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Gromov–Witten Invariants of Blow-ups Along Surfaces

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Abstract. In this paper, using the gluing formula of Gromov–Witten invariants for symplectic cutting developed by Li and Ruan, we established some relations between Gromov–Witten invariants of a semipositive symplectic manifold M and its blow-ups along a smooth surface.

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Key words. gluing formula, blowup, Gromov–Witten invariant, semipositive symplectic manifold.

1. Introduction

In 1995, Ruan and Tian [R1, RT1] first established the mathematical foundation of the theory of quantum cohomology or Gromov–Witten invariants (GW-invariant) for a semipositive symplectic manifold. Recently, the semipositivity condition has been removed by many authors. Now the focus is on the calculations and applications many Fano manifolds have been computed. We think it is important to study the change in GW-invariants under surgery.

According to McDuff [M1], the blow up operation in symplectic gemoetry amounts to the removal of an open symplectic ball followed by the collapse of some boundary directions. Lerman [L] gave a generalization of blowup construction, '*the symplectic cut*'.

Let M be a compact symplectic manifold of dimension $2n$ and $\tilde{M}$ be the blowup of M along a smooth surface. Denote by $p: \tilde{M} \longrightarrow M$ the natural projection and by $\Psi^M_A(\alpha_1, \ldots, \alpha_m)$ the genus zero GW-invariant. In this paper, we proved a formula relating the GW-invariants of M and $\tilde{M}$. Our results is the following theorem:

THEOREM. *Suppose that M is a semipositive compact symplectic manifold and S is a smooth surface in M. If $A \in H_2(M)$, $\alpha_i \in H^*(M)$, $1 \leqslant i \leqslant m$, satisfy either* $\deg\alpha_i > 2$

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or $\deg\alpha_i \leqslant 2$ *and support away from S, then*

$$\Psi_A^M(\alpha_1, \ldots, \alpha_m) = \Psi_{p!(A)}^{\tilde{M}}(p^*\alpha_1, \ldots, p^*\alpha_m),$$

where $p!(A) = PDp^*PD(A)$.

2. General Review of Gluing Formula

The proof of our result is an application of the gluing formula for symplectic cutting developed by Li and Ruan [LR]. Let $\bar{M}^+$, $\bar{M}^-$ be the symplectic cutting of M along $N \subset M$. Here N is the common boundary of $\bar{M}^\pm$ with a S^1-action and a generic level set of a local periodic Hamiltonian function. The quotient $Z = N/S^1$ is the famous symplectic reduction, see [LR], pp. 9–10 for details. Li and Ruan defined *log GW-invariants* $\Psi^{(\bar{M}^\pm, Z)}$ of $\bar{M}^\pm$ relative to Z. For these, the following holds:

Let $K^\pm = (0, \ldots, 0, k_1, \ldots, k_v)$ with $m^\pm$-many zeros be a set of nonnegative integers. Denote by $\mathcal{M}_{A^\pm}(\bar{M}^\pm, Z, K^\pm)$ the spaces of pseudo-holomorphic maps $u^\pm$ which represent the homology classes $A^\pm$ respectively, such that $u^\pm$ has marked points $(x_1, \ldots, x_{m^\pm}, x_{m^\pm+1}, \ldots, x_{m^\pm+v})$ with the property that $u^\pm$ is tangent to Z at $x_{m^\pm+i}$ with order k_i and $f(x_j) \notin Z$ for $0 \leqslant j \leqslant m^\pm$ and $\sum_{i=1}^v k_i = Z^*(A^\pm)$, where Z^* is the Poincaré dual to Z.

Given $u^\pm \in \mathcal{M}_{A^\pm}(\bar{M}^\pm, Z, K^\pm)$, we consider the linearization of $\bar{\partial}$-operator

$$D_{u^\pm} = D\partial_J(u^\pm)\colon C^\infty((u^\pm)^*TM^\pm) \longrightarrow \Omega^{0,1}((u^\pm)^*TM^\pm).$$

If we choose a proper weighted Sobolev norm, we can define the Fredholm index $\mathrm{Ind}D_{u^\pm}$, see section 3.4 of [LR] for details. We also can glue u^+ and u^- to obtain a pseudo-holomorphic maps u in M. Suppose that $[u] = A$. Then

PROPOSITION 2.1 ([LR], Theorem 3.23).

$$\mathrm{Ind}D_{u^+} + \mathrm{Ind}D_{u^-} = 2(n-1)v + 2C_1(A) + 2n - 6 + 2m,$$

where C_1 *is the first Chern class of* M *and* $m = m^+ + m^-$.

Suppose that the homology classes of u^+, u^-, u are A^+, A^-, A respectively. If (u^+, u^-) is another representative and glued to u', by Lemma 2.11 of [LR], we have $[u'] = [u]$. Denote by $C = \{A^+, K^+; A^-, K^-\}$ the gluing component. The gluing formula of Li and Ruan counted the contribution of the gluing components to the GW-invariant of M. Denote by Ψ_C the contribution of C.

Choose a homology basis $\{\beta_b\}$ of $H^*(Z, \mathbf{R})$. Let (δ_{ab}) be its intersection matrix. There is a map $\pi\colon M \longrightarrow \bar{M}^+ \cup_Z \bar{M}^-$, where $\bar{M}^+ \cup_Z \bar{M}^-$ is the union of $\bar{M}^\pm$ along Z. For the gluing component $C = \{A^+, K^+; A^-, K^-\}$, where $K^\pm = (0. \ldots, 0, k_1, \ldots, k_v)$ with $m^\pm$-many zeros, we have

PROPOSITION 2.2 ([LR], Theorem 5.8). *Let* $\alpha_i^\pm$ *be differential forms with* $\deg\alpha_i^+ = \deg\alpha_i^-$ *even. Suppose that* $\alpha_i^+|_Z = \alpha_i^-|_Z$ *and, hence,* $\alpha_i^+ \cup_Z \alpha_i^- \in$

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$H^*(\bar{M}^+ \cup_Z \bar{M}^-; \mathbf{R})$. *Let* $\alpha_i = \pi^*(\alpha_i^+ \cup_Z \alpha_i^-)$. *We have the following gluing formula:*

$$\Psi_C(\alpha_1, \ldots, \alpha_m) = |K| \sum_{I,J} \Psi^{(\bar{M}^+,Z)}_{(A^+,K^+)}(\alpha^+, \beta_I)\delta^{I,J}\Psi^{(\bar{M}^-,Z)}_{(A^-,K^-)}(\alpha^-, \beta_J),$$

where we associate $\beta_i\delta^{i,j}\beta_j$ *to every intersection with* Z *and put* $|K| = k_1 \ldots k_v$, $\delta^{I,J} = \delta^{i_1,j_1} \ldots \delta^{i_v,j_v}$, *and denote by* $\Psi^{(\bar{M}^\pm,Z)}_{(A^\pm,K^\pm)}(\alpha^\pm, \beta_I)$ *the product of log invariants corresponding to each component.*

PROPOSITION 2.3 ([LR], Remark 5.5). *For* $C = \{A^\pm, (0, \ldots, 0)\}$, *we have*

$$\Psi_C(\alpha_i^\pm) = \Psi^{(\bar{M}^\pm,Z)}_{(A^\pm,(0,\ldots,0))}(\alpha_i^\pm).$$

3. Proof of Theorem

In this section, we will consider the changes of the GW-invariants of a semipositive symplectic manifold under the blowup along a smooth surface and prove our result.

Proof of Theorem. We first perform the symplectic cutting as in [H] and [LR]. Since S is a smooth surface in M, the normal bundle N_S is a symplectic vector bundle and has a compatible complex structure. Therefore, we may consider it as a bundle with fiber $(\mathbf{C}^{n-1}, -\sqrt{-1}\sum \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i)$. Furthermore, we may consider N_S over S with the symplectic form

$$\omega_S = \omega|_S + (-\sqrt{-1}\sum \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i),$$

where $\omega|_S$ is the restriction of the symplectic form ω to S, and $z = (z_1, \ldots, z_{n-1})$ are the coordinates in the fiber. The Hamiltonian action is $H(x, z) = |z|^2 - \varepsilon$ and the S^1-action is given by $\mathrm{e}^{i\theta}(x, z) = (x, \mathrm{e}^{i\theta}z)$.

Consider the symplectic vector bundle $N_S \oplus \mathcal{O}$ with the symplectic form $\omega_S + (-\sqrt{-1}\mathrm{d}w \wedge \mathrm{d}\bar{w})$ and the momentum map $\mu(x, z, w) = H(x, z) + |w|^2$ arising from the action of S^1 on $N_S \oplus \mathcal{O}$. As in [H] and [LR], the manifold $M^+ := \{(x, z)|H(x, z) < 0\}$ embeds as an open dense submanifold into the reduced space

$$\overline{M}_S^+ := \{(x, z, w)||z|^2 + |w|^2 = \varepsilon\}/S^1$$

and the difference $\overline{M}_S^+ - M_S^+$ is symplectomorphic to the reduced space $H^{-1}(0)/S^1$.

A similar procedure defines

$$\overline{M}_S^- := \{(x, z, w)||z|^2 - |w|^2 = \varepsilon\}/S^1.$$

It is easy to see that the symplectic manifold $H^{-1}(0)/S^1$ is embedded on both $\overline{M}_S^+$ and $\overline{M}_S^-$ as a codimension 2 symplectic submanifold, but with opposite normal bundles. So the symplectic gluing of $\overline{M}_S^+$ and $\overline{M}_S^-$ along the reduced space $H^{-1}(0)/S^1$ recovers the neighborhood $\mathcal{N}_\delta(S)$, i.e. the normal bundle.

We define $\overline{M}^+ := \overline{M}_S^+$ and $\overline{M}^- := (M - \mathcal{N}_\delta(S)) \bigcup \overline{M}_S^-$. From the above description, we know that the symplectic gluing of $\overline{M}^+$ and $\overline{M}^-$ recovers the original manifold M. Lerman [L] called the operation that produces $\overline{M}^+$ and $\overline{M}^-$ *symplectic cutting*.

According to [H, L, LR], we have

$$\overline{M}^+ = \mathbf{P}(N_S \oplus \mathcal{O}), \qquad \overline{M}^- = \tilde{M}.$$

Because of the assumption of α_i, $1 \leqslant i \leqslant m$, we may assume that $\alpha_i^+ = 0$ if we choose a sufficiently small $\delta > 0$.

We first consider the contribution of each component to the GW-invariants. Therefore, we consider the component $C = \{A^+, K^+; A^-, K^-\}$, where $K^\pm = (0, \ldots, 0, k_1, \ldots, k_v)$ with $m^\pm$-many zeros. From Proposition 2.1, we have

$$\mathrm{Ind} D_{u^+} + \mathrm{Ind} D_{u^-} = 2(n-1)v + 2C_1(A) + 2n - 6 + 2m,$$

where $m = m^+ + m^-$.

According to our convention, $u^\pm: \sum^\pm \longrightarrow \bar{M}^\pm$ may have many connected components $u_i^\pm: \sum_i^\pm \longrightarrow \bar{M}^\pm$, $i = 1, 2, \ldots, l^\pm$. Therefore, we have

$$\begin{aligned} \mathrm{Ind} D_{u^+} &= \sum_{i=1}^{l^+} \mathrm{Ind} D_{u_i^+} \\ &= 2\sum_{i=1}^{l^+} C_1[u_i^+] + (2n-6)l^+ + 2v - 2\sum k_i + 2m^+, \end{aligned} \tag{3.1}$$

where C_1 is the first Chern class of $\overline{M}^+$.

Now we want to calculate $C_1[u_i^+]$. For this purpose, we first recall the first Chern class of projective bundle.

Let V be a complex rank r vector bundle over X and $\pi_1: \mathbf{P}(V) \longrightarrow X$ be the corresponding projective bundle. Let ξ_V be the first Chern class of the tautological line bundle in $\mathbf{P}(V)$. A simple calculation shows that

$$C_1(\mathbf{P}(V)) = \pi_1^* C_1(X) + \pi_1^* C_1(V) - r\xi_V. \tag{3.2}$$

Note that $\overline{M}^+ = \mathbf{P}(N_S \oplus \mathcal{O})$. Applying (3.2) to $\overline{M}^+$, we obtain

$$C_1(\overline{M}^+) = \pi_1^* C_1(S) + \pi_1^* C_1(N_S) - (n-1)\xi, \tag{3.3}$$

where ξ is the first Chern class of the tautological line bundles in $\mathbf{P}(N_S \oplus \mathcal{O})$.

We know that $\bar{M}^+$ is a projective bundle over S with fiber $\mathbf{P}^{n-2}$. Denote by $[u_i^+]^S = \pi_{1*}[u_i^+]$ the homology class of the projection to S of the curve u_i^+. From (3.3), we have the projection formula

$$\pi_1^* C_1(M) \cdot [u_i^+] = C_1(\bar{M}^+) \cdot [u_i^+]^S.$$

From the fact that ξ is Poincaré dual to minus the section at infinity, it follows that $\xi \cdot [u_i^+] = \sum_j k_j$, where the summation runs over the ends of u_i^+. Because of the positivity of M, by (3.3) we have

$$\sum_{i=1}^{l^+} C_1[u_i^+] \geqslant (n-1)\sum k_i, \tag{3.4}$$

where the summation runs over ends of the component u^+. Furthermore, we have

$$\begin{aligned}\mathrm{Ind}D_{u^+} &\geqslant (2n-6)l^+ + 2v - 2\sum k_i + 2(n-1)\sum k_i + 2m^+\\ &= (2n-6)l^+ + 2v + 2(n-2)\sum k_i + 2m^+.\end{aligned}$$

Therefore, we have

$$\mathrm{Ind}D_{u^-} \leqslant 2C_1(A) + (2n-6)(1-l^+) + 2(n-2)(v - \sum k_i) + 2m^+.$$

From the assumption of α_i, $1 \leqslant i \leqslant m$, we may assume all α_i *support away* from the neighborhood $\mathcal{N}_\delta(S)$ of the blowup surface S. So we have $\alpha_i^+ = 0$, $1 \leqslant i \leqslant m$. Therefore, if $m^+ > 0$, we have for any $\beta_b \in H^*(Z)$, $\Psi^{(\overline{M}^+, Z)}_{(A^+, K^+)}(\alpha_i^+, \beta_b) = 0$. This implies $\Psi_C = 0$ except $m^- = m$. Now we assume that $m^- = m$, i.e. $m^+ = 0$. On the other hand, if

$$\sum \deg\alpha_i \neq 2C_1(A) + 2n - 6 + 2m,$$

where C_1 denotes the first Chern class of M, by the definition of the GW-invariants, we have

$$\Psi^M_A(\alpha_1, \ldots, \alpha_m) = \Psi^{\tilde{M}}_{p!(A)}(p^*\alpha_1, \ldots, p^*\alpha_m) = 0.$$

We have proved the assertion of the theorem. Therefore, we also assume that

$$\sum \deg\alpha_i = 2C_1(A) + 2n - 6 + 2m.$$

Then we have

$$\sum \deg\alpha_i \geqslant 2C_1(A) + (2n-6)(1-l^+) + 2(n-2)(v - \sum k_i) + 2m^-,$$

since $l^+ \geqslant 0$, $v \leqslant \sum k_i$. Therefore, by the definition of log GW-invariants, we have for any $\beta_b \in H^*(Z)$, $\Psi^{(\overline{M}^-, Z)}_{(A^-, K^-)}(\alpha_i^+, \beta_b) = 0$ where $K^- = (0, \ldots, 0, k_1, \ldots, k_v)$ with m-many zeros. Therefore, $\Psi_C = 0$ except $C = \{A^-, (0, \ldots, 0)\}$.

Now it remains to prove

$$\Psi^{\tilde{M}}_{p!(A)}(p^*\alpha_1, \ldots, p^*\alpha_m) = \Psi^{(\overline{M}^-, Z)}_{(A^-, (0, \ldots, 0))}(\alpha_1^-, \ldots, \alpha_m^-).$$

To prove this, we perform the symplectic cutting for $\tilde{M}$ around the divisor E as in the first part of the proof. Since the normal bundle N_E of E in $\tilde{M}$ is a symplectic line bundle. Therefore, we have

$$\overline{\tilde{M}}^+ = \mathbf{P}(N_E \oplus \mathcal{O}), \qquad \overline{\tilde{M}}^- \cong \tilde{M}.$$

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Now we also use the gluing theorem to prove that the contribution of stable J-holomorphic curves in $\tilde{M}$ which touch the exceptional divisor E to the GW-invariant of $\tilde{M}$ is zero. We consider the component

$$C = \{p!(A)^+, K^+; p!(A)^-, K^-\}.$$

where $K^{\pm} = (0, \ldots, 0, k_1, \ldots, k_v)$ with $m^{\pm}$-many zeros.

Since $\alpha_i^+ = 0$, $1 \leqslant i \leqslant m$, we have $\Psi_C = 0$ except $K^+ = (k_1, \ldots, k_v)$ and $K^- = (0, \ldots, 0, k_1, \ldots, k_v)$ with m-many zeros.

Now we only consider the following components:

$$C = \{p!(A)^+, (k_1, \ldots, k_v); p!(A)^-, (0, \ldots, 0, k_1, \ldots, k_v)\}.$$

From Proposition 2.1, we have

$$\mathrm{Ind} D_{u^+} + \mathrm{Ind} D_{u^-} = 2(n-1)v + 2C_1(A) + 2n - 6 + 2m,$$

where C_1 denotes the first Chern class of M.

We assume that $u^{\pm}\colon \Sigma^{\pm} \longrightarrow M^{\pm}$ has $l^{\pm}$ connected components $u_i^{\pm}\colon \Sigma_i^{\pm} \longrightarrow M^{\pm}$, $i = 1, \ldots, l^{\pm}$. Then we have

$$\begin{aligned} \mathrm{Ind} D_{u^+} &= \sum_{i=1}^{l^+} \mathrm{Ind} D_{u_i^+} \\ &= (2n-6)l^+ + 2\sum_{i=1}^{l^+} C_1[u_i^+] + 2v - 2\sum k_i\,, \end{aligned} \tag{3.5}$$

where C_1 is the first Chern class of $\overline{\tilde{M}}^+$.

Note that $\overline{\tilde{M}}^+ = \mathbf{P}(N_E \oplus \mathcal{O})$ and $E = \mathbf{P}(N_S)$. Applying (3.2) to $\overline{\tilde{M}}^+$ and E, we obtain

$$\begin{aligned} C_1(\overline{\tilde{M}}^+) &= C_1(E) + C_1(N_E) - 2\xi \\ &= C_1(S) + C_1(N_S) - (n-2)\xi_1 + C_1(N_E) - 2\xi, \end{aligned}$$

where ξ_1 and ξ are the first Chern classes of the tautological line bundles in $\mathbf{P}(N_S)$ and $\mathbf{P}(N_E \oplus \mathcal{O})$ respectively. Here we denote the Chern class and its pullback by the same symbol. It is well known that the normal bundle to E in $\tilde{M}$ is just the tautological line bundle on $E \cong \mathbf{P}(N_S)$. Therefore, $C_1(N_E) = \xi_1$. So we have

$$C_1(\overline{\tilde{M}}^+) = C_1(M) - (n-3)\xi_1 - 2\xi.$$

A similar calculation to (3.4) shows

$$\sum_{i=1}^{l^+} C_1[u_i^+] \geqslant 2(n-2)\sum k_i.$$

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Therefore, we have

$$\begin{aligned}\mathrm{Ind}D_{u^+} &\geqslant (2n-6)l^+ + 2v + 2(2n-5)\sum k_i,\\ \mathrm{Ind}D_{u^-} &\leqslant 2C_1(A) + (2n-6)(1-l^+)+\\ &\quad + 2(n-2)(v-\sum k_i) - 2(n-3)\sum k_i + 2m.\end{aligned}$$

For the same reasons as in the first part of our proof, we may also assume

$$\sum \deg(p^*\alpha_i) = 2C_1(A) + 2n - 6 + 2m.$$

Then

$$\begin{aligned}\sum \deg(p^*\alpha_i) &= 2C_1(A) + 2n - 6 + 2m\\ &> 2C_1(A) + (2n-6)(1-l^+)+\\ &\quad + 2(n-2)\Big(v - \sum k_i\Big) - 2(n-3)\sum k_i + 2m\\ &\geqslant \mathrm{Ind}D_{u^-},\end{aligned}$$

since $v > 0$, $k_i > 0$. Therefore, by the definition of log GW-invariants, we have for any $\beta_b \in H^*(Z)$

$$\Psi^{(\overline{\tilde{M}}^-, Z)}_{(p!(A)^-, K^-)}((p^*\alpha_i)^-, \beta_b) = 0,$$

where $K^- = (0, \ldots, 0, k_1, \ldots, k_v)$ with m-many zeros. Therefore, the contribution of J-holomorphic curves to the GW-invariant is nonzero only if it does not touch the exceptional divisor E, i.e. $C = \{p!(A)^-, (0, \ldots, 0)\}$. So, from Proposition 2.3, we have

$$\Psi^{\tilde{M}}_{p!(A)}(p^*\alpha_1, \ldots, p^*\alpha_m) = \Psi^{(\overline{\tilde{M}}^-, Z)}_{(p!(A)^-, K^-)}((p^*\alpha_1)^-, \ldots, (p^*\alpha_m)^-),$$

where $K^- = (0, \ldots, 0)$ with m-many zeros. However, $\overline{\tilde{M}}^- = \tilde{M} = \overline{M}^-$. Hence, the theorem follows.

EXAMPLES. Here are three ways of producing semipositive symplectic manifolds.

(1) If $K_M = 0$ (the Calabi–Yau case), then (M, ω) is a semipositive for any symplectic form ω.
(2) If M is a complex projective manifold with $|-K_M|$ ample (a 'Fano variety'), then we can take $\omega = -K_M$ to produce a semipositive (M, ω).
(3) If $n \leqslant 3$, M is automatically semipositive.

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