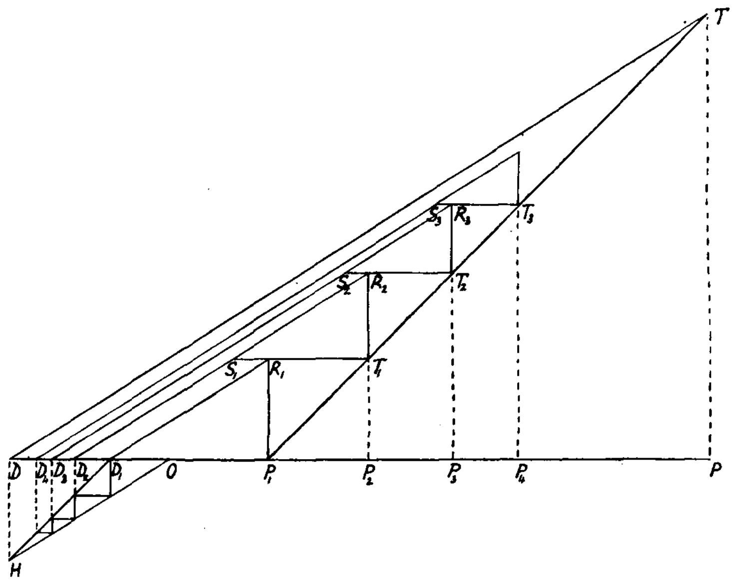
Geometrical summation of the series

$$a, \ (a+d)r, \ (a+2d)r^2, \ldots$$

The following method of summing the mixed series is an extension of the well-known method for the G.P. given by Dr Dougall in the April issue of the *Notes* for 1909.

Construction.

Lay off $D_1O = d$ and $OP = a$ in the same straight line. Draw D_1H, OH, P_1T making $45°$, $\tan^{-1}r$, $45°$ with the positive direction of D_1P_1. The figure shows the case $r < 1$ so that the first pair meet in H below D_1P_1. Next construct the G.P. $dr, dr^2, \ldots$ in the usual



way, giving $D_2, D_3, \ldots$. Through D_1 draw D_1R_1 to meet the perpendicular through P_1 in R_1 and draw R_1T_1 parallel to OP_1 to meet P_1T in T_1. Through D_2 draw D_2R_2 to meet T_1R_1 produced in S_1 and the perpendicular at T_1 in R_2. Repeat for $D_3, D_4, \ldots$ and project $T_1, T_2, \ldots$ to $P_2, P_3, \ldots$. Then $OP_n =$ sum of n terms.

Proof.

$$P_1P_2 = R_1T_1 = P_1R_1 = D_1P_1r = (a+d)r$$

$$P_2P_3 = R_2T_2 = T_1R_2 = S_1T_1r = (a+2d)r^2.$$

And generally since

$$\{a+(n-1)d\}r^{n-1}=[\{a+(n-2)d\}r^{n-2}+dr^{n-2}]r$$

we see that each segment on OP is got from the preceding segment by adding to it the appropriate segment of the G.P. D_1D_2, D_2D_3,… and multiplying the sum by r.

Sum to affinity.

Only D_1, O, P_1, H, D, T, P need be entered in the figure.

Now $$DO=\frac{d}{1-r} \text{ and } DP_1=a+\frac{d}{1-r}$$

$$\therefore \quad OP=\frac{DP_1}{1-r}=\frac{a}{1-r}+\frac{d}{(1-r)^2}.$$

Again OP is finite when DO is finite, that is when $|r|<1$. We thus have a visual proof of the limit theorem: if $|r|<1$

$$nr^n \rightarrow 0 \text{ when } n \rightarrow \infty.$$

The cases d or r negative require no modified construction or proof, as the above are quite general if the sign convention be applied.

Exactly analogous extensions apply to the constructions of Mr R. M. Milne (§ 291) and Mr F. J. W. Whipple (§ 292) in the *Mathematical Gazette*, 1909–11, p. 138.

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Note on Rational Right-Angled Triangles whose Legs are consecutive Whole Numbers.—Having given the sides of a rational right-angled triangle, to find from *them* the sides of other rational right-angled triangles.

Put a, b, c for the sides of the given right-angled triangle; then, of course,

$$a^2+b^2=c^2 \quad\text{.................................(1)}$$

Let $x+a$, $x+b$, and $2x-c$ denote the sides of the triangle sought; then

$$(x+a)^2+(x+b)^2=(2x-c)^2 \quad\text{.....................(2)}$$

Expanding and reducing, we get from (2)

$$x=a+b+2c,$$

remembering that $a^2+b^2=c^2$.