Triangles of Baumslag–Solitar Groups

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Abstract. Our main result is that many triangles of Baumslag–Solitar groups collapse to finite groups, generalizing a famous example of Hirsch and other examples due to several authors. A triangle of Baumslag–Solitar groups means a group with three generators, cyclically ordered, with each generator conjugating some power of the previous one to another power. There are six parameters, occurring in pairs, and we show that the triangle fails to be developable whenever one of the parameters divides its partner, except for a few special cases. Furthermore, under fairly general conditions, the group turns out to be finite and solvable of derived length ≤ 3. We obtain a lot of information about finite quotients, even when we cannot determine developability.
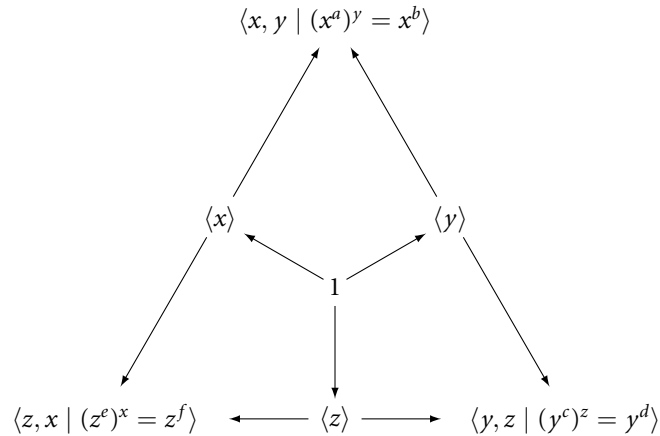
We study groups G of the form

$$(1) \qquad G(a, b; c, d; e, f) := \left\langle x, y, z \mid (x^a)^y = x^b, (y^c)^z = y^d, (z^e)^x = z^f \right\rangle,$$

where $a, \dots, f$ are nonzero integers. We prove that G collapses to a finite solvable group under a mild divisibility condition on the parameters. The motivation is that G is a triangle of groups in the language of [6] or [17], with the vertex groups being Baumslag–Solitar groups. Polygons of groups are an important means of constructing groups. For example, Haglund used them to construct hyperbolic buildings [7], and Ivanov and Meierfrankenfeld [9] used a triangle of groups to give the first computer-free construction of the sporadic finite simple group J_4. See [3], [4], and [19] for further discussion of polygons of groups. And the Baumslag–Solitar groups are famous for their "pathological" properties, like being non-Hopfian and (therefore) non-residually-finite and nonlinear.

These groups allow a simple construction (probably the first one) of a nondevelopable triangle of groups, because $G(1, 2; 1, 2; 1, 2)$ turns out to be trivial. This is a result of K. Hirsch, reported by Higman [8] and motivated by Higman's use of a square of $BS(1, 2)$'s to construct a finitely presented infinite group with no finite quotients. See also [14, Section 23]. The observation that it can be regarded as a nondevelopable triangle of groups seems to be due to K. Brown. Here the vertex groups are copies of $BS(1, 2)$, which is an atypical Baumslag–Solitar group, since it is solvable. It is natural to ask what is really causing the collapse; this led to our more general nondevelopability criterion.

Theorem 1 *Regard a and b as partners, and similarly for c and d and for e and f, and suppose one of $a, \dots, f$ divides its partner. Then the triangle of groups $G(a, b; c, d; e, f)$*

$$\langle x, y \mid (x^a)^y = x^b \rangle$$

$$\langle x \rangle \qquad\qquad \langle y \rangle$$

$$1$$

$$\langle z, x \mid (z^e)^x = z^f \rangle \quad\longleftarrow\quad \langle z \rangle \quad\longrightarrow\quad \langle y, z \mid (y^c)^z = y^d \rangle$$

is not developable, except in the special cases

(2) $G(a, -a; c, -c; e, -e)$,

(3) $G(a, \quad b; c, \quad c; e, \quad e)$,

(4) $G(a, \quad b; c, \quad c; e, -e)$, $a \equiv b$ mod 2,

(5) $G(a, \quad b; c, -c; e, \quad e)$,   *e even*,

(6) $G(a, \quad b; c, -c; e, -e)$,   *e even and* $a \equiv b$ mod 2,

all of which are developable.

We remind the reader that a triangle of groups is called developable if each of its vertex groups injects into the direct limit of the diagram, which in this case is $G(a, b; c, d; e, f)$. We will be informal and say that the group is developable when we mean that the triangle is. In the list of special cases we have left implicit other cases obtained from these by "trivial" transformations. These are cyclic permutation of the three pairs (corresponding to cyclic permutation of x, y, z), exchange of one of $a, \ldots, f$ with its partner (corresponding to inverting one of x, y, z), and simultaneous negation of one of $a, \ldots, f$ and its partner (corresponding to inverting a relation). We will apply these "moves" freely when it is convenient.

Of course, Theorem 1 begs the following question.

Question *If none of $a, \ldots, f$ divides its partner, is $G(a, b; c, d; e, f)$ ever developable? Always developable?*

Our work generalizes results of Post [16], who showed finiteness when $e = 1$ and the other parameters satisfy mild inequalities. His paper followed work by Mennicke [13] and Wamsley [18] concerning the case $a = c = e = 1$; see also Johnson and Robertson [11] and most recently Jabara [10]. The main claim of Neumann [15] is that G is infinite if $2 \le a \le |b|$, $2 \le c \le |d|$ and $2 \le e \le |f|$, but his proof

contains an error. (See the remarks after Lemma 5.) To our knowledge, the question of infiniteness of G remains open for every G not treated in this paper.

Not only does G collapse in the situation of Theorem 1, but we can say a great deal about what it collapses to. And with no more work, we also get information about the finite quotients of G in many cases not covered by Theorem 1.

Theorem 2 *Suppose $(a, b) = (c, d) = (e, f) = 1$ and none of the three pairs is $(\pm 1, \pm 1)$. Then there exists a quotient $Q = Q(a, b; c, d; e, f)$ of $G(a, b; c, d; e, f)$ which is universal among all quotients in which x, y and z have finite order; that is: any such quotient factors through Q. Furthermore, Q is finite and solvable, with its commutator subgroup Q' nilpotent of class ≤ 2. Finally, if any of $a, \ldots, f$ is 1 then $G = Q$.*

This immediately implies Post's result [16] that $G(a, a + 1; b, b + 1; 1, 2)$ is trivial, since it is a solvable group with trivial abelianization. In Section 3 we provide more detailed information, like a formula for the order of Q, exact up to a divisor of $(b - a)^2(d - c)^2(f - e)^2$, and a result showing that Q' is usually abelian, not just nilpotent. But Q' is not always abelian: a calculation using GAP [5] shows that $Q(1, 4; 1, 4; 1, 4)'$ is nonabelian.

The special cases in Theorem 1 indicate special behavior when $b = \pm a$, $d = \pm c$ or $f = \pm e$. This reflects properties of the Baumslag–Solitar groups

$$(7) \qquad BS(a, b) := \langle x, y \mid (x^a)^y = x^b \rangle,$$

which we recall here to help orient the reader. First, $BS(1, \pm 1) = \mathbb{Z} \rtimes \mathbb{Z}$, the quotient $\mathbb{Z}$ acting on the normal subgroup $\mathbb{Z}$ trivially or by $\{\pm 1\}$. Second, $BS(1, n \neq \pm 1)$ is $\mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z}$, a generator of the quotient $\mathbb{Z}$ acting on $\mathbb{Z}[\frac{1}{n}]$ by multiplication by n. Here $\mathbb{Z}[\frac{1}{n}]$ means the subring of $\mathbb{Q}$, or rather the underlying abelian group. Third, if $(a, b) = 1$ and $a, b \notin \{\pm 1\}$, then $BS(a, b)$ contains nonabelian free groups and is non-Hopfian, non-residually-finite, and nonlinear [2]. Finally, if a and b have a common divisor l, then $BS(a, b)$ is an amalgamated free product of $BS(a/l, b/l)$ and $\mathbb{Z}$.

1 The Relatively Prime Case

In this section we will prove Theorem 1 in the special case that $(a, b) = (c, d) = (e, f) = 1$. This is the basis for the general proof in the next section. Our convention for conjugation is that $x^y = \bar{y}xy$, where $\bar{y}$ means y^{-1}. Also, since some superscripts get very complicated, we sometimes write $x \uparrow \{y\}$ for x^y.

Our first step is to find the key relation that makes the triangles of Theorem 1 collapse; the exact form of the relation is not so important—the key is that some power of x lies in $\langle y, z \rangle$. The restriction to $a, \ldots, f > 0$ is minor, as we will see in the proof of Lemma 4.

Lemma 3 *Suppose $0 < a \leq b$, $0 < c \leq d$ and $1 = e \leq f$. Then for any $R, S, T > 0$, the relation*

$$(8) \quad x \uparrow \{Tb^{Sc^R}[b^{S(d^R - c^R)} - a^{S(d^R - c^R)}]\} = \bar{z} \uparrow \{Rf^{Ta^{S(d^R - c^R)}b^{Sc^R}}\} \bar{y}^{Sc^R} z \uparrow \{Rf^{Ta^{Sd^R}}\} y^{Sd^R}$$

holds in $G(a, b; c, d; e, f)$.

Proof We will evaluate $(x^P)\!\uparrow\!\{(y^Q)^{z^R}\}$ in two different ways, where P, Q, R are integers having whatever divisibility properties are needed for the following calculation to make sense. The underlines indicate where changes occur.

$$(x^P)\!\uparrow\!\{y^{Q(d/c)^R}\} = \bar{z}^R \bar{y}^Q \, \underline{z^R x^P} \, \bar{z}^R y^Q z^R$$

$$x\!\uparrow\!\{P(b/a)^{Q(d/c)^R}\} = \bar{z}^R \, \underline{\bar{y}^Q x^P} \, z\!\uparrow\!\{R(f/e)^P\}\underline{\bar{z}^R y^Q z^R}$$

$$= \underline{\bar{z}^R x\!\uparrow\!\{P(b/a)^Q\}} \bar{y}^Q z\!\uparrow\!\{R(f/e)^P\}y^{Q(d/c)^R}$$

$$= x\!\uparrow\!\{P(b/a)^Q\}\bar{z}\!\uparrow\!\{R(f/e)^{P(b/a)^Q}\}$$

$$\cdot \, \bar{y}^Q z\!\uparrow\!\{R(f/e)^P\}y^{Q(d/c)^R}.$$

(The second line uses $z^R x^P = x^P(z^R)^{x^P}$ and the fourth line is similar, while the third line uses $\bar{y}^Q x^P = (x^P)^{y^Q}\bar{y}^Q$.) The restrictions on P, Q and R come from considerations like this: $(y^Q)^{z^R} = y^{Q(d/c)^R}$ follows from $(y^c)^z = y^d$ provided that Q is divisible by $c^R/(c, d)^{R-1}$ if $R > 0$, or by $d^{|R|}/(c, d)^{|R|-1}$ if $R < 0$. The full set of conditions for the calculation to make sense are $c^R|Q$, $a^Q|P$ and $a^{Q(d/c)^R}\big|P$. Since $Q(d/c)^R \geq Q$, the third condition implies the second. We obtain (8) by taking $Q = Sc^R$ and $P = Ta^{Sd^R}$.

(One can show that there are no solutions for P, Q, R unless at least one of b/a, d/c and f/e is an integer or reciprocal integer. Each of these is ≥ 1, so we ignore the case of reciprocal integers. Also, if only one of b/a, d/c and f/e is an integer, then it must be f/e. This would explain a hypothesis $e|f$, and the stronger assumption $e = 1$ is enough for our applications.) ∎

Lemma 4 *Suppose $b \neq \pm a$ and $d \neq \pm c$. Then x has finite order in $G(a, b; c, d; 1, f)$. If $f \neq \pm 1$ then y and z also have finite order.*

Proof Note that x^2, y^2, z^2 satisfy the relations of $G(a^2, b^2; c^2, d^2; e^2, f^2)$, so it suffices to treat the case $a, \dots, f > 0$. So we may take $0 < a < b$, $0 < c < d$ and $1 = e \leq f$ without loss.

First suppose $f > 1$. Take $R = S = T = 1$ and write the relation (8) as $x^A = \bar{z}^B \bar{y}^C z^D y^E$. Being a word in y, z, x^A conjugates some power of y to another power. Namely,

$$(y^{d^B})^{x^A} = (y^{d^B})^{\bar{z}^B \bar{y}^C z^D y^E} = (y^{c^B})^{z^D y^E} = y^{c^{B-D}d^D}.$$

(The third equality is valid because $B > D$.) We write this relation as $(y^g)^X = y^h$ where $g = d^B$, $h = c^{B-D}d^D$ and $X = x^A$. Now we apply the relation $z^{f^A} = z^X$ to a large power of y. The conjugate of $y\!\uparrow\!\{hc^{f^A}\}$ by z^{f^A} is $y\!\uparrow\!\{hd^{f^A}\}$. So we have

$$y\!\uparrow\!\{hd^{f^A}\} = \bar{X}\bar{z}X \, y\!\uparrow\!\{hc^{f^A}\}\bar{X}zX$$

$$= \bar{X}\bar{z} \, y\!\uparrow\!\{gc^{f^A}\}zX$$

$$= \bar{X}y\!\uparrow\!\{gc^{f^A-1}d\}X$$

$$= y\!\uparrow\!\{hc^{f^A-1}d\}.$$

We conclude that y has order dividing $hd(d^{f^A-1} - c^{f^A-1})$. This is a nontrivial relation provided $d^{f^A-1} \neq c^{f^A-1}$. Since $d > c > 0$, the relation is nontrivial provided $f^A \neq 1$. Since $f > 1$, the relation is nontrivial provided $A \neq 0$. Recall that A is the exponent on the left side of (8) and that $b > a > 0$ and $d > c > 0$, so $A \neq 0$. Therefore y has finite order, say $y^n = 1$. Now the relation $x^{a^n} = (x^{a^n})^{y^n} = x^{b^n}$ implies that x has finite order (since $b > a > 0$), and repeating this argument shows that z also has finite order.

If $f = 1$ then $[x, z] = 1$ and the computation is similar but much easier. We conjugate x^{a^d} by the relation $(y^c)^z = y^d$, which leads to $x\!\uparrow\!\{a^{d-c}b^c\} = x\!\uparrow\!\{b^d\}$. Since $b > a > 0$ and $d > c > 0$, this is a nontrivial relation, so x has finite order. (Remark: y and z have infinite order, since adjoining the relation $x = 1$ reduces G to $BS_{yz}(c, d)$.) ∎

Lemma 5 *Assume the hypotheses of Theorem 1 and that $(a, b) = (c, d) = (e, f) = 1$. Then the triangle of groups there is developable in and only in the following cases:*

$$(9) \qquad\qquad G(1, -1; 1, -1; 1, -1),$$

$$(10) \qquad\qquad G(a, b; 1, 1; 1, 1),$$

$$(11) \qquad\qquad G(a, b; 1, 1; 1, -1), \quad a, b \text{ odd}.$$

Proof The hypothesis that one of $a, \dots, f$ divides its partner says that one is ± 1, say $e = 1$ without loss. If $b \neq \pm a$ and $d \neq \pm c$ then Lemma 4 shows that G is not developable. So suppose $b = \pm a$ or $d = \pm c$. Because of the relative primality, we are in one of the cases $G(1, \pm 1; 1, \pm 1; 1, f)$, $G(1, \pm 1; c, d \neq \pm c; 1, f)$ and $G(a, b \neq \pm a; 1, \pm 1; 1, f)$. In the last two cases, when $f \neq \pm 1$, G is nondevelopable by Lemma 4 (after cyclically permuting the variables). All remaining cases are now special cases of $G(a, b; 1, \pm 1; 1, \pm 1)$, after cyclic permutation of the variables. To begin with, $G(a, b; 1, 1; 1, 1) = BS_{xy}(a, b) \times \mathbb{Z}_z$ is obviously developable.

Next, in $G(a, b; 1, 1; 1, -1)$, $\langle z \rangle$ is normal, with quotient $BS(a, b)$, so $\langle x, y \rangle$ is a complement. So $G = \langle z \rangle \rtimes BS_{xy}(a, b)$, with y fixing z and x inverting it. If a and b have different parities then $(x^a)^y = x^b$ forces $z^2 = 1$, so G is not developable. On the other hand, if a and b have the same parity (so both are odd, because $(a, b) = 1$), then $\mathbb{Z}_z \rtimes BS_{xy}(a, b)$ satisfies all the relations of G, hence equals it. So $G(a, b; 1, 1; 1, -1)$ is developable if and only if a and b are both odd.

Next, in $G(a, b; 1, -1; 1, 1)$, $\langle x, y \rangle$ is normal, with a complementary $\mathbb{Z}$ generated by z. Conjugating $(x^a)^y = x^b$ by z gives $(x^a)^{\bar{y}} = x^b$, so $x^{a^2} = (x^{a^2})^{y\bar{y}} = x^{b^2}$. Therefore x has finite order unless $b = \pm a$, which by $(a, b) = 1$ leaves us with $G(1, \pm 1; 1, -1; 1, 1)$, which we treated in the previous case.

Finally we consider $G(a, b; 1, -1; 1, -1)$. Because x^2, y, z satisfy the relations of $G(a, b; 1, -1; 1, 1)$, the previous case shows that only $G(1, \pm 1; 1, -1; 1, -1)$ can be developable. The $+1$ case has already been treated, leaving only $G(1, -1; 1, -1; 1, -1)$, whose developability is due to Neumann [14, Section 5]. Observe that x^2, y^2 and z^2 generate a normal abelian group A, with quotient $(\mathbb{Z}/2)^3$. To see that G is developable, it suffices to prove $A \cong \mathbb{Z}^3$. This can be done by representing G by isometries of $\mathbb{R}^3$, with x acting by $(X, Y, Z) \mapsto (X + 1, Y, -Z)$ and the other

generators' actions defined similarly. In fact this action on $\mathbb{R}^3$ is free, realizing G as the fundamental group of a Euclidean 3-manifold. ∎

Remark The group $G = G(2, 3; 2, 3; 2, 4)$ is not developable, because x, y, z^2 satisfy the relations of $G(2, 3; 4, 9; 1, 2)$, which is nondevelopable by Lemma 5. This group is a counterexample to the main result (Theorem 4.4) of [15]. Neumann's argument relies on a complicated inductive definition of an action of G on a set of "normal matrices". Unfortunately, his operator $\rho(b^{-1})$ doesn't preserve the set of normal matrices: the right hand side of (3.53) is never a normal matrix because it violates (2.35) or (2.36), depending on the sign of $\gamma(n)$.

2   The General Case

In this section we derive Theorem 1 from the coprime case established in Lemma 5. The key idea is the following; consider $G := G(a, b; c, d; e, f)$ and suppose $l > 0$ is a common divisor of a and b. Then the elements $X = x^l$, y and z satisfy the relations of $H := G(a/l, b/l; c, d; e^l, f^l)$. Because of this change of variables, we will sometimes refer to (1) as $G_{xyz}(a, b; c, d; e, f)$ and (7) as $BS_{xy}(a, b)$. In this notation, $G_{xyz}(a, b; c, d; e, f)$ is the direct limit of the diagram

$$(12) \qquad G_{Xyz}(a/l, b/l; c, d; e^l, f^l) \longleftarrow BS_{zX}(e^l, f^l) \longrightarrow BS_{zx}(e, f).$$

We also sometimes write $\mathbb{Z}_x$ for a copy of $\mathbb{Z}$ with generator x. The right homomorphism of (12) is always injective; to see this, one may use the standard form for words in an HNN extension. In good cases, the left homomorphism is also injective, so that G is an amalgamated free product of H and $BS_{zx}(e, f)$. When this holds, we may reasonably hope to relate the developability of G to that of H. This hope is realized in the following lemma.

Lemma 6 *In the notation just established,*
(1) *If G is developable then so is H.*
(2) *Suppose that H is developable and that $\langle X, y \rangle \cap \langle z, X \rangle = \langle X \rangle$. Then G is developable.*

Proof (1) If H is not developable then $BS_{Xy}(a/l, b/l)$, $BS_{yz}(c, d)$ or $BS_{zX}(e^l, f^l)$ fails to inject into H. Since these are subgroups of $BS_{xy}(a, b)$, $BS_{yz}(c, d)$ and $BS_{zx}(e, f)$, at least one of these latter three fails to inject into G.

(2) The left arrow of (12) is injective, by the definition of developability of H. So (12) expresses G as a free product with amalgamation. Since $BS_{zx}(e, f)$ is a factor in this product, it injects into G. Also, $BS_{yz}(c, d)$ injects into H by developability, and then injects into G since H does. So it remains to check the injectivity of $BS_{xy}(a, b)$ into G.

We use the following assertion, whose proof is an easy exercise using the standard form for words in an amalgamated free product. Suppose we are given a commutative

diagram of inclusions of groups

$$A \longleftarrow I \longrightarrow B$$
$$\uparrow \qquad \uparrow \qquad \uparrow$$
$$C \longleftarrow J \longrightarrow D;$$

then $I \cap C = J = I \cap D$ implies that the natural map $C *_J D \to A *_I B$ is injective. The hypothesis in (2) is exactly what is needed to apply this to the diagram

$$H \longleftarrow BS_{zX}(e^l, f^l) \longrightarrow BS_{zx}(e, f)$$
$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$
$$BS_{Xy}(a/l, b/l) \longleftarrow \mathbb{Z}_X \longrightarrow \mathbb{Z}_x.$$

The amalgamation of the bottom row is $BS_{xy}(a, b)$ and that of the top is G. So the former injects into the latter and the proof is complete. ∎

In order to deduce the developability of G from that of H, we must verify the condition in (2). We will prove this in Lemma 8, by an argument that requires understanding certain centralizers in H.

Lemma 7 *In $BS_{xy}(a, b)$, the centralizer of y^n is*
(1) *$\langle x^a, y \rangle$ if $a = b$, or if $a = -b$ and n is even;*
(2) *$\langle y \rangle$ otherwise.*

Proof This is an exercise using the standard form for words in an HNN extension. Or one can apply the last part of the theorem stated in [12, 350–351]. ∎

Now we verify the condition in Lemma 6(2). Part (2) of the following lemma is needed for the inductive argument, but nowhere else. The important conclusion is (1).

Lemma 8 *Suppose G is developable. Then*
(1) *$\langle x, y \rangle \cap \langle y, z \rangle = \langle y \rangle$ and similarly for cyclic permutations of x, y, z;*
(2) *if $|a| = |b|$, $|c| = |d|$ and $|e| = |f|$ then some powers of x, y and z generate a group $\mathbb{Z}^3$.*

Proof Suppose G were a counterexample, with the number of pairs (a, b), (c, d), (e, f) of the form $(\pm 1, \pm 1)$ being maximal among all counterexamples. If it is conclusion (2) that fails for G, then $b = \pm a$, $d = \pm c$ and $f = \pm e$. We cannot have $a \ldots, f \in \{\pm 1\}$, because then we would be in one of the special cases $G = G(1, \pm 1; 1, \pm 1; 1, \pm 1)$, for which the lemma can be checked directly. (The only interesting case is $G(1, -1; 1, -1; 1, -1)$, for which see the proof of Lemma 5.) Suppose $a > 1$, so that G is the pushout of the diagram

$$(13) \qquad G_{Xyz}(1, \pm 1; c, d; e^a, f^a) \leftarrow BS_{zX}(e^a, f^a) \to BS_{zx}(e, f).$$

The developability of G implies that of the left term H (Lemma 6(1)), so (13) expresses G as a free product with amalgamation, so H injects into G. The lemma holds for H, by the maximality in the choice of counterexample G. So some powers of X, y, z generate a group $\mathbb{Z}^3$. Since X is a power of x, we have proven (2).

So it must be (1) that fails. Then $\langle x, y \rangle \cap \langle y, z \rangle$ is strictly larger than $\langle y \rangle$, so take w to be an element in the intersection but not in $\langle y \rangle$. Since $w \in \langle y, z \rangle$, it conjugates some power of y to another power (possibly the same), say $(y^m)^w = y^n$. On the other hand, since $w \in \langle x, y \rangle$, we see that y^m and y^n are conjugate in $\langle x, y \rangle = BS_{xy}(a, b)$. This forces $m = n$, so that w centralizes some power of y. Since $w \notin \langle y \rangle$, Lemma 7 forces $a = \pm b$ and $w \in \langle x^a, y \rangle = \langle x^a \rangle \rtimes \langle y \rangle$. Any subgroup of this $\mathbb{Z} \rtimes \mathbb{Z}$ that strictly contains $\langle y \rangle$ must contain a power of x. Therefore $\langle x, y \rangle \cap \langle y, z \rangle$ contains a power of x; we may even suppose without loss of generality that w is a power of x.

As a power of x, w conjugates some power of z to another, say $(z^p)^w = z^q$. We now essentially repeat the argument just used: since $w \in \langle y, z \rangle = BS_{yz}(c, d)$, we must have $p = q$, and this forces $f = \pm e$. Also, since w centralizes z^p and is not in $\langle z \rangle$, the centralizer of z^p in $BS_{yz}(c, d)$ must be larger than $\langle z \rangle$, which forces $c = \pm d$ by Lemma 7.

We have proven that $a = \pm b$, $c = \pm d$, $e = \pm f$ and that some power of x lies in the centralizer of z^p in $BS_{yz}(c, \pm c)$, which has structure $\langle y^c \rangle \rtimes \langle z \rangle$. But this contradicts the fact that some powers of x, y, z generate a copy of $\mathbb{Z}^3$, by (2). ∎

We summarize our results so far in the following lemma.

Lemma 9   *G is developable if and only if H is.*

Proof   We have already shown that developability of G implies that of H. For the converse, we apply Lemma 8 to H, and then conclusion (1) of that lemma allows us to apply Lemma 6 and deduce G's developability. ∎

Corollary 10   *Write*

$$(a, b; c, d; e, f) = (Al, Bl; Cm, Dm; En, Fn),$$

where $l, m, n > 0$ and $(A, B) = (C, D) = (E, F) = 1$. Then $G(a, b; c, d; e, f)$ is developable if and only if $G(A^m, B^m; C^{n^l}, D^{n^l}; E^l, F^l)$ is.

Proof   Consider the following four groups:

(14) $$G(Al, Bl; Cm, Dm; En, Fn)$$

(15) $$G\big(A, B; Cm, Dm; (En)^l, (Fn)^l\big)$$

(16) $$G\big(A^m, B^m; C, D; (En)^l, (Fn)^l\big)$$

(17) $$G(A^m, B^m; C^{n^l}, D^{n^l}; E^l, F^l).$$

By Lemma 9, each is developable if and only if the previous one is. ∎

Proof of Theorem 1 We suppose without loss that $a, c, e > 0$. If $b = \pm a$, $d = \pm c$ and $f = \pm e$, then we are in case

$$(18) \qquad G = G(a, \pm a; c, \pm c; e, \pm e)$$

and Corollary 10 and Lemma 5 imply that G is developable. If two of the equalities $b = \pm a$, $d = \pm c$, $f = \pm e$ fail, then that corollary and lemma prove that G is nondevelopable. The remaining case is when exactly one of the equalities fails, so suppose $b \neq \pm a$, $d = \pm c$, $f = \pm e$. We take $l, m, n, A, \dots, F$ as in Corollary 10. Since $a, c, e > 0$ we have $A, C, E > 0$. By that corollary, G is developable if and only if $G(A^m, B^m; C^{n^l}, D^{n^l}; E^l, F^l)$ is, which can be determined using the relatively-prime case, Lemma 5. So developability is equivalent to $(A^m, B^m; C^{n^l}, D^{n^l}, E^l, F^l)$ being equal to

$$(19) \qquad (A^m, B^m; 1, 1; 1, -1) \quad \text{with } A^m \text{ and } B^m \text{ odd, or}$$

$$(20) \qquad (A^m, B^m; 1, 1; 1, \quad 1).$$

In either case, we know $C = E = 1$ because $C, E > 0$.

In case (19), $F^l = -1$ is equivalent to $F = -1$ and l odd, and of course the oddness of A^m and B^m is equivalent to the oddness of A and B. The condition $D^{n^l} = 1$ is equivalent to: either $D = 1$, or else $D = -1$ and n is even. So we have

$$(A, B; C, D; E, F) = \begin{cases} (A, B; 1, \quad 1; 1, -1) & \text{with } A, B, l \text{ odd, or} \\ (A, B; 1, -1; 1, -1) & \text{with } A, B, l \text{ odd and } n \text{ even}; \end{cases}$$

note that $n = e$. This is equivalent to

$$(21) \qquad G = G(a, b; c, \quad c; e, -e) \quad \text{with } a, b \text{ odd, or}$$

$$(22) \qquad G(a, b; c, -c; e, -e) \quad \text{with } a, b \text{ odd and } e \text{ even.}$$

In case (20), the treatment of $D^{n^l} = 1$ is as before, and $F^l = 1$ is equivalent to: either $F = 1$, or else $F = -1$ and l is even. So we have

$$(A, B; C, D; E, F) = \begin{cases} (A, B; 1, \quad 1; 1, \quad 1), & \text{or} \\ (A, B; 1, \quad 1; 1, -1) & \text{with } l \text{ even, or} \\ (A, B; 1, -1; 1, \quad 1) & \text{with } n \text{ even, or} \\ (A, B; 1, -1; 1, -1) & \text{with } l \text{ and } n \text{ even}; \end{cases}$$

again $n = e$. This is equivalent to

$$(23) \qquad G = G(a, b; c, \quad c; e, \quad e) \quad \text{or}$$

$$(24) \qquad G(a, b; c, \quad c; e, -e) \quad \text{with } a, b \text{ even, or}$$

$$(25) \qquad G(a, b; c, -c; e, \quad e) \quad \text{with } e \text{ even, or}$$

$$(26) \qquad G(a, b; c, -c; e, -e) \quad \text{with } a, b, e \text{ even.}$$

Now, (21) and (24) together correspond to (4) in the statement of the theorem, and (22) and (26) correspond to (6). Also, (23) and (25) correspond to (3) and (5), and (3)–(6) contain every case of (18) except $G(a, -a; c, -e; e, -e)$, which we listed as (2). ∎

3   Finite Solvable Groups

We have shown that $G = G(a, b; c, d; e, f)$ is nondevelopable under fairly mild conditions, and in this section we study just how much G collapses. We first prove Theorem 2, which often says that G is a finite solvable group. We assume the hypotheses of Theorem 2 throughout this section, and without loss we suppose $a < b$, $c < d$, $e < f$. It is convenient to define $X = x^{b-a}$, $Y = y^{d-c}$ and $Z = z^{f-e}$.

Lemma 11   *The relation*

$$(27) \qquad\qquad x \uparrow \{(b-a)^2 (b^{d-c} - a^{d-c})\} = 1$$

and its cyclic permutations hold in any quotient of G in which x, y and z have finite order. In particular, G has a universal quotient Q in which x, y and z have finite order, which is got by imposing these relations.

Proof   Suppose $\bar{G}$ is a quotient of G in which x, y, z have finite order, and write n for the order of x. The orders of x^a and x^b are $n/(n, a)$ and $n/(n, b)$, which are equal since x^a and x^b are conjugate. Since $(a, b) = 1$, this forces $(n, a) = (n, b) = 1$. Therefore $\langle x^a \rangle = \langle x^b \rangle = \langle x \rangle$, so y normalizes $\langle x \rangle$. Similarly, z normalizes $\langle y \rangle$ and x normalizes $\langle z \rangle$.

Now let H be the subgroup generated by all the y^{x^i}, $i \in \mathbb{Z}$. We have $H = \langle y, x^{b-a} \rangle$, since $y^{x^a} = yx^{a-b}$ and x^a generates $\langle x \rangle$. Obviously x and y normalize H. And the fact that z normalizes $\langle y \rangle$ implies that z^{x^i} normalizes $\langle y^{x^i} \rangle$. Since $\langle z \rangle = \langle z^{x^i} \rangle$ normalizes every $\langle y^{x^i} \rangle$, it normalizes H. So H is normal in $\bar{G}$.

Next, the commutator subgroup H' is $\langle x^{(b-a)^2} \rangle$, which is characteristic in H, hence normal in $\bar{G}$. Now, the automorphism group of a cyclic group is abelian, so every commutator acts trivially, in particular y^{d-c}. This implies (27). ∎

Lemma 12   *Let α be a solution of $\alpha a = 1$ modulo $(b-a)^2 (b^{d-c} - a^{d-c})$. Then*

$$(28) \qquad\qquad y^x = yx \uparrow \{-\alpha(b-a)\} = yX^{-\alpha}$$

$$(29) \qquad\qquad Y^x = YX \uparrow \left\{ -\alpha^{d-c} \frac{b^{d-c} - a^{d-c}}{b - a} \right\}$$

$$(30) \qquad\qquad Y^X = YX \uparrow \{-\alpha^{d-c} (b^{d-c} - a^{d-c})\}.$$

Proof   The key property of α is that $(x^a)^\alpha = 1$. We may rewrite $(x^a)^y = x^b$ as

$y^{x^a} = yx^{a-b}$. Conjugating y by x^a, α many times, gives (28). For (29) we compute

$$
\begin{aligned}
Y^{x^a} &= (y^x)^{d-c} = (y\bar{X}^\alpha)^{d-c} \\
&= Y(\bar{X}^\alpha)^{y^{d-c-1}}(\bar{X}^\alpha)^{y^{d-c-2}}\cdots(\bar{X}^\alpha)^{y^0} \\
&= Y(\bar{X}^{\alpha\alpha^{d-c-1}a^{d-c-1}})^{y^{d-c-1}}\cdots(\bar{X}^{\alpha\alpha^{d-c-1}a^{d-c-1}})^{y^0} \\
&= Y\bar{X}\uparrow\{\alpha^{d-c}(b^{d-c-1} + b^{d-c-2}a + \cdots + a^{d-c-1})\} \\
&= YX\uparrow\left\{-\alpha^{d-c}\frac{b^{d-c} - a^{d-c}}{b-a}\right\}.
\end{aligned}
$$

Then (30) follows by applying (29) $b - a$ times. ∎

Proof of Theorem 2 We must show that Q' is nilpotent of class ≤ 2. It follows from (29) and its cyclic permutations that $\langle X, Y, Z\rangle$ is normal in Q. Since adjoining the relations $X = Y = Z = 1$ abelianizes Q, we see that $\langle X, Y, Z\rangle = Q'$. Then (30) shows that $[X, Y]$ lies in $\langle X^{b-a}\rangle$. We saw in the proof of Lemma 11 that $\langle X^{b-a}\rangle$ is central in Q'. Together with the cyclic permutations of this argument, we have proven that $[Q', Q']$ is central in Q', as desired.

For the final assertion of the theorem, just use Lemma 4, which assures us that x, y, z have finite order in G, so G must equal Q. ∎

Jabara [10] proved the stronger result that Q'' is central in Q, not just in Q'. He treated only the case $a = c = e = 1$, but there is no loss of generality because $\langle x\rangle = \langle x^a\rangle = \langle x^b\rangle$ in Q, and similarly for y and z.

Q' is abelian in almost all cases. The easiest way to address this question is to work one prime at a time, since the nilpotence of Q' implies that Q' is the direct product of its Sylow subgroups. So for a prime p we define Q_p as the quotient of Q by the Sylow subgroups of Q' for all primes except p. Obviously, Q' is abelian if and only if every $Q_p' := (Q_p)'$ is.

We said in the introduction that $Q(1, 4; 1, 4; 1, 4)'$ is nonabelian. We found this using GAP [5], but simply entering the presentation led to memory overflow during coset enumeration. Adjoining the relations $x^{81} = y^{81} = z^{81} = 1$, which reduce G to Q_3, let GAP perform the computation almost instantly.

Lemma 13 *Unless p divides $b - a$, $d - c$ and $f - e$, Q_p' is abelian.*

Proof Since a nonabelian p-group has noncyclic Frattini quotient, it suffices to show that $Q_p/\Phi(Q_p)$ is cyclic. This is an abelian group with generators X, Y, Z satisfying relations including $pX = pY = pZ = 0$ and $(b-a)X = (d-c)Y = (f-e)Z = 0$, in additive notation. Suppose $p \nmid d-c$, so $Y = 0$. If $p \nmid b-a$ then $X = 0$ and $Q_p/\Phi(Q_p)$ is generated by Z, hence cyclic. So suppose $p | b - a$. Conjugating the relation $Y = 0$ by x yields

$$
Y - \alpha^{d-c}\frac{b^{d-c} - a^{d-c}}{b-a}X = 0, \quad \text{hence} \quad \frac{b^{d-c} - a^{d-c}}{b-a}X = 0.
$$

The hypotheses $p \mid b - a$ and $p \nmid d - c$ imply that the p-part of the numerator is the same as that of the denominator. So this relation implies $X = 0$, and again $Q_p / \Phi(Q_p)$ is cyclic. ∎

Corollary 14   *If $b - a$, $d - c$ and $f - e$ have no common divisor then Q' is abelian.*   ∎

Mennicke [13] gave an order formula for $G(1, t; 1, t; 1, t)$, and Johnson and Robertson [11] gave an upper bound for the order of $G(1, b; 1, d; 1, f)$. In [1], Albar and Al-Shuaibi improve this bound and give a correction to Mennicke's paper. It seems that the exact order and structure of Q_p' depend very sensitively on the number times p divides $b - a$, $d - c$ and $f - e$. We offer upper and lower bounds on $|Q|$ that are fairly close to each other.

Theorem 15   *Suppose $a < b$, $c < d$ and $e < f$. Then the order of Q is*

$$(b^{d-c} - a^{d-c})(d^{f-e} - c^{f-e})(f^{b-a} - e^{b-a})$$

$$\times \text{ a divisor of } (b-a)^2 (d-c)^2 (f-e)^2.$$

Proof   Killing x reduces Q to a group in which y has order $d^{f-e} - c^{f-e}$. This shows that $d^{f-e} - c^{f-e}$ divides $[\langle x, y \rangle : \langle x \rangle]$, hence $[\langle X, y \rangle : \langle X \rangle]$. Similarly, killing z shows that the order of x is divisible by $b^{d-c} - a^{d-c}$, so the order of X is divisible by $\left(b^{d-c} - a^{d-c} \right) / (b - a)$. And killing y leaves a group of order $\left(f^{b-a} - e^{b-a} \right)(b - a)$. Putting all this together shows that

$$|Q| = [Q : \langle y, X \rangle] \cdot [\langle y, X \rangle : \langle X \rangle] \cdot [\langle X \rangle : 1]$$

is divisible by $(b^{d-c} - a^{d-c})(d^{f-e} - c^{f-e})(f^{b-a} - e^{b-a})$.

On the other hand, the structure of Q as a polycyclic group shows that $|Q|$ divides the product of the orders of x, y and z. Referring to (27) shows that $|Q|$ divides

$$(b^{d-c} - a^{d-c})(d^{f-e} - c^{f-e})(f^{b-a} - e^{b-a})(b-a)^2 (d-c)^2 (f-e)^2. \qquad \blacksquare$$

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