture

Parthasarathy–Ranga Rao–Varadarajan (PRV) conjectured in the sixties the following.

The PRV conjecture. Let G be a complex connected reductive group with associated Weyl group W. Let $V_G(\mu)$ and $V_G(\nu)$ be two irreducible G-modules with highest weights μ and ν respectively. Then, for any $w \in W$, the irreducible G-module $V_G(\overline{\mu + w\nu})$ with extremal weight $\mu + w\nu$, occurs with multiplicity at least one in $V_G(\mu) \otimes V_G(\nu)$.

This conjecture was proved independently by Kumar in [Kum88] and Mathieu in [Mat89]. The aim of this paper is to simplify the proof of the PRV conjecture and to generalize it in two directions.

1.2 Two generalizations

We now assume that G is a subgroup of a bigger connected reductive group $\hat{G}$. Fix a Borel subgroup $\hat{B}$ and a maximal torus $\hat{T} \subset \hat{B}$ of $\hat{G}$ such that $B = \hat{B} \cap G$ is a Borel subgroup of G and $T = \hat{T} \cap G$ is a maximal torus of G. Consider the restriction map $\rho : X(\hat{T}) \longrightarrow X(T)$ from the character group of $\hat{T}$ to the one of T. Let $\hat{\lambda}$ be a dominant weight of $\hat{T}$ and $V_{\hat{G}}(\hat{\lambda})$ be the irreducible $\hat{G}$-module of highest weight $\hat{\lambda}$. Let $\hat{w} \in \hat{W}$. The first aim of this paper is the following.

Question. Does the irreducible G-module $V_G(\overline{\rho(\hat{w}\hat{\lambda})})$ with extremal weight $\rho(\hat{w}\hat{\lambda})$ occur with multiplicity at least one in $V_{\hat{G}}(\hat{\lambda})$?

Although the answer may be NO (examples are given in § 2.4.3 or in § 3.1), the PRV conjecture exactly asserts that the answer is YES if G is diagonally embedded in $\hat{G} = G \times G$.

Let $\hat{G}/\hat{B}$ denote the complete flag variety of $\hat{G}$, $X^\circ_{\hat{w}}$ denote the G-orbit $G\hat{w}\hat{B}/\hat{B}$ and $X_{\hat{w}}$ denote its closure in $\hat{G}/\hat{B}$. If $X^\circ_{\hat{w}}$ is closed in $\hat{G}/\hat{B}$, we easily check that the answer is YES. We also answer positively the question under a topological assumption on $X_{\hat{w}}$.

Received 4 May 2010, accepted in final form 17 September 2010, published online 18 March 2011.
2000 Mathematics Subject Classification 22E46, 17B10, 14L24 (primary).
Keywords: tensor product decomposition, branching rules, PRV conjecture.
N.R. was partially supported by the French National Research Agency (ANR-09-JCJC-0102-01).

Theorem 1. *We assume $X_{\hat{w}}$ is multiplicity free. Then $V_G(\overline{\rho(\hat{w}\hat{\lambda})})$ is a G-submodule of $V_{\hat{G}}(\hat{\lambda})$.*

Here, $X_{\hat{w}}$ is said to be multiplicity free if its cycle class in the cohomology of $\hat{G}/\hat{B}$ is a linear combination with coefficients 0 or 1 of Schubert classes. This assumption, which can be hard to check, is fulfilled for example if G is a spherical subgroup of $\hat{G}$ of minimal rank (see [Res10a] for the complete list of such subgroup). In particular, G is a spherical subgroup of $G \times G$ of minimal rank and Theorem 1 implies the original PRV conjecture.

Our second generalization of the PRV conjecture deals with the decomposition of tensor products: we exhibit new components.

Theorem 2. *Let λ, μ, ν be three dominant weights of T. We assume that there exist $v,\ w \in W$, a simple root α and an integer k such that:*

(i) $l(s_\alpha v) = l(v) + 1$, $l(s_\alpha w) = l(w) + 1$;

(ii) $\lambda = v\mu + w\nu - k\alpha$;

(iii) $0 \leqslant k \leqslant \langle v\mu, \alpha^\vee \rangle$, *and* $0 \leqslant k \leqslant \langle w\nu, \alpha^\vee \rangle$.

Then $V_G(\lambda)$ is a submodule of $V_G(\mu) \otimes V_G(\nu)$.

Here, $\alpha^\vee$ denotes the coroot associated to α, and $\langle \cdot, \cdot \rangle$ denotes the pairing between weights and coroots. We obtain the original PRV conjecture by applying Theorem 2 with extremal values of k in condition (iii).

1.3 About proofs

The two key ingredients in our proofs are the normality of $X_{\hat{w}}$, and the fact that for any $\hat{G}$-linearized and globally generated line bundle $\mathcal{L}$ on $\hat{G}/\hat{B}$, the restriction map $\mathrm{H}^0(\hat{G}/\hat{B}, \mathcal{L}) \longrightarrow \mathrm{H}^0(X_{\hat{w}}, \mathcal{L})$ is surjective (see Theorem 6 below). An analogous version of these two results was already stated by Demazure in the case of any Schubert varieties in flag varieties [Dem74], but there were gaps in the proofs. Correct proofs were obtained combining several works of Andersen, Joseph, Ramanan–Ramanathan and Seshadri (see [And85, Jos85, RR85, Ses87]). The version we used for $\hat{G} = G \times G$ was proved by Kumar in [Kum88]. We also use the generalization due to Brion for any G, $\hat{G}$ and multiplicity free $X_{\hat{w}}$ (see [Bri03]). These two ingredients also play a central role in Kumar's proof. However, Kumar's proof also uses a complete description of $\mathrm{H}^0(X_{\hat{w}}, \mathcal{L})$ mainly due to Bott and the Joseph filtration. We make these two latter ingredients superfluous by using an argument of semistability.

1.4 Link with a saturation problem

In the general situation $G \subset \hat{G}$, we consider the set $\mathrm{LR}(G, \hat{G})$ of pairs $(\lambda, \hat{\lambda})$ of dominant weights of T and $\hat{T}$ such that $V_G(\lambda)$ occurs in $V_{\hat{G}}(\hat{\lambda})$. By a theorem due to Brion and Knop, $\mathrm{LR}(G, \hat{G})$ is a finitely generated semigroup. From a theoretic viewpoint, the convex cone $\mathcal{LR}(G, \hat{G})$ generated by $\mathrm{LR}(G, \hat{G})$ is well understood: the complete and minimal list of inequalities is parametrized by explicit cohomological conditions (see [Res10b]). There are so many inequalities that it is not obvious to concretely describe this cone and especially to construct points in this cone. A starting point in the proof of Theorem 1 is the following well-known proposition.

Proposition 1. *Let $\hat{\lambda}$ be a dominant character of $\hat{T}$ and $\hat{w} \in \hat{W}$. Then there exists a positive integer n such that $V_G(n\overline{\rho(\hat{w}\hat{\lambda})})$ is a G-submodule of $V_{\hat{G}}(n\hat{\lambda})$. In other words, $(\overline{\rho(\hat{w}\hat{\lambda})}, \hat{\lambda})$ belongs to $\mathcal{LR}(G, \hat{G})$.*

With the additional assumption that $X_{\hat{w}}$ is multiplicity free, Theorem 1 asserts that $(\overline{\rho(\hat{w}\hat{\lambda})}, \hat{\lambda})$ belongs to $\mathrm{LR}(G, \hat{G})$. The question of understanding the difference between $\mathcal{LR}(G, \hat{G})$ and $\mathrm{LR}(G, \hat{G})$ is known as a saturation problem. Let Λ be the subgroup of $X(T) \times X(\hat{T})$ generated by $\mathrm{LR}(G, \hat{G})$. The semigroup $\mathrm{LR}(G, \hat{G})$ is said to be *saturated along a half-line* if the first non-zero point of Λ on this half-line belongs to $\mathrm{LR}(G, \hat{G})$ (and $\mathrm{LR}(G, \hat{G})$ is said to be *saturated* if it is along any half-line in $\mathcal{LR}(G, \hat{G})$). Theorem 1 shows that if $X_{\hat{w}}$ is multiplicity free, $\mathrm{LR}(G, \hat{G})$ is saturated along all the half-lines given by Proposition 1.

Knutson and Tao proved in [KT99] that $\mathrm{LR}(\mathrm{SL}_n, \mathrm{SL}_n \times \mathrm{SL}_n)$ is saturated. Belkale and Kumar proved in [BK10] that $\mathrm{LR}(\mathrm{Sp}_{2n}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n})$ and $\mathrm{LR}(\mathrm{Spin}_{2n+1}, \mathrm{Spin}_{2n+1} \times \mathrm{Spin}_{2n+1})$ are saturated up to a factor 2: the second point of Λ on any half-line belongs to LR. Kapovich, Leeb and Millson obtained important results on the saturation question for semigroups $\mathrm{LR}(G, G \times G)$ (see [KLM08]).

We can now explain Theorem 2 in this context. Fix two dominant weights μ and ν of T. The intersection of $\mathcal{LR}(G, G \times G)$ with $X(T) \otimes \mathbb{Q} \times \{\mu\} \times \{\nu\}$ is a polytope $P(\mu, \nu)$ (namely, a moment polytope). The original PRV conjecture gives finitely many points in $P(\mu, \nu)$ that generate saturated half-lines. Theorem 2 gives finitely many segments in $P(\mu, \nu)$ all whose rational points generate saturated half-lines (see § 4.3.2 for examples).

1.5 Link with Wahl's conjecture

Solving Wahl's conjecture, Kumar proved in [Kum92] the surjectivity of the Gaussian map for flag varieties. The consequence in terms of tensor product decomposition is the following.

THEOREM 3 (See [Kum11]). *Let μ and ν be two dominant weights of T and α be a positive root. We set $\lambda = \mu + \nu - \alpha$ and assume that the following hold.*

(i) *The weight λ is dominant.*

(ii) *For all simple roots β such that $\langle \mu, \beta^\vee \rangle = 0$ or $\langle \nu, \beta^\vee \rangle = 0$, $\alpha - \beta$ is neither a root nor 0.*

Then $V_G(\lambda)$ is a submodule of $V_G(\mu) \otimes V_G(\nu)$.

The case when α is simple in Theorem 3 can also be obtained applying Theorem 2 with $v = w = e$ and $k = 1$. Condition (ii) in Theorem 3 asserts that condition (iii) in Theorem 2 is satisfied. Nevertheless, the conclusion of Theorem 2 does not hold if α is only assumed to be positive (see § 4.3.1). Our Theorem 2 is not a strict generalization of Theorem 3; for example, take $G = \mathrm{Sp}_4$, $\mu = \nu = \omega_1 + \omega_2$ and $\alpha = \alpha_1 + \alpha_2$ (with notation of § 4.3.1 below). Note that our proof does not work in this example because the conclusion of Lemma 6 is not satisfied (the corresponding space has dimension three instead of one).

2. Restriction to a subgroup

2.1 Setting

Let G be a complex connected reductive group, with a fixed Borel subgroup B and maximal torus $T \subset B$. Let $X(T)$ denote the character group of T. For any dominant weight $\lambda \in X(T)$, let $V_G(\lambda)$ denote the irreducible G-module with highest weight λ. Let W be the Weyl group of (G, T). For any character λ, the orbit $W \cdot \lambda$ intersects the dominant chamber in one point denoted by $\bar{\lambda}$. We will denote by w_0 the longest element of the Weyl group W.

https://doi.org/10.1112/S0010437X10005233 Published online by Cambridge University Press

We now assume that G is a subgroup of a connected reductive group $\hat{G}$. Let $\hat{T}$ and $\hat{B}$ be a maximal torus and a Borel subgroup of $\hat{G}$ such that $T \subset \hat{T} \subset \hat{B} \supset B$. We will use hats to denote objects relative to $\hat{G}$ instead of G; for example we will write $\hat{W}$, $\hat{w}_0, \ldots$. For a given dominant character $\hat{\lambda}$ of $\hat{T}$, we are interested in the following.

Problem. Find irreducible G-submodules of $V_{\hat{G}}(\hat{\lambda})$.

2.2 G-orbits in the complete flag manifold of $\hat{G}$

For any $\hat{w} \in \hat{W}$, we set $X^\circ_{\hat{w}} = G\hat{w}\hat{B}/\hat{B}$ and set $X_{\hat{w}}$ to be its closure. We also denote by $\sigma_{\hat{w}}$ the cycle class of the Schubert variety $\overline{\hat{B}\hat{w}\hat{B}/\hat{B}}$ in $\hat{G}/\hat{B}$. It is well-known that

$$\mathrm{H}^*(\hat{G}/\hat{B}, \mathbb{Z}) = \bigoplus_{\hat{w} \in \hat{W}} \mathbb{Z} \cdot \sigma_{\hat{w}}. \tag{1}$$

Let V be an irreducible subvariety of $\hat{G}/\hat{B}$. The cycle class $[V]$ of V in $\mathrm{H}^*(\hat{G}/\hat{B}, \mathbb{Z})$ can be expanded as follows

$$[V] = \sum_{\hat{w} \in \hat{W}} c_{\hat{w}}(V)\sigma_{\hat{w}}, \tag{2}$$

where the $c_{\hat{w}}(V)$ are non-negative integers. The variety V is said to be multiplicity free if for any $\hat{w} \in \hat{W}$, $c_{\hat{w}}(V) = 0$ or 1.

2.3 The statement

Consider the restriction map $\rho : X(\hat{T}) \longrightarrow X(T)$. We now state a slightly more general version of Theorem 1.

Theorem 4. *With above notation, let $\hat{\lambda}$ be a dominant character of $\hat{T}$ and $\hat{w} \in \hat{W}$. We assume that one of the following assumption holds.*

- (i) *The orbit $X^\circ_{\hat{w}}$ is closed.*
- (ii) *The subgroup G is spherical of minimal rank in $\hat{G}$.*
- (iii) *The variety $X_{\hat{w}}$ is multiplicity free.*
- (iv) *The variety $X_{\hat{w}\hat{w}_0}$ is multiplicity free.*

Then $V_G(\overline{\rho(\hat{w}\hat{\lambda})})$ is a G-submodule of $V_{\hat{G}}(\hat{\lambda})$.

The first case is easy and certainly well known.

Proof in case (i). Since $X^\circ_{\hat{w}}$ is complete, the isotropy group of $\hat{w}\hat{B}/\hat{B}$ in G is a parabolic subgroup of G. However, it is contained in $\hat{w}\hat{B}\hat{w}^{-1}$, so it is solvable. It follows that $B' := \hat{w}\hat{B}\hat{w}^{-1} \cap G$ is a Borel subgroup of G containing T. Then there exists $w \in W$ such that $w^{-1}Bw = B'$.

Let v be a non-zero vector of $V_{\hat{G}}(\hat{\lambda})$ of highest weight $\hat{\lambda}$. It is clear that $\hat{w}v$ is an eigenvector of weight $\rho(\hat{w}\hat{\lambda})$ for B' (here, we identify $X(T)$ and $X(B')$ by the restriction morphism). It follows that $w\hat{w}v$ is an eigenvector of weight $w\rho(\hat{w}\hat{\lambda})$ for B, so that $w\rho(\hat{w}\hat{\lambda})$ is dominant and $w\rho(\hat{w}\hat{\lambda}) = \overline{\rho(\hat{w}\hat{\lambda})}$. The theorem follows. $\square$

We now prove case (iv) assuming that case (iii) is known.

Proof in case (iv). We apply the theorem in case (iii) to the dominant weight $-\hat{w}_0\hat{\lambda}$ and the element $\hat{w}\hat{w}_0$ of $\hat{W}$. We obtain that $V_G(\overline{\rho(-\hat{w}\hat{\lambda})})$ is contained in $V_{\hat{G}}(-\hat{w}_0\hat{\lambda}) = V_{\hat{G}}(\hat{\lambda})^*$.

Since $-\rho(\hat{w}\hat{\lambda})$ is an extremal weight of $V_G(\overline{\rho(\hat{w}\hat{\lambda})})^*$, we deduce that $V_G(\overline{\rho(\hat{w}\hat{\lambda})})^*$ is a G-submodule of $V_{\hat{G}}(\hat{\lambda})^*$. The theorem follows by duality. $\square$

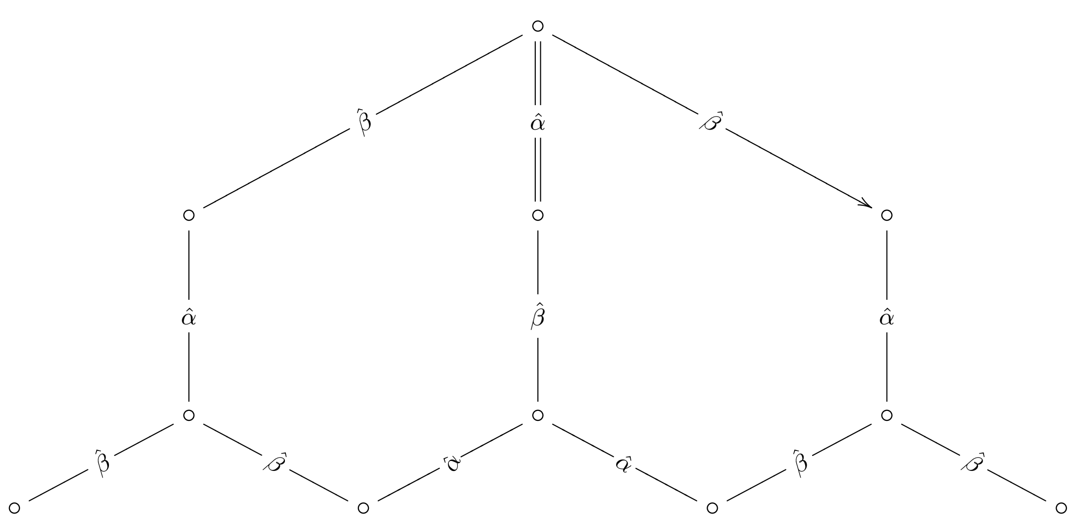
2.4 The spherical case

In this subsection, we look at a situation where we can check when assumption (iii) is fulfilled. This will allow us to discuss the various assumptions on examples and to include case (ii) in case (iii).

2.4.1 Assume that G is a spherical subgroup of $\hat{G}$; i.e. G acts on $\hat{G}/\hat{B}$ with finitely many orbits. In [Bri01], Brion defined an oriented graph $\Gamma(\hat{G}/G)$ whose vertices are the G-orbit closures in $\hat{G}/\hat{B}$. The edges, which can be simple or double, are labeled by the simple roots of $\hat{G}$. The assumption '$X_{\hat{w}}$ is multiplicity free' can be easily read off this graph: $X_{\hat{w}}$ is multiplicity free if and only if for any path from $X_{\hat{w}}$ to $\hat{G}/\hat{B}$ there is no double edge. In particular, by [Res10a, Proposition 2.1], if G is spherical of minimal rank, any G-orbit closure in $\hat{G}/\hat{B}$ is multiplicity free. In particular, case (ii) of Theorem 4 is a consequence of case (iii).

We now study two examples where G is spherical, which illustrate Theorem 4.

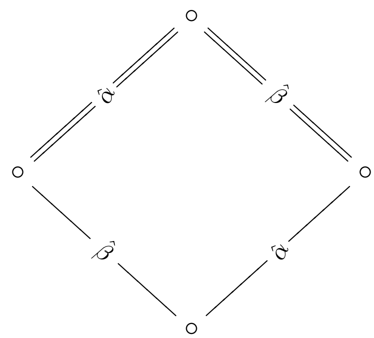
2.4.2 Let $\hat{G} = \mathrm{Sp}_4$ and $G = \mathrm{Gl}_2$ be the Levi subgroup of a maximal parabolic subgroup of Sp_4 that stabilizes an isotropic plane in $\mathbb{C}^4$. Then G is a spherical subgroup of $\hat{G}$ and the oriented graph $\Gamma(\hat{G}/G)$ (with arrows pointed down) is the following ($\hat{\alpha}$ and $\hat{\beta}$ denote respectively the short and the long simple roots of Sp_4).



In this example, the varieties $X_{\hat{w}}$ correspond to the four vertices at the bottom of the graph $\Gamma(\hat{G}/G)$ and they are in fact the four closed G-orbits in $\hat{G}/G$. Hence Theorem 4 can be applied here for all $\hat{w} \in \hat{W}$. This gives an example where we need to use hypothesis (i) of Theorem 4 to apply it, because two of the closed G-orbits above are not multiplicity free.

2.4.3 Let $\hat{G} = \mathrm{SL}_3$ and $G = \mathrm{SO}_3$ be naturally embedded in SL_3. Let α, $\hat{\alpha}$ and $\hat{\beta}$ denote the simple roots of SO_3 and SL_3. Also denote by ω_α, $\omega_{\hat{\alpha}}$ and $\omega_{\hat{\beta}}$ the corresponding fundamental weights. Then G is a spherical subgroup of $\hat{G}$ and the oriented graph $\Gamma(\hat{G}/G)$ is the following diagram.

https://doi.org/10.1112/S0010437X10005233 Published online by Cambridge University Press



We can read on the graph that there exist exactly two not-closed G-orbits in $\hat{G}/\hat{B}$ with multiplicity, namely $X^{\circ}_{s_{\hat{\alpha}}}$ and $X^{\circ}_{s_{\hat{\beta}}}$. An easy computation gives us that $\rho(s_{\hat{\alpha}}\omega_{\hat{\alpha}}) = \rho(s_{\hat{\beta}}\omega_{\hat{\beta}}) = 0$. However, we can also check that $V_G(0)$ is neither in $V_{\hat{G}}(\omega_{\hat{\alpha}})$ nor in $V_{\hat{G}}(\omega_{\hat{\beta}})$, so that the conclusion of Theorem 4 is not satisfied in these two cases.

We have just seen that $(0, \omega_{\hat{\alpha}})$ is not in the semigroup $\mathrm{LR}(G, \hat{G})$ defined in the introduction. However, we can remark that $(0, 2\omega_{\hat{\alpha}}) \in \mathrm{LR}(G, \hat{G})$, while $(0, \omega_{\hat{\alpha}})$ is in the subgroup of $X(T) \times X(\hat{T})$ generated by $\mathrm{LR}(G, \hat{G})$ (because we can compute that $(2\omega_\alpha, \omega_{\hat{\beta}})$ and $(2\omega_\alpha, \omega_{\hat{\alpha}} + \omega_{\hat{\beta}})$ are in $\mathrm{LR}(G, \hat{G})$). Then $\mathrm{LR}(G, \hat{G})$ is not saturated along the half-line generated by $(0, 2\omega_{\hat{\alpha}})$.

The rest of § 2 is devoted to the proof of Theorem 4.

2.5 A result of geometric invariant theory

Let X be an irreducible projective G-variety. As in [MFK94], we denote by $\mathrm{Pic}^{\mathrm{G}}(\mathrm{X})$ the group of G-linearized line bundles on X. Let $\mathcal{L} \in \mathrm{Pic}^{\mathrm{G}}(\mathrm{X})$ and let $\mathrm{H}^0(X, \mathcal{L})$ denote the G-module of regular sections of $\mathcal{L}$. A point $x \in X$ is said to be *semistable with respect to* $\mathcal{L}$ if there exists $n > 0$ and $\tau \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G$ such that $\tau(x) \neq 0$.

Remark. Note that this definition of semistable points is not standard. Indeed, it is usually agreed that the open subset defined by the non-vanishing of τ is affine. This property, which is useful to construct a good quotient, is automatic only if $\mathcal{L}$ is ample; hence, our definition coincides with the usual one if $\mathcal{L}$ is ample.

A line bundle $\mathcal{L}$ on X is said to be *semiample* if a positive power of $\mathcal{L}$ is base point free. If $\mathcal{L}$ is a line bundle on X and x is a point in X, $\mathcal{L}_x$ denotes the fiber in $\mathcal{L}$ over x. We will need the following lemma mainly due to Kostant.

Lemma 1. *Let $\mathcal{L} \in \mathrm{Pic}^{\mathrm{G}}(\mathrm{X})$ be semiample and $x \in X$ be a T-fixed point. We assume that T acts trivially on $\mathcal{L}_x$.*

Then x is semistable with respect to $\mathcal{L}$.

Proof. Let n be a positive integer, such that the natural morphism

$$\varphi : X \longrightarrow \mathbb{P}(\mathrm{H}^0(X, \mathcal{L}^{\otimes n})^*)$$

is well defined. Set $V = \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^*$. Let $v \in V$ be a non-zero vector on the line $\varphi(x)$. The assumption implies that v is fixed by T.

Let U be the unipotent radical of B. Then, as an orbit of an unipotent group in an affine variety, $U \cdot v$ is closed in V (see [Ros61, Theorem 2]); and, $B \cdot v = U \cdot v$. Since G/B is complete, it follows that $G \cdot v$ is closed in V. We deduce that there exists a G-invariant homogeneous

https://doi.org/10.1112/S0010437X10005233 Published online by Cambridge University Press

polynomial P of degree d on V such that $P(\varphi(x)) \neq 0$. It follows that there exists a G-invariant section τ of $\mathcal{L}^{\otimes nd}$ such that $\tau(x) \neq 0$. $\square$

2.6 The Borel–Weil theorem

Let P be a parabolic subgroup of G. Let ν be a character of P. Let $\mathbb{C}_\nu$ denote the field $\mathbb{C}$ endowed with the action of P defined by $p \cdot \tau = \nu(p)\tau$ for all $\tau \in \mathbb{C}_\nu$ and $p \in P$. We define the line bundle $G \times_P \mathbb{C}_{-\nu}$ on G/P as the quotient of $G \times \mathbb{C}_{-\nu}$ by the following equivalent relation

$$\forall g \in G, \forall \tau \in \mathbb{C}_\nu \text{ and } \forall p \in P, \quad (g, \tau) \sim (gp, p^{-1} \cdot \tau).$$

It is a G-linearized line bundle on G/P, denoted by $\mathcal{L}_\nu$. In fact, the map

$$\begin{aligned} X(P) &\longrightarrow \mathrm{Pic^G(G/P)} \\ \nu &\longmapsto \mathcal{L}_\nu \end{aligned}$$

is an isomorphism.

We assume that P contains B (in that case, P is said to be *standard*). Then $X(P)$ identifies with a subgroup of $X(T)$. For $\nu \in X(P)$, $\mathcal{L}_\nu$ is semiample if and only if it has non-zero sections if and only if ν is dominant. Moreover, $\mathrm{H}^0(G/P, \mathcal{L}_\nu)$ maps onto $\mathrm{H}^0(P/P, \mathcal{L}_\nu) \simeq \mathbb{C}_{-\nu}$ and is the irreducible G-module of extremal weight $-\nu$, that is $V_G(\nu)^*$. For ν dominant, $\mathcal{L}_\nu$ is ample if and only if ν cannot be extended to a subgroup of G bigger than P.

2.7 The Brion theorem

We will need the following theorem, due to Brion, on multiplicity free subvarieties of G/B.

THEOREM 5 [Bri03, Theorem 1]. *Let V be a multiplicity free subvariety of G/B and $\mathcal{L}$ be any semiample G-linearized line bundle on G/B, then the following hold.*

(i) *The subvariety V is normal.*

(ii) *The restriction map $\mathrm{H}^0(G/B, \mathcal{L}) \longrightarrow \mathrm{H}^0(V, \mathcal{L})$ is surjective.*

2.8 Proof of Theorem 4

2.8.1 We first prove an **asymptotic version** of Theorem 4, that is Proposition 1 of the introduction.

Proof of Proposition 1. Set $X = \hat{G}/\hat{B}$. By the Borel–Weil theorem, we have

$$\mathrm{H}^0(X, \mathcal{L}_{\hat{\lambda}}) = V_{\hat{G}}(\hat{\lambda})^*.$$

It remains to prove that, for some $n > 0$, $\mathcal{L}_{\hat{\lambda}}^{\otimes n}$ admits a non-zero section that is an eigenvector of weight $-n\overline{\rho(\hat{w}\hat{\lambda})}$ for the opposite Borel subgroup B^- of G. This is made more precisely in Lemma 2 below. $\square$

LEMMA 2. *There exists $n > 0$ such that $\mathcal{L}_{\hat{\lambda}}^{\otimes n}$ admits a section τ which is an eigenvector of weight $-n\overline{\rho(\hat{w}\hat{\lambda})}$ for B^- such that the restriction of τ to $X_{\hat{w}}^\circ$ is non-zero.*

Proof. Consider the variety $Y = X \times G/B^-$ endowed with the diagonal action of G given by $g' \cdot (\hat{g}\hat{B}/\hat{B}, gB^-/B^-) = (g'\hat{g}\hat{B}/\hat{B}, g'gB^-/B^-)$. Let $\mathcal{L}^-_{\overline{-\rho(\hat{w}\hat{\lambda})}}$ be the G-linearized line bundle on G/B^- such that B^- acts on the fiber over B^- by the character $\overline{\rho(\hat{w}\hat{\lambda})}$. We also consider the line

https://doi.org/10.1112/S0010437X10005233 Published online by Cambridge University Press

bundle $\mathcal{M} := \mathcal{L}_{\hat\lambda} \boxtimes \mathcal{L}^-_{\overline{-\rho(\hat w\hat\lambda)}}$ on Y. Note that $\mathcal{M}$ is semiample because $\hat\lambda$ is dominant and $-\overline{\rho(\hat w\hat\lambda)}$ is dominant with respect to B^-.

By definition of $\overline{\rho(\hat w\hat\lambda)}$, there exists $v \in W$ such that $\overline{\rho(\hat w\hat\lambda)} = v\rho(\hat w\hat\lambda)$. Then it is clear that T acts trivially on the fiber in $\mathcal{M}$ over the point $y := (v\hat w\hat B/\hat B, B^-/B^-)$. Now, applying Lemma 1, we obtain, for some $n > 0$, a section $\tau_Y \in H^0(Y, \mathcal{M}^{\otimes n})^G$ such that $\tau_Y(y) \neq 0$.

Define τ as the restriction of τ_Y to $X \times B^-/B^-$ seen as a section of $\mathcal{L}_{\hat\lambda}$ on X. Since τ_Y is G-invariant, τ is B^--equivariant of weight $-n\overline{\rho(\hat w\hat\lambda)}$. It is clear that $\tau(v\hat w\hat B/\hat B) \neq 0$, so that the restriction of τ to $X^\circ_{\hat w}$ is non-zero. The lemma is proved. □

2.8.2 We have already seen that it is sufficient to prove Theorem 4 under assumption (iii). By Theorem 5, it is sufficient to prove the following.

Theorem 6. *Let $\hat\lambda$ be a dominant character of $\hat T$ and $\hat w \in \hat W$. We assume the following.*

(i) *The variety $X_{\hat w}$ is normal.*

(ii) *The restriction map $\mathrm{H}^0(\hat G/\hat B, \mathcal{L}_{\hat\lambda}) \longrightarrow \mathrm{H}^0(X_{\hat w}, \mathcal{L}_{\hat\lambda})$ is surjective.*

Then $V_G(\overline{\rho(\hat w\hat\lambda)})$ is a G-submodule of $V_{\hat G}(\hat\lambda)$.

Proof. Consider the following restriction maps:

$$\mathrm{H}^0(\hat G/\hat B, \mathcal{L}_{\hat\lambda}) \twoheadrightarrow \mathrm{H}^0(X_{\hat w}, \mathcal{L}_{\hat\lambda}) \longrightarrow \mathrm{H}^0(X^\circ_{\hat w}, \mathcal{L}_{\hat\lambda}).$$

Since the first one is surjective and G-equivariant, it is sufficient to find $V_G(\overline{\rho(\hat w\hat\lambda)})^*$ in $\mathrm{H}^0(X_{\hat w}, \mathcal{L}_{\hat\lambda})$. We will first prove that $V_G(\overline{\rho(\hat w\hat\lambda)})^*$ is a submodule of $\mathrm{H}^0(X^\circ_{\hat w}, \mathcal{L}_{\hat\lambda})$ without multiplicity. Next, we will prove that the corresponding B^--equivariant section on $X^\circ_{\hat w}$ extends to $X_{\hat w}$ using both the asymptotic version and the normality of $X_{\hat w}$.

By Lemma 3 below, there exists a (unique up to scalar multiplication) non-zero regular section σ of $\mathcal{L}_{\hat\lambda}$ on $X^\circ_{\hat w}$ which is B^--equivariant of weight $\overline{-\rho(\hat w\hat\lambda)}$.

Let $n > 0$ and τ be as in Lemma 2. Then $\tau_{|X^\circ_{\hat w}}$ and $\sigma^{\otimes n}$ are two non-zero regular sections of $\mathcal{L}_{n\hat\lambda}$ on $X^\circ_{\hat w}$ which are B^--equivariant of weight $-n\overline{\rho(\hat w\hat\lambda)}$. By Lemma 3, it follows that $\tau_{|X^\circ_{\hat w}}$ and $\sigma^{\otimes n}$ are proportional. In particular, $\sigma^{\otimes n}$ extends to a regular section of $\mathcal{L}_{n\hat\lambda}$ on $X_{\hat w}$. Since $X_{\hat w}$ is normal, it follows that σ also extends to a regular section of $\mathcal{L}_{\hat\lambda}$ on $X_{\hat w}$. The theorem is proved. □

Notation. If H is an algebraic affine group, χ is a character of H and V is a H-module, we set

$$V^{(H)_\chi} = \{v \in V \mid \forall h \in H, h \cdot v = \chi(h)v\}.$$

Lemma 3. *The G-module $V_G(\overline{\rho(\hat w\hat\lambda)})^*$ has multiplicity exactly one in $\mathrm{H}^0(X^\circ_{\hat w}, \mathcal{L}_{\hat\lambda})$.*

Proof. Let $G_{\hat w} \subset G$ be the isotropy group of $\hat w\hat B/\hat B$ so that $X^\circ_{\hat w}$ is isomorphic to the homogeneous space $G/G_{\hat w}$. Let us define $\mu = \rho(\hat w\hat\lambda)$. Since $G_{\hat w}$ acts on the fiber $(\mathcal{L}_{\hat\lambda})_{\hat w\hat B/\hat B}$ by the character $-\mu$, the line bundle $\mathcal{L}$ on $X^\circ_{\hat w}$ is isomorphic to $G \times_{G_{\hat w}} \mathbb{C}_{-\mu}$.

https://doi.org/10.1112/S0010437X10005233 Published online by Cambridge University Press

Then, by using the Frobenius decomposition, the space of global sections $H^0(G/G_{\hat{w}}, G \times_{G_{\hat{w}}} \mathbb{C}_{-\mu})$ can be identified with

$$\bigoplus_{\chi} V_G^*(\chi) \otimes (V_G(\chi))^{(G_{\hat{w}})_\mu},$$

where the sum is over the set of dominant weights of G. Hence we have to prove that the vector space $V_G(\overline{\mu})^{(G_{\hat{w}})_\mu}$ is one-dimensional. First, since $G_{\hat{w}} = G \cap \hat{w}\hat{B}\hat{w}^{-1}$ contains T, the dimension of $V_G(\overline{\mu})^{(G_{\hat{w}})_\mu}$ is less than or equal to one.

The dimension is exactly one if $G_{\hat{w}}$ is contained in the parabolic group $P_G(\mu)$ associated to the weight μ. By Lemma 2, there exist an integer n and a section $\tau \in H^0(X_{\hat{w}}, \mathcal{L}_{n\hat{\lambda}})^{(B^-)_{-n\overline{\mu}}}$ such that the restriction of τ to $X_{\hat{w}}^\circ$ is non-zero. So the dimension of $H^0(X_{\hat{w}}^\circ, \mathcal{L}_{n\hat{\lambda}})^{(B^-)_{-n\overline{\mu}}}$ is at least one. By using the Frobenius decomposition as above, we deduce that the dimension of $V_G(\overline{n\mu})^{(G_{\hat{w}})_{n\mu}}$ is at least (and so equal to) one, and that the parabolic group $P_G(n\mu)$ associated to the weight $n\mu$ contains the group $G_{\hat{w}}$. We conclude by saying that $P_G(n\mu) = P_G(\mu)$. □

3. Applications

3.1 Applications to the Kronecker product

The aim of this section is to detail our results for $\mathrm{Gl}(E) \times \mathrm{Gl}(F) \subset \mathrm{Gl}(E \otimes F)$. This problem is equivalent to the question on the decomposition of tensor products of representations for the symmetric group.

A partition π is a sequence $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ of weakly decreasing non-negative integers. By convention, we allow partitions with some zero parts, and two partitions that differ by zero parts are the same. If several parts are equal we denote the multiplicity of this part by an exponent. For example $(3^2, 2^4, 1)$ means the partition $(3, 3, 2, 2, 2, 2, 1)$. For any partition π, we define $|\pi| = \pi_1 + \pi_2 + \cdots + \pi_k$ and define $l(\pi)$ as the number of non-zero parts of π.

Recall that if V is a finite-dimensional vector space, then the $\mathrm{Gl}(V)$-irreducible polynomial representations are in bijection with the partitions π such that $l(\pi) \leqslant \dim V$: we denote by $S_\pi V$ the representation associated to π.

Let E, F be two vector spaces of respective dimensions m, n, and consider $G = \mathrm{Gl}(E) \times \mathrm{Gl}(F)$ and $\hat{G} = \mathrm{Gl}(E \otimes F)$. Let γ be a partition such that $l(\gamma) \leqslant mn$. We can decompose the irreducible representation $S_\gamma(E \otimes F)$ as a G-representation:

$$S_\gamma(E \otimes F) = \sum_{\alpha,\beta} N_{\alpha\beta\gamma} S_\alpha E \otimes S_\beta F,$$

where the sum is taken over partitions α, β such that $|\alpha| = |\beta| = |\gamma|$, $l(\alpha) \leqslant m$ and $l(\beta) \leqslant n$.

Remark. The irreducible representations of the symmetric group $\mathfrak{S}_n$ correspond bijectively with the partitions π such that $|\pi| = n$; we denote by $[\pi]$ the representation corresponding to π. By using the Schur–Weyl duality, we can show that $N_{\alpha\beta\gamma}$ is also the multiplicity of $[\gamma]$ in $[\alpha] \otimes [\beta]$ (see for example [FH91, ch. 6]). Now, the fact that the representations of $\mathfrak{S}_n$ are self-dual implies that $N_{\alpha\beta\gamma}$ is symmetric in α, β and γ.

We fix bases of E and F and we denote by T_E and T_F the maximal tori of $\mathrm{Gl}(E)$ and $\mathrm{Gl}(F)$ consisting of diagonal matrices. For $i = 1, \ldots, m$, denote by η_i the character that maps an element of T_E to its ith diagonal coefficient. Similarly, we define the characters δ_j's of T_F. The basis of E and F induce a natural basis of $E \otimes F$ indexed by pairs (i, j). Let $\hat{T}$ denote the corresponding maximal torus of $\hat{G}$ and $\hat{\varepsilon}_{i,j}$ the character of $\hat{T}$ corresponding to (i, j). Note that $\rho(\hat{\varepsilon}_{i,j}) = (\eta_i, \delta_j)$.

The coordinates of characters of $\hat{T}$ in the basis $\hat{\varepsilon}_{i,j}$, which are indexed by pairs (i,j), will be represented in tableaux of m rows and n columns. For any tableau t (identified with the corresponding character of $\hat{T}$), $\rho(t)$ is obtained by summing along columns, to obtain the coordinates of a character of T_E, and along rows to obtain the coordinates of a character of T_F.

In Theorem 4, the weights of the form $\hat{w}\hat{\lambda}$ are exactly the extremal weights of $V_{\hat{G}}(\hat{\lambda})$. In particular, they do not depend on the choice of a Borel subgroup of $\hat{G}$ but only on $\hat{T}$ and the representation $V_{\hat{G}}(\hat{\lambda})$. Here, we have fixed the torus and the representation: the extremal weights of $\hat{T}$ in $S_\gamma(E \otimes F)$ are the tableaux $m \times n$ filled by the parts of γ.

For example, suppose that $m = n = 3$ and the two following tableaux correspond to extremal weights of $S_{1^4}(E \otimes F)$:

$$\begin{array}{|c|c|c|c}
\cline{1-3} 1 & 1 & 1 & \;3 \\
\cline{1-3} 1 & & & \;1 \\
\cline{1-3} & & & \;0 \\
\cline{1-3} \multicolumn{1}{c}{2} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{1} &
\end{array}
\qquad\qquad
\begin{array}{|c|c|c|c}
\cline{1-3} 1 & & 1 & \;2 \\
\cline{1-3} & 1 & & \;1 \\
\cline{1-3} & & 1 & \;1 \\
\cline{1-3} \multicolumn{1}{c}{1} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} &
\end{array}$$

where the boxes corresponding to zero coordinates are left empty.

In the first tableau, $\rho(t) = \overline{\rho(t)} = ((3,1,0),(2,1,1))$. We can easily check that the irreducible representation $[1^4]$ (which is the one-dimensional representation given by the signature of $\mathfrak{S}_4$) appears in the tensor product $[3,1] \otimes [2,1^2]$.

In the second tableau, $\rho(t) = ((2,1,1),(1,1,2))$ and $\overline{\rho(t)} = ((2,1,1),(2,1,1))$. We can check that $[2^4]$ appears in $[4,2^2] \otimes [4,2^2]$, which matches with our asymptotic result (Proposition 1). However, the irreducible representation $[1^4]$ does not appear in the tensor product $[2,1^2] \otimes [2,1^2]$. Then Theorem 4 shows that some $\mathrm{Gl}(E) \times \mathrm{Gl}(F)$-orbit closures of the form $X_{\hat{w}}$ of the complete flag variety of $E \times F$ are not multiplicity free. A natural but probably difficult question appears here: which orbit closures $X_{\hat{w}}$ (for $\hat{w} \in \hat{W}$) are multiplicity free?

We can prove that the $\hat{w}$ in $\hat{W}$ such that the orbit $X^\circ_{\hat{w}}$ is closed, correspond bijectively to standard tableaux $m \times n$. Now case (i) of Theorem 4 gives the following rule to compute some components of the tensor product of two representations of the symmetric group. We do not know if this rule is already known.

Rule. (1) Fill the tableau $m \times n$ by the parts of γ in weakly decreasing order along rows and columns.

(2) Sum along rows and columns to obtain α and β.

Then $[\gamma]$ appears in $[\alpha] \otimes [\beta]$.

For example, the tableaux

$$\begin{array}{|c|c|c|c}
\cline{1-3} 2 & 1 & 1 & \;4 \\
\cline{1-3} 1 & 1 & & \;2 \\
\cline{1-3} & & & \\
\cline{1-3} \multicolumn{1}{c}{3} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{1} &
\end{array}
\qquad\qquad
\begin{array}{|c|c|c|c}
\cline{1-3} 4 & 3 & 3 & \;10 \\
\cline{1-3} 2 & 1 & & \;3 \\
\cline{1-3} 1 & & & \;1 \\
\cline{1-3} \multicolumn{1}{c}{7} & \multicolumn{1}{c}{4} & \multicolumn{1}{c}{3} &
\end{array}
\qquad\qquad
\begin{array}{|c|c|c|c}
\cline{1-3} 4 & 3 & 2 & \;9 \\
\cline{1-3} 3 & 2 & 1 & \;6 \\
\cline{1-3} 3 & 2 & 1 & \;6 \\
\cline{1-3} \multicolumn{1}{c}{10} & \multicolumn{1}{c}{7} & \multicolumn{1}{c}{4} &
\end{array}$$

show that the representations $[2,1^4]$, $[4,3^2,2,1^2]$, $[4,3^3,2^3,1^2]$ appear in the respective tensor products: $[4,2] \otimes [3,2,1]$, $[7,4,3] \otimes [10,3,1]$, $[9,6,6] \otimes [10,7,4]$.

https://doi.org/10.1112/S0010437X10005233 Published online by Cambridge University Press

3.2 Application to a branching rule

Here we apply Theorem 4 to the subgroup $G=\mathrm{Sp}(2n)$ of $\hat{G}=\mathrm{Gl}(2n)$. This subgroup is spherical of minimal rank, so that Theorem 4 applies for any $\hat{\lambda}$ and $\hat{w}$.

We define G as the subgroup of $\mathrm{Gl}(2n)$ which preserves the alternate form given by the matrix

$$I=\begin{pmatrix} J & 0 & \dots & 0 \\ 0 & J & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J \end{pmatrix}$$

where $J=\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then we choose for $\hat{T}$ the group of invertible diagonal matrices, and for any $i\in\{1,\dots,2n\}$, we denote by $\hat{\varepsilon}_i$ the usual character of $\hat{T}$. Set $\hat{\lambda}=\hat{\lambda}_1\hat{\varepsilon}_1+\dots+\hat{\lambda}_{2n}\hat{\varepsilon}_{2n}$. The Weyl group $\hat{W}$ is the symmetric group $\mathfrak{S}_{2n}$ and $\hat{w}^{-1}\hat{\lambda}=\hat{\lambda}_{\hat{w}(1)}\hat{\varepsilon}_1+\dots+\hat{\lambda}_{\hat{w}(2n)}\hat{\varepsilon}_{2n}$, for $\hat{w}\in\mathfrak{S}_{2n}$.

Set $T=G\cap\hat{T}$ and define, for any $i\in\{1,\dots,n\}$, the restriction $\varepsilon_i=\rho(\hat{\varepsilon}_{2i-1})$. Then $(\varepsilon_1,\dots,\varepsilon_n)$ is a basis of characters of T and we have

$$\rho(\hat{w}^{-1}\hat{\lambda})=(\hat{\lambda}_{\hat{w}(1)}-\hat{\lambda}_{\hat{w}(2)},\hat{\lambda}_{\hat{w}(3)}-\hat{\lambda}_{\hat{w}(4)},\dots,\hat{\lambda}_{\hat{w}(2n-1)}-\hat{\lambda}_{\hat{w}(2n)}).$$

The Weyl group W acts on the characters of T by permuting coordinates and by multiplying some coordinates by -1. So $\overline{\rho(\hat{w}^{-1}\hat{\lambda})}$ is obtained by arranging in a weak decreasing order the absolute values $|\hat{\lambda}_{\hat{w}(2i-1)}-\hat{\lambda}_{\hat{w}(2i)}|$, for $i\in\{1,\dots,n\}$. We summarize this in the following.

Rule. (1) Consider a permutation $(\hat{\lambda}_{\hat{w}(1)},\dots,\hat{\lambda}_{\hat{w}(2n)})$ of the coordinates of a dominant weight $\hat{\lambda}$ of $\mathrm{Gl}(2n)$.

(2) Order the n absolute values $|\hat{\lambda}_{\hat{w}(2i-1)}-\hat{\lambda}_{\hat{w}(2i)}|$ to obtain a dominant weight μ of G.

Then the multiplicity of $V_G(\mu)$ in $\hat{V}_{\hat{G}}(\hat{\lambda})$ is non-zero.

There exist combinatorial models computing the decomposition as a $\mathrm{Sp}(2n)$-module of a given $\mathrm{Gl}(2n)$-module. Our rule only gives some irreducible $\mathrm{Sp}(2n)$-components of a given $\mathrm{Gl}(2n)$-module, but more directly. It seems that our rule cannot be directly deduced from known models like the one explained in [Sun90].

4. Tensor product decomposition

The aim of this section is to prove Theorem 2 stated in the introduction. We also give, at the end, two examples.

Remark. (1) Condition (i) of Theorem 2 implies that $\langle v\mu,\alpha^\vee\rangle\geqslant 0$ and $\langle w\nu,\alpha^\vee\rangle\geqslant 0$.

(2) Theorem 2 asserts that the half-line generated by (λ,μ,ν) is saturated in the Littlewood–Richardson semigroup.

Indeed, assume that $\lambda=v\mu+w\nu+k\alpha$ with a rational number k satisfies $(-w_0\lambda+\mu+\nu)_{|Z(G)}=0$. We obtain that $-w_0\lambda+\mu+\nu=(\lambda-w_0\lambda)+(\mu-v\mu)+(\nu-w\nu)+k\alpha$. However, $\lambda-w_0\lambda$, $\mu-v\mu$ and $\nu-w\nu$ belong to the root lattice. It follows that $k\alpha$ has to belong to the root lattice and so k is an integer.

The strategy of the proof of Theorem 2 is similar to that of Theorem 6. Hence, we first prove adaptations of Proposition 1 and of Lemma 3.

4.1 Asymptotic version

To prove Proposition 1, we used Lemma 1 mainly due to Kostant; here, in order to prove Lemma 5 below, we will need to use the following strong result of semistability mainly due to Luna.

Lemma 4. *Consider the variety $Y=(G/B)^3$. Let λ,μ and ν be three dominant weights of T. Let β be a root of (G,T). Denote by S the neutral component of the Kernel of β in T. Let $(u,v,w)\in W^3$ and C be the irreducible component of Y^S containing $(uB/B, vB/B, wB/B)$. We assume that $u\Phi^+\cap v\Phi^+\cap w\Phi^+$ contains β.*

The following are equivalent:

(i) *C contains semistable points with respect to $\mathcal{L}_\lambda\boxtimes\mathcal{L}_\mu\boxtimes\mathcal{L}_\nu$;*

(ii) $$\begin{cases}(u\lambda+v\mu+w\nu)_{|S}=0\\ \langle u\lambda,\beta^\vee\rangle+\langle v\mu,\beta^\vee\rangle-\langle w\nu,\beta^\vee\rangle\geqslant 0,\\ \langle u\lambda,\beta^\vee\rangle-\langle v\mu,\beta^\vee\rangle+\langle w\nu,\beta^\vee\rangle\geqslant 0,\\ -\langle u\lambda,\beta^\vee\rangle+\langle v\mu,\beta^\vee\rangle+\langle w\nu,\beta^\vee\rangle\geqslant 0.\end{cases}$$

Proof. Let L be the centralizer of S in G; it is a Levi subgroup of G of semisimple rank one. The variety C is isomorphic to the product of three copies of the complete flag manifold of L, i.e. $(\mathbb{P}^1)^3$. Moreover, $(\mathcal{L}_\lambda\boxtimes\mathcal{L}_\mu\boxtimes\mathcal{L}_\nu)_{|C}$ is isomorphic as an abstract line bundle to $\mathcal{O}(\langle u\lambda,\beta^\vee\rangle)\boxtimes\mathcal{O}(\langle v\mu,\beta^\vee\rangle)\boxtimes\mathcal{O}(\langle w\nu,\beta^\vee\rangle)$. Note that $\langle u\lambda,\beta^\vee\rangle$, $\langle v\mu,\beta^\vee\rangle$ and $\langle w\nu,\beta^\vee\rangle$ are non-negative integers, because $\beta\in u\Phi^+\cap v\Phi^+\cap w\Phi^+$.

It is not difficult to check that $(\mathbb{P}^1)^3$ has semistable points for the action of SL_2 or PSL_2 with respect to $\mathcal{O}(a)\boxtimes\mathcal{O}(b)\boxtimes\mathcal{O}(c)$ (where a, b and c are non-negative integers) if and only if we have

$$\begin{cases}a+b-c\geqslant 0,\\ a-b+c\geqslant 0,\\ -a+b+c\geqslant 0.\end{cases}$$

Now, the first equation of condition (ii) means that S acts trivially on $(\mathcal{L}_\lambda\boxtimes\mathcal{L}_\mu\boxtimes\mathcal{L}_\nu)_{|C}$; and so, induces a L/S-linearized line bundle on C. The three inequalities of condition (ii) are equivalent to the fact that C contains semistable points for the action of L/S (which is isomorphic to SL_2 or PSL_2) with respect to $(\mathcal{L}_\lambda\boxtimes\mathcal{L}_\mu\boxtimes\mathcal{L}_\nu)_{|C}$. Now, it is clear that condition (i) implies condition (ii).

The converse implication is a direct application of [Lun75, Corollary 2, Remark 1] (see also [Res10b, Proposition 8] for a formulation that can be directly applied here). □

We use notation from § 2 with $\hat{G}=G\times G$. In particular, $X^\circ_{v,w}$ is the G-orbit of $(vB/B, wB/B)$ in $X=(G/B)^2$.

We now prove the adaptation of Lemma 2.

Lemma 5. *With the assumptions of Theorem 2, there exist $n>0$ and a section τ of $(\mathcal{L}_\mu\boxtimes\mathcal{L}_\nu)^{\otimes n}$ of weight $-n\lambda$ for B^- whose restriction to $X_{v,s_\alpha w}$ is non-zero.*

Proof. We apply Lemma 4 with the dominant weights $-w_0\lambda$, μ and ν, the root α and $(s_\alpha w_0,\ v,\ w)\in W^3$. Then the first equation of condition (ii) of Lemma 4 is clearly satisfied and the three inequalities of condition (ii) are respectively equivalent to

$$\begin{cases}k\leqslant\langle v\mu,\alpha^\vee\rangle,\\ k\leqslant\langle w\nu,\alpha^\vee\rangle,\\ k\geqslant 0.\end{cases}$$

We now remark, because of condition (i) of Theorem 2, that $\{w_0B/B\}\times X_{v,s_\alpha w}$ intersects $C\cap(\{w_0B/B\}\times X)$ along an open subset, and we conclude the proof of the lemma, using the same arguments as in Lemma 2. □

https://doi.org/10.1112/S0010437X10005233 Published online by Cambridge University Press

4.2 Proof of Theorem 2

In this section, we suppose that all assumptions of Theorem 2 are fulfilled. We also set $\bar{w} = s_\alpha w$ and we denote by $G_{u,\bar{w}}$ the isotropy subgroup of $(vB/B, \bar{w}B/B)$ in G, i.e. $G_{v,\bar{w}} = vBv^{-1} \cap \bar{w}B\bar{w}^{-1}$.

We now prove the equivalent of Lemma 3.

Lemma 6. *The space $\mathbb{C}[G]^{(B^-)_{-\lambda}\times(G_{v,\bar{w}})_{v\mu+\bar{w}\nu}}$ has dimension one.*

Proof. We first prove that

$$\mathbb{C}[G]^{(B^-)_{-\lambda}\times(T)_{v\mu+\bar{w}\nu}}$$

has dimension one. Let us recall a classical property of some characters of the representation $V_G(\lambda)$: the weights $\lambda - l\alpha$ with $l \in \{0, \ldots, \langle\lambda, \alpha^\vee\rangle\}$ have exactly multiplicity one for T. Frobenius' theorem implies that $\mathbb{C}[G]^{(B^-)_{-\lambda}\times(T)_{v\mu+\bar{w}\nu}}$ is isomorphic to $V_G(\lambda)^{(T)_{v\mu+\bar{w}\nu}}$. Assumption (ii) of Theorem 2 implies that $v\mu + \bar{w}\nu = \lambda - (\langle w\nu, \alpha^\vee\rangle - k)\alpha$. Assumption (iii) of the same theorem implies that $0 \leqslant \langle w\nu, \alpha^\vee\rangle - k \leqslant \langle\lambda, \alpha^\vee\rangle$. We obtain the dimension of $\mathbb{C}[G]^{(B^-)_{-\lambda}\times(T)_{v\mu+\bar{w}\nu}}$ from the above-mentioned classical property.

Let s be a non-zero element of $\mathbb{C}[G]^{(B^-)_{-\lambda}\times(T)_{v\mu+\bar{w}\nu}}$. Since T is contained in $G_{v,\bar{w}}$, it is sufficient to prove that for any $h \in G_{v,\bar{w}}$, we have

$$hs = (v\mu + \bar{w}\nu)(h)s.$$

By Lemma 5, there exist n and a non-zero $s_n \in \mathbb{C}[G]^{(B^-)_{-n\lambda}\times(G_{v,\bar{w}})_{nv\mu+n\bar{w}\nu}}$. Consider the algebra $\mathcal{A} = \bigoplus_{n\geqslant 0} \mathbb{C}[G]^{(B^-)_{-n\lambda}}$. Now, in $\mathcal{A}$, $s^{\otimes n}$ is a non-zero element of $\mathbb{C}[G]^{(B^-)_{-n\lambda}\times(G_{v,\bar{w}})_{nv\mu+n\bar{w}\nu}}$. By the first part of the proof, $s^{\otimes n}$ and s_n have to be proportional. It follows that for any h in $G_{v,\bar{w}}$

$$(hs)^{\otimes n} = h \cdot s^{\otimes n} = (nv\mu + n\bar{w}\nu)(h)s^{\otimes n} = ((v\mu + \bar{w}\nu)(h)s)^{\otimes n}.$$

Since $\mathcal{A}$ is the algebra of regular sections of powers of an ample line bundle over a $P\backslash G$, it is integrally closed. However, for any $h \in G_{v,\bar{w}}$, $(hs/s)^{\otimes n}$ and $(s/hs)^{\otimes n}$ belong to $\mathcal{A}$. Hence, hs and s are proportional. There exists a regular map $\theta : H \longrightarrow \mathbb{C}^*$ such that

$$hs = \theta(h)s$$

for any $h \in G_{v,\bar{w}}$. We easily check that θ must be a character of $G_{v,\bar{w}}$. However, the restriction of θ to T equals $v\mu + \bar{w}\nu$ so that $\theta = v\mu + \bar{w}\nu$. The lemma follows. □

We now prove Theorem 2.

Proof. It remains to prove that $V_G(\lambda)^*$ is a submodule of $V_G(\mu)^* \otimes V_G(\nu)^*$. We interpret the latter module as the space of sections of $\mathcal{L}_\mu \boxtimes \mathcal{L}_\nu$ on X and consider the following sequence of morphisms.

$$\begin{array}{c} \mathrm{H}^0((G/B)^2, \mathcal{L}_\mu \boxtimes \mathcal{L}_\nu) \\ \downarrow \\ \mathrm{H}^0(X_{v,\bar{w}}, \mathcal{L}_\mu \boxtimes \mathcal{L}_\nu) \\ \big\downarrow \\ \mathrm{H}^0(X^\circ_{v,\bar{w}}, \mathcal{L}_\mu \boxtimes \mathcal{L}_\nu) \\ \big\downarrow \\ \mathbb{C}[G]^{\{1\}\times(G_{v,\bar{w}})_{v\mu+\bar{w}\nu}} \end{array}$$

The surjectivity of the first map is a particular case (known before) of Theorem 5. The injectivity of the second map is obvious. Also, the next isomorphism is obtained by applying Frobenius' theorem.

Now, by Lemma 6, there exists a non-zero section σ of $\mathcal{L}_\mu \boxtimes \mathcal{L}_\nu$ on $X^\circ_{v,\bar{w}}$ of weight $-\lambda$ for B^-. Then, for some $n>0$, $\sigma^{\otimes n}$ extends to $X_{v,\bar{w}}$ by Lemmas 5 and 6 together. Since $X_{v,\bar{w}}$ is normal, it follows that σ also extends to a regular section of $\mathcal{L}_\mu \boxtimes \mathcal{L}_\nu$ on $X_{v,\bar{w}}$. Thus, the theorem is proved. □

4.3 Examples

4.3.1 In the following example, we will see that the hypothesis on α to be simple, in Theorem 2, is necessary. Consider $G=\mathrm{Sp}_4$. Denote by α_1 and α_2 respectively the short and the long simple roots, and ω_1 and ω_2 the associated fundamental weights. Let $\mu=\nu=\omega_2$ (and $v=w=\mathrm{Id}$). Then we can compute that

$$V_G(\mu)\otimes V_G(\nu)=V_G(0)\oplus V_G(2\omega_1)\oplus V_G(2\omega_2).$$

Define $\lambda:=v\mu+w\nu-(\alpha_1+\alpha_2)=\omega_2$. Note that λ satisfies the conditions (ii) and (iii) of Theorem 2 with $\alpha=\alpha_1+\alpha_2$, because $\langle\omega_2,(\alpha_1+\alpha_2)^\vee\rangle=2$. We cannot apply Theorem 2 just because $\alpha_1+\alpha_2$ is not a simple root. Also, in fact, $V_G(\lambda)$ is not a submodule of $V_G(\mu)\otimes V_G(\nu)$.

4.3.2 In this section, we look at the positions of the dominant weights λ obtained in Theorem 2 for fixed μ, ν, α and varying k. We prove that, by this way, we obtain an 'integral segment' with at least one extremity corresponding to an original PRV component.

Proposition 2. *Let λ be a dominant weight as in Theorem 2. Suppose, for convenience, that $\langle v\mu,\alpha^\vee\rangle\leqslant\langle w\nu,\alpha^\vee\rangle$. Set $k_{\max}=\langle v\mu,\alpha^\vee\rangle$ and $\lambda_k=v\mu+w\nu-k\alpha$. Let k_0 be such that $\lambda=\lambda_{k_0}$.*

Then, for any $k_0\leqslant k\leqslant k_{\max}$, λ_k is a dominant weight. Moreover, $V_G(\lambda_{k_{\max}})=V_G(s_\alpha v\mu+w\nu)$ is an original PRV component of $V_G(\mu)\otimes V_G(\nu)$.

Proof. Denote by S the set of simple roots of (G,B) and by ω_γ the fundamental weight corresponding to the simple root γ. Then, for all $0\leqslant k\leqslant k_{\max}$, we can write $\lambda_k=\sum_{\gamma\in S}a_{\gamma,k}\omega_\gamma$, with the $(a_{\gamma,k})$ in $\mathbb{Z}$. Note that

$$a_{\alpha,k_{\max}}=\langle\lambda_{k_{\max}},\alpha^\vee\rangle=-\langle v\mu,\alpha^\vee\rangle+\langle w\nu,\alpha^\vee\rangle\geqslant 0.$$

Remark also that

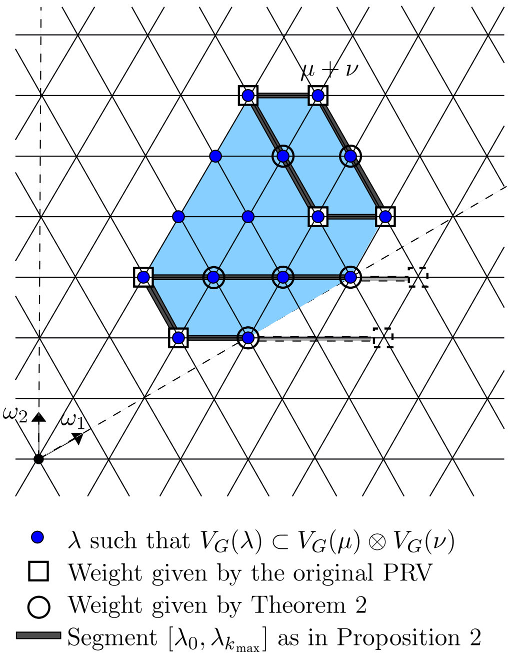
$$\alpha=\sum_{\gamma\in S}\langle\alpha,\gamma^\vee\rangle\omega_\gamma=\sum_{\gamma\in S}b_\gamma\omega_\gamma,\quad \text{with } b_\alpha=2 \text{ and } b_\gamma\leqslant 0,\forall\gamma\neq\alpha.$$

Then $a_{\alpha,k}$ decreases when k increases, and for any $\gamma\neq\alpha$, $a_{\gamma,k}$ increases with k. Moreover, since $a_{\alpha,k_{\max}}\geqslant 0$, $a_{\alpha,k}$ is non-negative for all $0\leqslant k\leqslant k_{\max}$. This implies that, as soon as λ_k is dominant, it stays dominant when k increases up to $k_{\max}$. Now, the proposition follows from the fact that $\lambda=\lambda_{k_0}$ is dominant. □

We now illustrate this proposition by the following example. Consider $G=\mathrm{SL}_3$ with simple roots α_1 and α_2. Let $\mu=7\omega_1+2\omega_2$ and $\nu=\omega_1+3\omega_2$. Then the following picture represents the set of dominant weights λ such that $V_G(\lambda)$ is a submodule of $V_G(\mu)\otimes V_G(\nu)$. In this example, $\mu+\nu$ is an element of the root lattice so that all weights of $V_G(\mu)\otimes V_G(\nu)$ are in the root lattice.

https://doi.org/10.1112/S0010437X10005233 Published online by Cambridge University Press

Then, in order to make the picture nicer, we only draw the root lattice instead of the weight lattice.



Acknowledgement

The authors would like to thank S. Kumar for his particular attention to this paper and also for his judicious remarks.

References

And85 H. H. Andersen, *Schubert varieties and Demazure's character formula*, Invent. Math. **79** (1985), 611–618.

BK10 P. Belkale and S. Kumar, *Eigencone, saturation and horn problems for symplectic and odd orthogonal groups*, J. Algebraic Geom. **19** (2010), 199–242.

Bri01 M. Brion, *On orbit closures of spherical subgroups in flag varieties*, Comment. Math. Helv. **76** (2001), 263–299.

Bri03 M. Brion, *Multiplicity-free subvarieties of flag varieties*, in *Commutative algebra (Grenoble/Lyon, 2001)*, Contemporary Mathematics, vol. 331 (American Mathematical Society, Providence, RI, 2003), 13–23.

Dem74 M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 53–88.

FH91 W. Fulton and J. Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129 (Springer, New York, 1991).

Jos85 A. Joseph, *On the Demazure character formula*, Ann. Sci. École Norm. Sup. (4) **18** (1985), 389–419.

KLM08 M. Kapovich, B. Leeb and J. J. Millson, *The generalized triangle inequalities in symmetric spaces and buildings with applications to algebra*, Mem. Amer. Math. Soc. **192** (2008), viii and 83.

KT99 A. Knutson and T. Tao, *The honeycomb model of* $\mathrm{GL}_n(\mathbf{C})$ *tensor products. I. Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), 1055–1090.

Kum88 S. Kumar, *Proof of the Parthasarathy–Ranga Rao–Varadarajan conjecture*, Invent. Math. **93** (1988), 117–130.

Kum92 S. Kumar, *Proof of Wahl's conjecture on surjectivity of the Gaussian map for flag varieties*, Amer. J. Math. **114** (1992), 1201–1220.

Kum11 S. Kumar, *Tensor product decomposition* in *Proceedings of the international congress of mathematicians 2010, Hyderabad, India* (World Scientific, 2011), available at http://www.math.unc.edu/Faculty/kumar/papers/kumar60.pdf.

Lun75 D. Luna, *Adhérences d'orbite et invariants*, Invent. Math. **29** (1975), 231–238.

Mat89 O. Mathieu, *Construction d'un groupe de Kac–Moody et applications*, Compositio Math. **69** (1989), 37–60.

MFK94 D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, third edition (Springer, New York, 1994).

RR85 S. Ramanan and A. Ramanathan, *Projective normality of flag varieties and Schubert varieties*, Invent. Math. **79** (1985), 217–224.

Res10a N. Ressayre, *Spherical homogeneous spaces of minimal rank*, Adv. Math. **224** (2010), 1784–1800.

Res10b N. Ressayre, *Geometric invariant theory and generalized eigenvalue problem*, Invent. Math. **180** (2010), 389–441.

Ros61 M. Rosenlicht, *On quotient varieties and the affine embedding of certain homogeneous spaces*, Trans. Amer. Math. Soc. **101** (1961), 211–223.

Ses87 C. S. Seshadri, *Line bundles on Schubert varieties*, in *Vector bundles on algebraic varieties, Bombay, 1984*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 11 (Tata Institute of Fundamental Research Studies Bombay, Bombay, 1987), 499–528.

Sun90 S. Sundaram, *Tableaux in the representation theory of the classical Lie groups*, in *Invariant theory and tableaux, Minneapolis, MN, 1988*, The IMA Volumes in Mathematics and its Applications, vol. 19 (Springer, New York, 1990), 191–225.

P. L. Montagard pierre-louis.montagard@math.univ-montp2.fr
Université Montpellier II, CC 51-Place Eugène Bataillon, 34095 Montpellier Cedex 5, France

B. Pasquier boris.pasquier@math.univ-montp2.fr
Université Montpellier II, CC 51-Place Eugène Bataillon, 34095 Montpellier Cedex 5, France

N. Ressayre nicolas.ressayre@math.univ-montp2.fr
Université Montpellier II, CC 51-Place Eugène Bataillon, 34095 Montpellier Cedex 5, France