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RESEARCH ARTICLE

Relationships between cumulative entropy/extropy, Gini mean difference and probability weighted moments

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Abstract

In this work, we establish a connection between the cumulative residual entropy and the Gini mean difference (GMD). Some relationships between the extropy and the GMD, and the truncated GMD and dynamic versions of the cumulative past extropy are also established. We then show that several entropy and extropy measures discussed here can be brought into the framework of probability weighted moments, which would facilitate finding estimators of these measures.

1. Introduction

The Gini mean difference (GMD) is a prominent measure that is used extensively in economics. Let X be a non-negative random variable with absolutely continuous distribution function F and finite mean μ, and X_1 and X_2 be two independent random variables from F. Then, the GMD is defined as

$$\text{GMD} = E|X_1 - X_2|.$$

For more details on GMD and other measures derived from it, interested readers may refer to Yitzhaki and Schechtman [33].

The right-truncated version of the GMD has also been discussed for examining inequality between the poor and affluent groups, while the left-truncated version has found application in reliability analysis. For more details, see Nair *et al.* [21] and Behdani *et al.* [5]. Recently, Nair and Vineshkumar [22] discussed some relationships between the cumulative past entropy and income gap ratio, Lorenz curve, Gini index, Bonferroni curve and Zenga curve.

In many practical situations, measuring the uncertainty associated with a random variable is quite important, and many measures have been introduced for this purpose. The seminal work on information theory started with the concept of Shannon entropy or differential entropy introduced by Shannon [26]. Since then, different measures of entropy have been discussed, each one being suitable for some specific situation. Some of the widely used measures of entropy are cumulative residual entropy [24], cumulative past entropy [12], the corresponding weighed measures by Mirali *et al.* [19] and Mirali and Baratpour [18], and a general measure of cumulative residual entropy [29].

One important generalization of Shannon entropy is due to Tsallis [31], known as generalized Tsallis entropy of order α. Many extensions and modifications have been developed for it as well. For example, Rajesh and Sunoj [23] proposed cumulative residual Tsallis entropy of order α, while Chakraborty and Pradhan [7] defined weighted cumulative residual Tsallis entropy (WCRTE) of order α and its dynamic

version. Calì *et al.* [6] introduced cumulative past Tsallis entropy of order α. Chakraborty and Pradhan [8] introduced weighted cumulative past Tsallis entropy (WCTE) of order α, and also studied its dynamic version.

An alternative measure of uncertainty, called extropy, was introduced by Lad *et al.* [16] as a complementary dual of entropy. Jahanshahi *et al.* [14] and Tahmasebi and Toomaj [30] studied cumulative residual extropy and negative cumulative extropy, while Balakrishnan *et al.* [4] and Chakraborty and Pradhan [9] discussed different weighted versions of extropy. Sudheesh and Sreedevi [28] established some relationships between different extropy measures and reliability concepts, and Sudheesh *et al.* [29] established some relationships between entropy and extropy measures.

The probability weighted moments (PWMs) generalize the concept of moments of a probability distribution, and they have been used effectively for estimating the parameters of different distributions. The PWM was introduced by Greenwood *et al.* [13]. Here, we show that extropy measures mentioned above can be expressed in terms of PWM.

The rest of this article is organized as follows. In Section 2, using a generalized cumulative residual entropy, we derive connections between some entropy measures and GMD. In Section 3, we establish some relationships between GMD and extropy measures. In Section 4, we show that several extropy measures can be brought into the framework of PWM. Finally, we make some concluding remarks in Section 5.

2. Connection between entropy and GMD

Let X be a non-negative random variable with absolute continuous distribution function F and let $\bar{F}(x) = P(X > x)$ denote the survival function. Let us further assume that the mean $\mu = E(X) < \infty$.

2.1. Cumulative residual entropy and GMD

We now consider the generalized cumulative residual entropy measure introduced recently by Sudheesh *et al.* [29].

Definition 1. *Let X be a non-negative random variable with absolute continuous distribution function F. Furthermore, let $\phi(\cdot)$ be a function of X and $w(\cdot)$ be a weight function. Then, the generalized cumulative residual entropy is defined as*

$$\mathcal{GE}(X) = \int_0^\infty w(u)E\left[\phi(X) - \phi(u) \mid X > u\right] dF(u). \tag{1}$$

Sudheesh *et al.* [29] showed that several measures of entropy available in the literature can be deduced from (1) with different choices of $w(\cdot)$ and $\phi(\cdot)$. In particular, for the choice of $w(u) = 1$ and $\phi(x) = x$, (1) reduces to the cumulative residual entropy [3].

We now show that GMD is a special case of $\mathcal{GE}(X)$.

Theorem 1. *For the choice of $w(u) = \bar{F}(u)$ and $\phi(x) = 2x$, $\mathcal{GE}(X) = $ GMD.*

Proof. Let X_1 and X_2 be independent and identical random variables having the distribution function F. With GMD $= E|X_1 - X_2|$, and because $|X_1 - X_2| = \max(X_1, X_2) - \min(X_1, X_2)$, the GMD can alternatively be expressed as

$$\text{GMD} = E(\max(X_1, X_2) - \min(X_1, X_2)).$$

Now, for the choice of $w(u) = \bar{F}(u)$ and $\phi(x) = 2x$, the expression in (1) becomes

$$
\begin{aligned}
\mathcal{GE}(X) &= 2 \int_0^\infty \bar{F}(u) E(X - u \mid X > u) \, dF(u) \\
&= 2 \int_0^\infty \bar{F}(u) \left(\frac{1}{\bar{F}(u)} \int_u^\infty (y - u) \, dF(y) \right) dF(u) \\
&= 2 \int_0^\infty \int_u^\infty y \, dF(y) \, dF(u) - 2 \int_0^\infty u \int_u^\infty dF(y) \, dF(u) \\
&= \int_0^\infty y \cdot 2F(y) \, dF(y) - \int_0^\infty y \cdot 2\bar{F}(y) \, dF(y) \\
&= E(\max(X_1, X_2)) - E(\min(X_1, X_2)),
\end{aligned}
$$

which is precisely the GMD. $\qquad\square$

Remark 1. In general, for the choices of $w(u) = \bar{F}(u)$ and $\phi(x) = 2x^v$ for some positive integer v, we have

$$
\mathcal{GE}(X) = E(\max(X_1, X_2)^v - \min(X_1, X_2)^v).
$$

Also, for $w(x) = \bar{F}^{k-1}(u)$, $k = 2, 3, \ldots$ and $\phi(x) = kx^v$, $v = 1, 2, \ldots$, we get

$$
\mathcal{GE}(X) = E(\max(X_1, \ldots, X_k)^v - \min(X_1, \ldots, X_k)^v).
$$

In the particular case of $v = 1$, we have

$$
\begin{aligned}
\mathcal{GE}(X) &= E(\max(X_1, \ldots, X_k) - \min(X_1, \ldots, X_k)) \\
&= E(\max(X_1, \ldots, X_k)) - E(X_1) + E(X_1) - (E(\min(X_1, \ldots, X_k))) \\
&= EG_k(-X) - EG_k(X),
\end{aligned}
$$

where $EG_k(X)$ and $EG_k(-X)$ are the risk-premium and the gain-premium of X of order k, respectively. These two quantities have found key applications in bid and ask prices in finance; see Agouram and Lakhnati [2]. The above expression can alternatively be expressed as

$$
\begin{aligned}
\mathcal{GE}(X) &= E(\max(X_1, \ldots, X_k)) - E(X_1) + E(X_1) - (E(\min(X_1, \ldots, X_k))) \\
&= k \operatorname{Cov}(X, F^{k-1}(X)) - k \operatorname{Cov}(X, \bar{F}^{k-1}(X)).
\end{aligned}
$$

In particular, when $k = 2$, we obtain

$$
\mathcal{GE}(X) = 4\operatorname{Cov}(X, F(X)),
$$

which is yet another representation of GMD [17].

2.2. *Generalized cumulative past entropy and GMD*

We now consider the generalized cumulative past entropy and discuss its relationship with GMD.

Definition 2 [29]**.** *Let X be a non-negative random variable with absolute continuous distribution function F. Furthermore, let $\phi(\cdot)$ be a function of X and $w(\cdot)$ be a weight function. Then, the generalized cumulative past entropy is defined as*

$$
\mathcal{GCE}(X) = \int_0^\infty w(u) E[\phi(u) - \phi(X) \mid X \le u] \, dF(u). \tag{2}
$$

In particular, for the choice of $w(u) = 1$ and $\phi(x) = x$, (2) reduces to $\mathcal{CE}(X)$ [12].

Theorem 2. *For the choice of $w(u) = F(u)$ and $\phi(x) = x$, $\mathcal{GCE}(X)$ reduces to the GMD.*

Proof. Consider

$$
\begin{aligned}
\mathcal{GE}(X) &= 2 \int_0^\infty F(u) E(u - X \mid X < u) \, dF(u) \\
&= 2 \int_0^\infty F(u) \left(\frac{1}{F(u)} \int_0^u (u - y) \, dF(y) \right) dF(u) \\
&= 2 \int_0^\infty \int_0^u u \, dF(y) \, dF(u) - 2 \int_0^\infty \int_0^u y \, dF(y) \, dF(u) \\
&= E(\max(X_1, X_2)) - \int_0^\infty y \cdot 2\bar{F}(y) \, dF(y) \\
&= E(\max(X_1, X_2) - \min(X_1, X_2)),
\end{aligned}
$$

as required. □

Furthermore, for the choice of $w(u) = F^k(u)$ and $\phi(x) = (k+1)x$, where k is a positive integer, we have

$$
\mathcal{GCE}(X) = E(\max(X_1, \ldots, X_{k+1}) - \min(X_1, \ldots, X_{k+1})).
$$

We now show that GMD is a special case of the cumulative residual Tsallis entropy of order α defined by Rajesh and Sunoj [23] as

$$
\mathrm{CRT}_\alpha(X) = \frac{1}{\alpha - 1} \int_0^\infty (\bar{F}(x) - \bar{F}^\alpha(x)) \, dx, \quad \alpha > 0, \ \alpha \neq 1.
$$

For $\alpha = 2$, the above expression becomes

$$
\mathrm{CRT}_2(X) = \int_0^\infty (\bar{F}(x) - \bar{F}^2(x)) \, dx.
$$

For a non-negative random variable X, we have $\mu = E(X) = \int_0^\infty \bar{F}(x) \, dx$. Noting that $\bar{F}^2(x)$ is the survival function of $\min(X_1, X_2)$, we simply obtain

$$
\mathrm{CRT}_2(X) = E(X_1) - E(\min(X_1, X_2)).
$$

Now, because $|X_1 - X_2| = X_1 + X_2 - 2\min(X_1, X_2)$, we obtain GMD $= 2\mathrm{CRT}_2(X)$.

3. Connection between extropy and GMD

Tahmasebi and Toomaj [30] discussed a relationship between GMD and cumulative past extropy. Now upon using the generalized entropy measures introduced by Sudheesh *et al.* [29], we establish here some relationships between different dynamic extropy measures and the right- and left-truncated GMD.

First, we express the GMD in terms of cumulative residual extropy and cumulative past extropy. For a non-negative random variable X, Jahanshahi *et al.* [14] defined the cumulative residual extropy as

$$
\mathcal{CRJ}(X) = -\frac{1}{2} \int_0^\infty \bar{F}^2(x) \, dx, \tag{3}
$$

while the cumulative past extropy is defined as Tahmasebi and Toomaj [30]

$$CJ(X) = -\frac{1}{2} \int_0^\infty (1 - F^2(x)) \, dx.$$

Various applications of $CRJ(X)$ and $CJ(X)$ have been discussed by Jahanshahi *et al.* [14] and Tahmasebi and Toomaj [30].

Using the survival functions of $\max(X_1, X_2)$ and $\min(X_1, X_2)$, we easily find that

$$
\begin{aligned}
2CRJ(X) - 2CJ(X) &= \int_0^\infty (1 - F^2(x)) \, dx - \int_0^\infty \bar{F}^2(x) \, dx \\
&= E(\max(X_1, X_2) - \min(X_1, X_2)) \\
&= \text{GMD}.
\end{aligned}
$$

Sudheesh and Sreedevi [28] discussed the non-parametric estimation of $CRJ(X)$ and $CJ(X)$ for right censored data, and so using the above relationship, we can readily obtain an estimator of the GMD based on right censored data.

Sathar and Nair [25] defined dynamic survival extropy as

$$J_t(X) = -\frac{1}{2\bar{F}^2(t)} \int_t^\infty \bar{F}^2(x) \, dx, \tag{4}$$

while Kundu [15] introduced dynamic cumulative extropy as

$$H_t(X) = -\frac{1}{2F^2(t)} \int_0^t F^2(x) \, dx.$$

Sudheesh and Sreedevi [28] proposed simple alternative expressions for different extropy measures, and then used them to establish some relationships between different dynamic and weighted extropy measures and reliability concepts.

The left-truncated GMD is given by

$$\text{GMD}_L = \frac{1}{\bar{F}^2(t)} \int_t^\infty (\bar{F}(t) - 2\bar{F}(x))x \, dF(x)$$

while the right-truncated GMD is defined as

$$\text{GMD}_R = \frac{1}{F^2(t)} \int_0^t (2F(x) - F(t))x \, dF(x).$$

Theorem 3. *Let $m(x) = E(X - x \mid X > x)$ and $r(x) = E(x - X \mid X \le x)$. Then, the relationships between the GMD and the dynamic survival extropy and/or the dynamic cumulative extropy are*

(i)

$$\text{GMD}_L = m(x) + 2J_t(X), \tag{5}$$

(ii)

$$\text{GMD}_R = -2H_t(x) - r(x). \tag{6}$$

Proof. Consider the left-truncated GMD given by

$$
\begin{aligned}
\mathrm{GMD}_L &= \frac{1}{\bar{F}^2(t)} \int_t^\infty (\bar{F}(t) - 2\bar{F}(x))x\, dF(x) \\
&= \frac{1}{\bar{F}^2(t)} \int_t^\infty (\bar{F}(t) - t + t - 2\bar{F}(x))x\, dF(x) \\
&= \frac{1}{\bar{F}(t)} \int_t^\infty (x - t)\, dF(x) - \frac{1}{\bar{F}^2(t)} \int_t^\infty (x - t)2\bar{F}(x)\, dF(x) \\
&= m(t) - E(\min(X_1, X_2) - t \mid \min(X_1, X_2) > t).
\end{aligned}
\tag{7}
$$

Sudheesh and Sreedevi [28] expressed $J_t(X)$ in (4) as

$$
J_t(X) = -\tfrac{1}{2} E(\min(X_1, X_2) - t \mid \min(X_1, X_2) > t).
\tag{8}
$$

Substituting (8) in (7), we obtain the relationship in (5).

Next, consider the right-truncated GMD given by

$$
\mathrm{GMD}_R = \frac{1}{F^2(t)} \int_0^t (2F(x) - F(t))x\, dF(x).
\tag{9}
$$

After some algebraic manipulations, we obtain from (9) that

$$
\mathrm{GMD}_R = E(t - \min(X_1, X_2) \mid \min(X_1, X_2) \le t) - r(x).
\tag{10}
$$

Sudheesh and Sreedevi [28] expressed $H_t(X)$ as

$$
H_t(X) = -\tfrac{1}{2} E(t - \max(X_1, X_2) \mid \max(X_1, X_2) \le t).
\tag{11}
$$

Substituting (11) in (10), we obtain the relationship in (6).   □

Theorem 4. *Let X_1 and X_2 be two independent and identical random variables having distribution function F, and $Z = \min(X_1, X_2)$ be the lifetime of a series system comprising two components. Then, the generalized residual entropy associated with Z is the weighted average of the difference between the left-truncated GMD and the mean residual life.*

Proof. Using (1), we define the generalized residual entropy associated with Z as

$$
\mathcal{GE}(Z) = \int 2w(u)E[\phi(Z) - \phi(u) \mid Z > u]\bar{F}(u)\, dF(u),
\tag{12}
$$

where $\phi(\cdot)$ is a function of Z and $w(\cdot)$ is a weight function. Now, for the choice of $w(u) = -\tfrac{1}{2}$ and $\phi(z) = z$, from (12), we obtain

$$
\mathcal{GE}(Z) = \int (\mathrm{GMD}_L - m(u))\bar{F}(u)\, dF(u).
$$

□

Theorem 5. *Let X_1 and X_2 be two independent and identical random variables having distribution function F, and $Z = \max(X_1, X_2)$ be the lifetime of a parallel system comprising two components. Then, the generalized cumulative past entropy associated with Z is the weighted average of the sum of the right-truncated GMD and the mean past life.*

Proof. The generalized cumulative past entropy associated with Z is defined as

$$\mathcal{GCE}(Z) = \int_0^\infty 2w(u)E[\phi(u) - \phi(X) \mid X \le u]F(u)\,dF(u). \tag{13}$$

Again, for the choice of $w(u) = \frac{1}{2}$ and $\phi(z) = z$, from (13), we obtain

$$\mathcal{GCE}(Z) = -\int_0^\infty (\text{GMD}_R + r(x))F(u)\,dF(u).$$

So, the generalized cumulative past entropy associated with Z is the weighted average of sum of the right-truncated GMD and the mean past life. $\qquad\square$

4. Connections with PWMs

We now show that many of the measures discussed in the preceding sections can be represented in terms of PWM.

The PWM of a random variable X with distribution function F is defined as [13]

$$\mathcal{M}_{p,r,s} = \mathbb{E}\{X^p F^r(X)(1 - F(X))^s\}, \tag{14}$$

where p, r and s are any real numbers, for which the involved expectation exists.

First, we consider the GMD and related measures. From the proof of Theorem 1, we can express GMD as

$$\text{GMD} = \int_0^\infty 2yF(y)\,dF(y) - \int_0^\infty 2y\bar{F}(y)\,dF(y),$$

and so we have

$$\text{GMD} = 2E(XF(X)) - 2E(X\bar{F}(X)),$$

which, when compared with (14), yields

$$\text{GMD} = 2\mathcal{M}_{1,1,0} - 2\mathcal{M}_{1,0,1}.$$

Several income inequality measures are derived from the GMD by choosing different weights in the expectation and one among them is the S-Gini family of indices [33]. The absolute S-Gini index is defined as

$$S_v = -\text{Cov}(X, \bar{F}^{v-1}(X)), \quad v > 0 \text{ and } v \ne 1,$$

which can be rewritten as

$$S_v = -E(X\bar{F}^{v-1}(X)) + \frac{1}{v}E(X).$$

Hence, the absolute S-Gini index can be represented as

$$S_v = \frac{1}{v}\mathcal{M}_{1,0,0} - \mathcal{M}_{1,0,v-1}.$$

For $v = 2$, S_v reduces to GMD and hence GMD can also be represented in terms of PWM as

$$\text{GMD} = \frac{1}{v}\mathcal{M}_{1,0,0} - \mathcal{M}_{1,0,1}.$$

Next, we consider different extropy measures. The cumulative past extropy and the weighted cumulative past extropy of X can be expressed as [28]

$$\mathcal{CRJ}(X) = -\tfrac{1}{2}E(\max(X_1, X_2)),$$
$$\mathcal{CRJW}(X) = -\frac{1}{4}E(\max(X_1, X_2)^2).$$

Observing that $2F(x)f(x)$ is the density function of $\max(X_1, X_2)$, the extropy measures given above can be rewritten as

$$\mathcal{CRJ}(X) = -E(XF(X)),$$
$$\mathcal{CRJW}(X) = -\tfrac{1}{2}E(X^2F(X)),$$

so that

$$\mathcal{CRJ}(X) = -\mathcal{M}_{1,1,0} \quad \text{and} \quad \mathcal{CRJW}(X) = -\tfrac{1}{2}\mathcal{M}_{2,1,0}.$$

The cumulative residual entropy and the weighted cumulative residual entropy of X can be expressed as [28]

$$\mathcal{CJ}(X) = -\tfrac{1}{2}E(\min(X_1, X_2)),$$
$$\mathcal{WCJ}(X) = -\tfrac{1}{4}E(\min(X_1, X_2)^2).$$

Here again, we observe that $2\bar{F}(x)f(x)$ is the density function of $\min(X_1, X_2)$, and so

$$\mathcal{CJ}(X) = -E(X\bar{F}(X))$$
$$\mathcal{WCJ}(X) = -\tfrac{1}{2}E(X^2\bar{F}(X)),$$

from which we readily obtain

$$\mathcal{CJ}(X) = -\mathcal{M}_{1,0,1} \quad \text{and} \quad \mathcal{WCJ}(X) = -\tfrac{1}{2}\mathcal{M}_{2,0,1}.$$

Next, we show that the cumulative residual Tsallis entropy of order α can be written in terms of PWM. Let us assume that α is a positive integer with $\alpha \neq 1$. Recall that

$$\mathrm{CRT}_\alpha(X) = \frac{1}{\alpha - 1} \int_0^\infty (\bar{F}(x) - \bar{F}^\alpha(x)) \, dx. \tag{15}$$

We can show that

$$\int_0^\infty x\alpha\bar{F}^{\alpha-1}(x) \, dF(x) = \int_0^\infty \bar{F}^\alpha(x) \, dx,$$

and so we can write from (15) that

$$\mathrm{CRT}_\alpha(X) = \frac{1}{\alpha - 1}E(X) - \frac{\alpha}{\alpha - 1}E(X\bar{F}^\alpha(X)),$$

which yields

$$\mathrm{CRT}_\alpha(X) = \frac{1}{\alpha - 1}\mathcal{M}_{1,0,0} - \frac{\alpha}{\alpha - 1}\mathcal{M}_{1,0,(\alpha-1)}.$$

For a non-negative continuous random variable X, the weighted cumulative residual Tsallis entropy (WCRTE) of order α is defined as [7]

$$\mathrm{WCRT}_\alpha(X) = \frac{1}{\alpha - 1} \int_0^\infty x(\bar{F}(x) - \bar{F}^\alpha(x)) \, dx.$$

After some simple algebra, we can show that

$$\int_0^\infty x^2 \alpha \bar{F}^{\alpha-1}(x) \, dF(x) = \int_0^\infty 2x \bar{F}^\alpha(x) \, dx,$$

so that

$$\mathrm{WCRT}_\alpha(X) = \frac{1}{2(\alpha-1)} E(X^2) - \frac{\alpha}{2(\alpha-1)} E(X^2 \bar{F}^\alpha(X))$$

from which we readily obtain

$$\mathrm{WCRT}_\alpha(X) = \frac{1}{2(\alpha-1)} \mathcal{M}_{2,0,0} - \frac{\alpha}{2(\alpha-1)} \mathcal{M}_{2,0,(\alpha-1)}.$$

Calì *et al.* [6] introduced the cumulative past Tsallis entropy of X as

$$CT_\alpha(X) = \frac{1}{\alpha-1} \int_0^\infty (F(x) - F^\alpha(x)) \, dx, \quad \alpha > 0, \ \alpha \neq 1,$$

while Chakraborty and Pradhan [8] defined the weighted cumulative past Tsallis entropy (WCTE) of order α as

$$WCT_\alpha(X) = \frac{1}{\alpha-1} \int_0^\infty x(F(x) - F^\alpha(x)) \, dx, \quad \alpha > 0, \ \alpha \neq 1.$$

As in the case of CRTE and WCRTE, we obtain

$$CT_\alpha(X) = \frac{1}{(\alpha-1)} \mathcal{M}_{1,0,0} - \frac{\alpha}{(\alpha-1)} \mathcal{M}_{1,(\alpha-1),0},$$

$$\mathrm{WCRT}_\alpha(X) = \frac{1}{2(\alpha-1)} \mathcal{M}_{2,0,0} - \frac{\alpha}{2(\alpha-1)} \mathcal{M}_{2,(\alpha-1),0}.$$

Aaberge [1] introduced a family of inequality measures based on the moments of the Lorenz curve as

$$D_k(F) = \frac{1}{k\mu} \int_0^\infty F(x)(1 - F^k(x)) \, dx, \quad k = 1, 2, \ldots.$$

We note that $D_k(F)$ can be expressed in term of cumulative past Tsallis entropy as $CT_{k+1}(X)/\mu$. Hence, we can express

$$D_k(F) = \frac{1}{k\mu} \mathcal{M}_{1,0,0} - \frac{k+1}{k\mu} \mathcal{M}_{1,k,0}.$$

Sharma and Taneja [27] and Mittal [20] independently introduced the STM entropy of X as

$$S_{\alpha,\beta} = \frac{1}{\beta-\alpha} \int_0^\infty (f^\alpha(x) - f^\beta(x)) \, dx. \tag{16}$$

Some entropy measures discussed in the literature can be derived from (16) by making different choices for α and β.

Sudheesh *et al.* [29] introduced a cumulative residual STM entropy of X as

$$SR_{\alpha,\beta} = \frac{1}{\beta-\alpha} \int_0^\infty (\bar{F}^\alpha(x) - \bar{F}^\beta(x)) \, dx,$$

and the cumulative past STM entropy of X as

$$SP_{\alpha,\beta} = \frac{1}{\beta - \alpha} \int_0^\infty (F^\alpha(x) - F^\beta(x))\, dx.$$

They also introduced weighted versions of these measures as

$$SRW_{\alpha,\beta} = \frac{1}{\beta - \alpha} \int_0^\infty x(\bar{F}^\alpha(x) - \bar{F}^\beta(x))\, dx$$

and

$$SPW_{\alpha,\beta} = \frac{1}{\beta - \alpha} \int_0^\infty x(F^\alpha(x) - F^\beta(x))\, dx.$$

Sudheesh *et al.* [29] showed that $SR_{\alpha,\beta}$ and $SRW_{\alpha,\beta}$ are special cases of (1), while $SP_{\alpha,\beta}$ and $SPW_{\alpha,\beta}$ are special cases of (2).

We now express these measures in terms of PWM as follows, presented without proofs, for the sake of conciseness:

$$SR_{\alpha,\beta} = \frac{1}{\beta - \alpha}(\mathcal{M}_{1,0,(\alpha-1)} - \mathcal{M}_{1,0,(\beta-1)}),$$

$$SP_{\alpha,\beta} = \frac{1}{\beta - \alpha}(\mathcal{M}_{1,(\alpha-1),0} - \mathcal{M}_{1,(\beta-1),0}),$$

$$SRW_{\alpha,\beta} = \frac{1}{2(\beta - \alpha)}(\mathcal{M}_{2,0,(\alpha-1)} - \mathcal{M}_{2,0,(\beta-1)}),$$

$$SPW_{\alpha,\beta} = \frac{1}{2(\beta - \alpha)}(\mathcal{M}_{2,(\alpha-1),0} - \mathcal{M}_{2,(\beta-1),0}).$$

These representations in terms of PWM would enable us to develop inferential methods for all these measures using the known results on PWM.

5. Concluding remarks

The GMD is a well-known measure of dispersion that is used extensively in economics. We have established here some relationships between the information measures and the GMD. We have shown that the GMD is a special case of the generalized cumulative residual entropy proposed recently by Sudheesh *et al.* [29]. We have also established some relationships between the GMD and extropy measures. We have further presented some relationships between the dynamic versions of the cumulative residual/past extropy and the truncated GMD.

The PWMs generalize the concept of moments of a probability distribution. The estimates derived using PWMs are often superior than the standard moment-based estimates. For discussion on different methods of estimation of the PWM, we refer to David and Nagaraja [10], Vexler *et al.* [32] and Deepesh *et al.* [11] and the references therein. We have shown here that many of the information measures can be expressed in terms of PWM, which would facilitate the development of inferential methods for these measures using the well-known results on PWM.

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