SOME APPLICATIONS OF SPACES OF ENTIRE FUNCTIONS

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A Hilbert space, whose elements are entire functions, is of particular interest if it has these properties:

(H1) Whenever $F(z)$ is in the space and has a non-real zero w, the function $F(z)(z - \bar{w})/(z - w)$ is in the space and has the same norm as $F(z)$.

(H2) For each non-real number w, the linear functional defined on the space by $F(z) \to F(w)$ is continuous.

(H3) Whenever $F(z)$ is in the space, $F^*(z) = \bar{F}(\bar{z})$ is in the space and has the same norm as $F(z)$. If $E(z)$ is an entire function satisfying

$$|E(\bar{z})| < |E(z)| \tag{1}$$

for $y > 0$ $(z = x + iy)$, we write $E(z) = A(z) - iB(z)$, where $A(z)$ and $B(z)$ are entire functions which are real for real z, and

$$K(w, z) = [B(z)\bar{A}(w) - A(z)\bar{B}(w)]/[\pi(z - \bar{w})].$$

Let $\mathfrak{H}(E)$ be the Hilbert space of entire functions $F(z)$ such that

$$||F||^2 = \int |F(t)/E(t)|^2 \, dt < \infty,$$

with integration on the real axis, and

$$|F(z)|^2 \leqslant ||F||^2 \, K(z, z)$$

for all complex z. For each complex number w, $K(w, z)$ belongs to $\mathfrak{H}(E)$ as a function of z, and

$$F(w) = \langle F(t), K(w, t)\rangle$$

for every $F(z)$ in $\mathfrak{H}(E)$. As shown in **(9)**, a Hilbert space, whose elements are entire functions, which satisfies (H1), (H2), and (H3), and which contains a non-zero element, is equal isometrically to some such $\mathfrak{H}(E)$.

A useful quantity associated with a given $E(z)$ is the corresponding phase function $\phi(x)$. Suppose for simplicity that $E(z)$ has no real zeros and that $E(0) = 1$. Then $\phi(x)$ is to be the continuous determination of the phase of $\bar{E}(x)$ which vanishes at the origin. The points where $\phi(x) \equiv 0 \pmod{\pi}$ are just the zeros of $B(z)$, which are real and simple. The points where $\phi(x) \equiv \pi/2 \pmod{\pi}$ are the zeros of $A(z)$, which are real and simple and are lodged between zeros of $B(z)$. The location of these zeros is significant in the formula of **(8)** for mean squares of entire functions. A phase function is essentially determined by a knowledge of these points.

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THEOREM I. *Let* $E_1(z)$ *and* $E_2(z)$ *be entire functions which satisfy* (1), *have no real zeros, and have value* 1 *at the origin. Let* $\phi_1(x)$ *and* $\phi_2(x)$ *be the corresponding phase functions. If* $\phi_1(x) = \phi_2(x)$ *whenever* $\phi_1(x) \equiv 0 \pmod{\pi/2}$ *or* $\phi_2(x) \equiv 0 \pmod{\pi/2}$, *then* $\tan \phi_2(x)/\tan \phi_1(x)$ *is a positive constant. There is an entire function* $G(z)$ *which is real for real* z *and has no zeros, such that* $F(z) \to G(z)F(z)$ *is a linear isometric transformation of* $\mathfrak{H}(E_1)$ *onto* $\mathfrak{H}(E_2)$.

Under these conditions, the spaces $\mathfrak{H}(E_1)$ and $\mathfrak{H}(E_2)$ are equivalent as far as any of the deeper structural properties are concerned. Therefore, the phase function $\phi(x)$ associated with a given $E(z)$ may be regarded as essentially determining $\mathfrak{H}(E)$, and $\phi(x)$ is essentially determined by a knowledge of the points where $\phi(x) \equiv 0 \pmod{\pi/2}$. This set contains the origin and has no limit points, but is otherwise arbitrary.

THEOREM II. *Let* $\psi(x)$ *be a continuous, increasing function which has value* 0 *at the origin. Then there is an entire function* $E(x)$ *which satisfies* (1), *has no real zeros, and has value* 1 *at the origin, for which the phase function* $\phi(x)$ *agrees with* $\psi(x)$ *whenever* $\phi(x) \equiv 0 \pmod{\pi/2}$ *or* $\psi(x) \equiv 0 \pmod{\pi/2}$. *If*

$$\int (1 + t^2)^{-1}\, d\psi(t) < \infty, \tag{2}$$

$E(z)$ *may be chosen of genus* 0 *or* 1. *In this case, if* $W(z)$ *is the choice of a function analytic for* $y > 0$ *such that*

$$\frac{\partial}{\partial y} \log |W(x + iy)| = \frac{y}{\pi} \int \frac{d\psi(t)}{(t - x)^2 + y^2} \tag{3}$$

and $W(0) = 1$, *we may choose* $E(z)$ *so that* $B'(0) = 1$ *and* $E(z)/W(z)$ *is never negative for* $y > 0$. *Then* $E(z)$ *is uniquely determined by* $W(z)$.

Entire functions which are real for real z and have only real zeros frequently occur in analysis, and then usually in pairs $F(z)$ and $G(z)$ whose zeros separate each other. In this case, $F(z) - iG(z)$ need not satisfy (1) since, instead, $F(z) + iG(z)$ may satisfy (1). Furthermore, (1) implies a growth restriction on the ratio $G(z)/F(z)$.

THEOREM III. *If* $E(z)$ *is an entire function which satisfies* (1), *then*

$$\log |B(z)/A(z)| = \frac{|y|}{\pi} \int \frac{\log |B(t)/A(t)|}{(t - x)^2 + y^2}\, dt \tag{4}$$

when $y \neq 0$, *with absolute convergence of the integral.*

But if (4) is satisfied, there is always a space of entire functions present.

THEOREM IV. *Let* $F(z)$ *and* $G(z)$ *be entire functions which are real for real* z *and have only real simple zeros. Suppose that there is a continuous, increasing function* $\psi(x)$ *of real* x *such that the zeros of* $F(z)$ *are just the points* t *where* $\psi(t) \equiv \pi/2 \pmod{\pi}$, *and the zeros of* $G(z)$ *are just the points* t *where* $\psi(t) \equiv 0 \pmod{\pi}$. *If* $F(z)$ *or* $G(z)$ *has at least one zero, and if*

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(5) $$\log |G(z)/F(z)| = \frac{|y|}{\pi} \int \frac{\log |G(t)/F(t)|}{(t-x)^2 + y^2} \, dt$$

for $y \neq 0$, with absolute convergence of the integral, then either $F(z) - iG(z)$ or $F(z) + iG(z)$ satisfies (1).

The construction of Theorem II is of special interest for particular choices of $\psi(x)$.

THEOREM V. *Let $\psi(x)$ be a uniformly continuous increasing function which has value* 0 *at the origin. Then, there is an entire function $E(z)$ of genus* 0 *or* 1 *which satisfies* (1), *has no real zeros, and has value* 1 *at the origin, for which the phase function $\phi(x)$ agrees with $\psi(x)$ whenever $\phi(x) \equiv 0 \pmod{\pi}$ or $\psi(x) \equiv 0 \pmod{\pi}$, and such that $\phi'(x)$ is bounded.*

THEOREM VI. *Let $\tau \geqslant 0$ be given in Theorem II. A sufficient condition that $E(z)$ can be chosen of exponential type τ, and not of smaller type, such that*

(6) $$\int (1 + t^2)^{-1} \log^+ |E(t)| \, dt < \infty$$

is that

(7) $$\int (1 + t^2)^{-1} |\psi(t) - \tau t|^2 \, dt < \infty .$$

Applications of these constructions depend on a theorem of Beurling and Malliavin **(1)**, which has this consequence for Hilbert spaces of entire functions.

THEOREM VII. *Let $E(b, z)$ be an entire function of exponential type $\tau(b) > 0$ and not of smaller type, which satisfies* (1) *and* (6) *and has no real zeros. If the phase function $\phi(b, x)$ has a bounded derivative, and if $0 < \tau(a) < \tau(b)$, then there is an entire function $E(a, z)$ of exponential type $\tau(a)$, which satisfies* (1) *and has no real zeros, such that $\mathfrak{H}(E(a))$ is contained isometrically in $\mathfrak{H}(E(b))$.*

Now let us consider a problem in Fourier analysis. If $c > 0$ is given and if X is a given closed subset of the real line, does there exist a measure of finite total variation, supported in X, whose Fourier transform $\int e^{ixt} \, d\mu(t)$ vanishes in $[-c, c]$ and does not vanish identically? Since this is a difficult question to answer for general sets, we shall reformulate the problem using methods from the Bernstein problem **(7)**.

THEOREM VIII. *Let $c > 0$ be given and let X be a closed subset of the real line. A necessary and sufficient condition that there exist a measure μ of finite total variation, supported in X, whose Fourier transform $\int e^{ixt} \, d\mu(t)$ vanishes in $[-c, c]$ and does not vanish identically, is that there exist an entire function $S(z)$ of exponential type c, and not of smaller type, which is real for real z, has only real simple zeros all in X, satisfies* (6) *and*

(8) $$\sum_{S(t)=0} |S'(t)|^{-1} < \infty .$$

In this case, μ may be taken with support at the zeros of $S(x)$ and with mass $S'(t)^{-1}$ at each such zero t.

This theorem is interesting because it makes us look at discrete subsets of X which have a density. However, (8) is not easily verified from a knowledge of the zeros of $S(z)$. Fortunately, there is another formulation of the problem.

THEOREM IX. *Let $E(z)$ be an entire function which satisfies* (1), *has no real zeros, and has value* 1 *at the origin. Let $\phi(x)$ be the corresponding phase function and let $c > 0$. Then, condition* (B) *below is a sufficient condition for* (A). *If*

$$\sum_{\sin \phi(t)=0} (1 + t^2)^{-1} \phi'(t) < \infty, \tag{9}$$

(B) *is also a necessary condition for* (A).

(A) *Whenever $0 < a < c$, there is a measure μ of finite total variation, supported in the set of points t where $\phi(t) \equiv 0 \pmod{\pi}$, whose Fourier transform $\int e^{ixt} d\mu(t)$ vanishes in $[-a, a]$ and does not vanish identically.*

(B) *Whenever $0 < b < c$, there is an entire function $E_b(z)$, which satisfies* (1) *and has no real zeros, such that $\mathfrak{H}(E_b)$ is contained isometrically in $\mathfrak{H}(E)$ and*

$$\lim y^{-1} \log|E(iy)/E_b(iy)| = b \tag{10}$$

as $y \to +\infty$.

Condition (10) has been discussed previously from a different point of view in Theorems VIII, IX, and X of **(13)**. A partial answer to these questions may now be obtained by applying Theorem VII.

THEOREM X. *Let $\tau > 0$ be given and let $\psi(x)$ be a uniformly continuous, increasing function of real x which satisfies* (7). *Whenever $0 < a < \tau$, there is a measure μ of finite total variation, supported in the set of points t where $\psi(t) \equiv 0 \pmod{\pi}$, whose Fourier transform $\int e^{ixt} d\mu(t)$ vanishes in $[-a, a]$ and does not vanish identically.*

The theorem has applications to a problem of Levinson **(17**, chapters VIII and IX**)**. If X is a given closed subset of the real line, does there exist an entire function of minimal exponential type which remains bounded on X and is not a constant?

THEOREM XI. *Let $a > 0$ and let μ be a measure of finite total variation on the Borel sets of the real line whose Fourier transform $\int e^{ixt} d\mu(t)$ vanishes in $[-a, a]$ and does not vanish identically. If $K(z)$ is an entire function of minimal exponential type which remains bounded on the support of μ, then $K(z)$ is a constant.*

Therefore if $\psi(x)$ is a uniformly continuous, increasing function which satisfies (7) for some $\tau > 0$, an entire function of minimal exponential type, which remains bounded at the points t where $\psi(t) \equiv 0 \pmod{\pi}$, is a constant. An existence theorem for entire functions of minimal type, bounded on certain sets, can be obtained from a theorem of Levinson as it is formulated in **(5)**.

THEOREM XII. *Let* $((a_n, b_n))$ *be a sequence of disjoint intervals to the right of* $x = 1$ *such that*

$$\sum a_n^{-1} b_n^{-1} < \infty \tag{11}$$

and

$$\sum (b_n - a_n)^2 a_n^{-1} b_n^{-1} = \infty. \tag{12}$$

Then, there is an entire function of minimal exponential type which remains bounded on the real complement of $\bigcup (a_n, b_n)$ *and is not a constant.*

Theorem III will be proved first since it provides an estimate used in the proof of Theorem I. Otherwise, the theorems are proved in numerical order.

Proof of Theorem III. Since $E(z)$ satisfies (1) by hypothesis, the function $f(z) = B(z)/A(z)$ is defined and analytic for $y > 0$, and $\operatorname{Re}(-if(z)) \geqslant 0$, and $|f(z) + i| \geqslant 1$. By the Poisson representation of a function positive and harmonic in a half-plane, there is a non-negative measure μ on the Borel sets of the real line and a number $a \geqslant 0$ such that

$$\log|f(z) + i| = ay + \frac{y}{\pi}\int \frac{d\mu(t)}{(t-x)^2 + y^2}$$

for $y > 0$. From a similar representation of $\operatorname{Re}(-if(z))$, we obtain $f(iy) = O(y)$ as $y \to +\infty$, by the Lebesgue dominated convergence theorem, and so $a = 0$. Since $f(z)$ is analytic across the real axis except for poles, μ is an absolutely continuous measure with density $\log|f(x) + i|$. We have shown that

$$\log|f(z) + i| = \frac{y}{\pi}\int \frac{\log|f(t) + i|}{(t-x)^2 + y^2}\, dt \tag{13}$$

for $y > 0$, where the integrand is positive since $f(z) = f^*(z)$ is real for real z. Estimates from this representation will show that $f(z)$ satisfies (6) and that

$$\lim r^{-1} \int_0^\pi \log^+ |f(re^{i\theta})| \sin\theta\, d\theta = 0 \tag{14}$$

as $r \to \infty$. The argument of Boas (**2**, pp. 92–93) now allows us to establish the representation

$$\log|f(x + iy)| = \frac{|y|}{\pi}\int \frac{\log|f(t)|}{(t-x)^2 - y^2}\, dt \tag{15}$$

for $y > 0$, with absolute convergence of the integral. Since $f^*(z) = f(z)$, the same formula holds for $y < 0$, and this establishes (4).

Proof of Theorem I. By Theorem III, (4) holds with absolute convergence of the integral if we take $E(z) = E_1(z)$ or $E(z) = E_2(z)$. Therefore, the function

$$f(z) = [B_2(z)/A_2(z)]/[B_1(z)/A_1(z)]$$

satisfies (15) when $y \neq 0$, with absolute convergence of the integral. Since $A_2(z)/A_1(z)$ and $B_2(z)/B_1(z)$ are entire functions which have no zeros by

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hypothesis, $f(z)$ is an entire function and it has no zeros. Because of (15), $f(z)$ has minimal exponential type by an estimate of M. G. Krein (cf. Boas **2**, p. 132). Since this function has no zeros, it is a constant. Since $E_1(0) = 1$ and $E_2(0) = 1$ by hypothesis, $f(z) = B_2'(0)/B_1'(0)$, where $B_1'(0)$ and $B_2'(0)$ are positive (cf. **8**, Lemma 2). If $f(z) = c^2$, where $c > 0$, then

$$G(z) = c^{-1} B_2(z)/B_1(z) = cA_2(z)/A_1(z)$$

is an entire function which is real for real z, has no zeros, and satisfies

$$K_2(w, z) = K_1(w, z)G(z)\bar{G}(w)$$

for all complex z and w. If $F(z)$ is a finite linear combination of the functions $K_1(w, z)\bar{G}(w)$, with w complex, this identity implies that $G(z)F(z)$ does belong to $\mathfrak{H}(E_2)$, where it has the same norm as $F(z)$ in $\mathfrak{H}(E_1)$. Since such combinations are dense in $\mathfrak{H}(E_1)$, $F(z) \to G(z)F(z)$ is a linear isometric transformation of $\mathfrak{H}(E_1)$ into $\mathfrak{H}(E_2)$. Similarly, $F(z) \to G(z)^{-1} F(z)$ is a linear isometric transformation of $\mathfrak{H}(E_2)$ into $\mathfrak{H}(E_1)$ and each transformation is onto.

Proof of Theorem II. For each $n = 1, 2, 3, \ldots$, let $\psi_n(x) = \psi(x)$ whenever $|\psi(x)| \leqslant |n \arctan x|$ and let $\psi_n(x) = n \arctan x$ otherwise. Let $A_n(z)$ be the unique polynomial with value 1 at the origin, whose zeros are real and simple and occur at the points t where $\psi_n(t) \equiv \pi/2 \pmod{\pi}$. Let $B_n(z)$ be the unique polynomial with derivative 1 at the origin, whose zeros are real and simple and occur at the points t where $\psi_n(t) \equiv 0 \pmod{\pi}$. These polynomials are real on the real axis, and $A_n(x)$ always has the same sign as $\cos \psi_n(x)$, and $B_n(x)$ always has the same sign as $\sin \psi_n(x)$. A study of sign changes will show that

$$B_n'(x)A_n(x) - A_n'(x)B_n(x) > 0$$

whenever $\psi_n(x) \equiv 0 \pmod{\pi/2}$. If the degree of $B_n(z)$ is no more than that of $A_n(z)$, there is a real number k_n such that

$$B_n(z)\, A_n(z)^{-1} = k_n + \sum_{A_n(t)=0} B_n(t)\, A_n'(t)^{-1}\, (z - t)^{-1}$$

and so

$$\operatorname{Re}(-iB_n(z)/A_n(z)) = y \sum_{A_n(t)=0} -B_n(t)\, A_n'(t)^{-1}\, |t - z|^{-2} > 0$$

for $y > 0$. This inequality implies that $A_n(z) - iB_n(z)$ satisfies (1). The same conclusion can be reached by a partial fraction decomposition of $A_n(z)/B_n(z)$ if the degree of $B_n(z)$ is larger. In any case,

$$\operatorname{Re}(-iB_n(z)/A_n(z)) > 0$$

for $y > 0$ by (**8**, Lemma 1). By the Poisson representation of a function positive and harmonic in a half-plane, there is a number $p_n \geqslant 0$ such that

$$\operatorname{Re}(-iB_n(z)/A_n(z)) = p_n y + y \sum_{A_n(t)=0} -B_n(t)\, A_n'(t)^{-1}\, |t - z|^{-2}$$

when $A_n(z) \neq 0$. Since $A_n(0) = 1$ and $B_n'(0) = 1$ by construction, we obtain

$$1 = p_n + \sum_{A_n(t)=0} -t_n^{-2} B_n(t)/A_n'(t)$$

on dividing by y and letting $z \to 0$. It follows that the functions $B_n(z)/A_n(z)$ are uniformly bounded on any bounded set at a positive distance from the real points t where $\psi(t) \equiv \pi/2 \pmod{\pi}$. By compactness, there is a sub-sequence $(f_n(z))$ of these functions which converges uniformly on any bounded set at a positive distance from these real points. The limit function $f(z)$ is defined and analytic in the complement of the same set of points. Since $\mathrm{Re}(-if_n(z)) \geqslant 0$ for $y > 0$ and every n, we have $\mathrm{Re}(-if(z)) \geqslant 0$ for $y > 0$. By the Poisson representation of a function positive and harmonic in a half-plane, there is a non-negative measure μ supported in the real points t where $\psi(t) \equiv \pi/2 \pmod{\pi}$ and there is a number $p \geqslant 0$ such that

$$\mathrm{Re}(-if(z)) = py + y\int |t - z|^{-2}\, d\mu(t)$$

for $y > 0$. If $\psi(t) \equiv 0 \pmod{\pi}$, $f_n(t) = 0$ for large values of n, and so $f(t) = 0$. We know that $f(z)$ is real for real z since this is true of every $f_n(z)$. Since $f_n'(0) = 1$ for every n, $f'(0) = 1$ and $f(z)$ does not vanish identically. Since

$$f'(x) = \int (t - x)^{-2}\, d\mu(t) > 0,$$

the function $f(x)$ increases in each interval of continuity. Since $f(x)$ is known to vanish on the left and right of each point t where $\psi(t) \equiv \pi/2 \pmod{\pi}$, each such point must be a singularity for $f(z)$ and carries a positive mass for μ. The uniqueness of the function $f(z)$ with the said properties is clear from the proof of Theorem I. Therefore, it was unnecessary to pass to a sub-sequence, and

$$f(z) = \lim B_n(z)/A_n(z)$$

as $n \to \infty$. By the Weierstrass factorization, there exists an entire function $A(z)$ which is real for real z and has real simple zeros, all at the points t where $\psi(t) \equiv \pi/2 \pmod{\pi}$. Then, $B(z) = f(z)A(z)$ is an entire function which is real for real z and has only real simple zeros, the points t where $\psi(t) \equiv 0 \pmod{\pi}$. Since $\mathrm{Re}(-iB(z)/A(z)) > 0$ for $y > 0$ by construction, $E(z) = A(z) - iB(z)$ satisfies (1). This function clearly has the required properties.

If (2) holds, $A(z)$ may be chosen of genus 0 or 1 by the Hadamard factorization. There is a sequence (h_n) of real numbers such that

$$A(z) = \lim A_n(z)\exp(h_n z),$$

$$B(z) = \lim B_n(z)\exp(h_n z)$$

uniformly on bounded sets. Then, $E_n(z) = A_n(z) - iB_n(z)$ is a polynomial which satisfies (1) for every n and

$$E(z) = \lim E_n(z)\exp(h_n z)$$

uniformly on bounded sets. Since $|E_n(x+iy)|$ is a non-decreasing function of $y > 0$ for each fixed x by (**11**, Lemma 2), $|E(x+iy)|$ is a non-decreasing function of $y > 0$ for each fixed x. By the same lemma,

$$E(z) = F(z) \exp(-az^2),$$

where $F(z)$ is an entire function of genus 0 or 1 which satisfies (1), and $a \geqslant 0$. Since

$$B(iy)/A(iy) = O(y)$$

as $y \to +\infty$ by the Poisson representation, and since

$$\log |A(iy)| = o(y^2)$$

by the choice of $A(z)$ of genus 0 or 1, we have

$$\log|E(iy)| = o(y^2) \tag{16}$$

as $y \to +\infty$. Therefore, $a = 0$ and $E(z) = F(z)$ has genus 0 or 1.

In the absence of additional conditions, the function $E(z)$ so constructed contains an undetermined factor of $\exp(hz)$, h real. Let $W(z)$ be the choice of a function, defined and analytic for $y > 0$, which satisfies (3). Since $\psi(x)$ is a continuous, increasing function by hypothesis, the boundary value function $W(x) = \lim W(x+iy)$ is defined for all real x, as $y \searrow 0$, and $|W(x)| > 0$. The function $W(z)$ is chosen by hypothesis so that $W(0) = 1$. We have seen that $|E(x+iy)|$ is a non-decreasing function of $y > 0$ for each fixed x and that (16) holds. By the Poisson representation of a function positive and harmonic in a half-plane,

$$\frac{\partial}{\partial y} \log |E(x+iy)| = \pi^{-1} y \int |t - z|^{-2}\, d\phi(t) \tag{17}$$

for $y > 0$. Because of (3), we have

$$\frac{\partial}{\partial y} \log |E(x+iy)/W(x+iy)| = \pi^{-1} y \int |t-z|^{-2}\, d[\phi(t) - \psi(t)]$$

for $y > 0$, where $-\pi < \phi(x) - \psi(x) < \pi$ for all real x. If the arbitrary exponential factor in $E(z)$ is chosen so that

$$\mathrm{Re}(i \log[E(x+iy)/W(x+iy)]) = \pi^{-1} y \int |t-z|^{-2} [\phi(t) - \psi(t)]\, dt$$

for $y > 0$, it follows that $E(z)/W(z)$ is never negative. Of course, this formula refers to the continuous determination of the said logarithm, for $y \geqslant 0$, which vanishes at the origin. Conversely, if $E(z)/W(z)$ is never negative, this last formula certainly must hold and uniqueness follows.

Proof of Theorem IV. By interchanging $F(z)$ and $G(z)$ and making a translation, if necessary, we may suppose that $\psi(0) = 0$. By multiplying these functions by real constants, we may restrict ourselves to the case in which $F(0) = 1$ and $G'(0) = 1$. These normalizations do not affect the conclusion of the theorem since (1) for $F(z) - iG(z)$ is equivalent to the positivity of $\operatorname{Re}(-iG(z)/F(z))$ for $y > 0$, and since (1) for $F(z) + iG(z)$ is equivalent to the positivity of $\operatorname{Re}(iG(z)/F(z))$ for $y > 0$. Let $E(z)$ be defined for $\psi(x)$ as in Theorem II, with $A(0) = 1$ and $B'(0) = 1$ as in the proof of the theorem. Then

$$f(z) = [G(z)/F(z)]/[B(z)/A(z)]$$

is an entire function with no zeros, which is real for real z and has value 1 at the origin. Since (4) holds by Theorem III and since (5) holds by hypothesis, formula (15) is valid with absolute convergence of the integral when $y \neq 0$. As in the proof of Theorem I, $f(z)$ is a constant, equal to 1 by its value at the origin. Since $E(z)$ satisfies (1) by construction, and since $G(z)/F(z) = B(z)/A(z)$ for all complex z, $F(z) - iG(z)$ satisfies (1).

Proof of Theorem V. Since $\psi(x)$ is uniformly continuous, it satisfies (2), and there is an entire function $B(z)$ of genus 0 or 1 which is real for real z, has only real simple zeros all at the points t where $\psi(t) \equiv 0 \pmod{\pi}$, and has a derivative of 1 at the origin. By Laguerre's theorem (cf. Boas **2**, p. 23) the function $A(z) = B'(z)$ is real for real z and has only real simple zeros which are distinct from the zeros of $B(z)$. In proving the theorem, Boas shows that

$$\operatorname{Re}(iA(z)/B(z)) = y \sum_{B(t)=0} |t - z|^{-2} > 0 \tag{18}$$

for $y > 0$. Therefore, $E(z) = A(z) - iB(z)$ is an entire function which satisfies (1), has no real zeros, and has value 1 at the origin. The associated phase function $\phi(x)$ clearly has the property that $\phi(x) = \psi(x)$ whenever $\phi(x) \equiv 0 \pmod{\pi}$ or $\psi(x) \equiv 0 \pmod{\pi}$, since such points x are just the zeros of $B(z)$. Since $\tan\phi(x) = B(x)/A(x)$, the choice of $A(z)$ implies that $\phi'(x) = 1$ whenever $\phi(x) \equiv 0 \pmod{\pi}$. From (18), we obtain

$$\phi'(x) = \sum_{\sin\phi(t)=0} (t - x)^{-2} \sin^2\phi(x). \tag{19}$$

Since $\psi(x)$ is uniformly continuous by hypothesis, there is a number $\delta > 0$ such that $|\psi(x) - \psi(t)| < \pi$ whenever $|x - t| < \delta$. Our calculations are simplified if we choose $\delta < \pi/2$. Then $|x - t| \geqslant \delta$ whenever $|\psi(x) - \psi(t)| \geqslant \pi$. By the construction of $E(z)$, we have $|x - t| \geqslant \delta$ whenever $|\phi(x) - \phi(t)| = \pi$ and $\phi(x) \equiv \phi(t) \equiv 0 \pmod{\pi}$. Let a be any fixed real number such that $\phi(a) \equiv 0 \pmod{\pi}$. When $|a - x| \leqslant \frac{1}{2}\delta$, (19) yields the estimate

$$\begin{aligned}\phi'(x) &\leqslant \sum \sin^2\phi(x)\ (x - a + n\delta)^{-2} \\ &\leqslant \pi^2\,\delta^{-2}\sin^2\phi(x)\sin^{-2}[\pi(x - a)/\delta],\end{aligned}$$

which implies that

$$\cot\phi(x) - (\pi/\delta)\cot[\pi(x - a)/\delta]$$

is non-decreasing in this interval. Since this function is continuous at $x = a$ and has value 0 there,

$$\cot^2 \phi(x) \geqslant (\pi/\delta)^2 \cot^2[\pi(x - a)/\delta]$$

for $|x - a| \leqslant \frac{1}{2}\delta$, and therefore $\phi'(x) \leqslant (\pi/\delta)^2$ for these values of x. From (19) we see that the same estimate holds when x is at a distance of $\frac{1}{2}\delta$ or more from the zeros of $B(z)$. Therefore, the same estimate holds for all real x.

Proof of Theorem VI. Since $\psi(x)$ is increasing, the convergence of (7) implies that

$$\lim \psi(x)/x = \tau \tag{20}$$

as $|x| \to \infty$, and therefore $\psi(x)$ satisfies (2). Let $E(z)$ be chosen of genus 0 or 1. If $\log E(z)$ is defined continuously for $y \geqslant 0$ so as to vanish at the origin, we have the inequality

$$\mathrm{Re}(iz^{-1} \log E(z)) \geqslant 0 \tag{21}$$

for $y \geqslant 0$. This inequality is generally true for entire functions of genus 0 or 1 which satisfy (1), have no real zeros, and have value 1 at the origin. It is most easily verified for polynomials; the general case follows on passing to a limit by (**11**, Lemma 1). In the polynomial case, the inequality follows from the fact that $1 - z/\bar{w}$ satisfies (21) if $i(\bar{w} - w) > 0$. By (16) and the Poisson representation of a function positive and harmonic in a half-plane,

$$\mathrm{Re}(iz^{-1} \log E(z)) = \pi^{-1} y \int |t - z|^{-2} \phi(t)/t\, dt \tag{22}$$

for $y > 0$. By (7) and the theory of the Hilbert transform, there is a function $f(x)$ in $L^2(-\infty, \infty)$ such that

$$\mathrm{Re} f(x) = [\phi(x) - \tau x]/x$$

for almost all real x, and

$$0 = \int (t - z)^{-1} f(t)\, dt \tag{23}$$

for $y < 0$. If we define

$$f(z) = (2\pi i)^{-1} \int (t - z)^{-1} f(t)\, dt \tag{24}$$

for $y > 0$, we then have

$$f(z) = \pi^{-1} y \int |t - z|^{-2} f(t)\, dt$$

for $y > 0$, and hence

$$\mathrm{Re} f(z) = \mathrm{Re}(i[\log E(z) + i\tau z]/z).$$

By the Cauchy-Riemann equations, there is a real number c such that

$$[\log E(z) + i\tau z]/z = c - if(z).$$

By altering $E(z)$ by an exponential factor, we may suppose that $c = 0$. In this case, our choice of $f(z)$ implies that

$$(25) \qquad \int (1 + t^2)^{-1} \log^2 |E(t)| \, dt < \infty$$

and

$$(26) \qquad \log |E(x + iy)| = \tau y + \pi^{-1} y \int |t - z|^{-2} \log |E(t)| \, dt$$

for $y > 0$. By the Krein estimate (cf. Boas **2**, p. 132), $E(z)$ has exponential type. Since (23) implies (6), we have

$$(27) \qquad \lim r^{-1} \log |E(re^{i\theta})| = \tau \sin \theta$$

as $r \to \infty$ for $0 < \theta < \pi$, by the Lebesgue dominated convergence theorem. Since $E(z)$ satisfies (1), it has type τ and does not have smaller type.

Proof of Theorem VII. Let $\mathfrak{M}(a)$ be the set of all entire functions $F(z)$ of exponential type at most $\tau(a)$ such that

$$\int |F(t)/E(b, t)|^2 \, dt < \infty.$$

If $F(z)$ is in $\mathfrak{M}(a)$, then

$$\int (1 + t^2)^{-1} \log^+ |F(t)/E(b, t)| \, dt < \infty$$

by Jensen's inequality. By the absolute convergence of (26) for $E(b, z)$, $F(z)$ satisfies (6) and hence

$$\lim \sup |x|^{-1} \log |F(x)| = 0$$

as $|x| \to \infty$ by Boas (**2**, p. 97). By (**11**, Lemma 11), $F(z)$ belongs to $\mathfrak{H}(E(b))$. If

$$\tau(a) - \tau(b) < h < \tau(b) - \tau(a),$$

the same argument will show that $e^{ihz} F(z)$ belongs to $\mathfrak{H}(E(b))$. By taking the limit in the metric of $\mathfrak{H}(E(b))$, we see that the same conclusion holds for

$$\tau(a) - \tau(b) \leqslant h \leqslant \tau(b) - \tau(a).$$

Conversely, a function $G(z)$ in $\mathfrak{H}(E(b))$ such that $e^{ihz} G(z)$ belongs to $\mathfrak{H}(E(b))$ whenever

$$\tau(a) - \tau(b) \leqslant h \leqslant \tau(b) - \tau(a)$$

is of exponential type $\tau(a)$ because of (26) for $E(b, z)$ and the defining inequality of $\mathfrak{H}(E(b))$. It follows that $\mathfrak{M}(a)$ is a closed subspace of $\mathfrak{H}(E(b))$. It clearly satisfies the axioms (H1), (H2), and (H3) when considered in the metric of $\mathfrak{H}(E(b))$. If there is a non-zero element in $\mathfrak{M}(a)$, then $\mathfrak{M}(a)$ is equal isometrically to $\mathfrak{H}(E(a))$, where $E(a, z)$ is an entire function which satisfies (1). Since $F(z)/(z - w)$ belongs to $\mathfrak{M}(a)$ whenever $F(z)$ belongs to $\mathfrak{M}(a)$ and $F(w) = 0$, $E(a, z)$ has no real zeros. Since every $F(z)$ in $\mathfrak{H}(E(a))$ has exponential type at most a, $E(a, z)$ has type a. The problem now is to show that $\mathfrak{M}(a)$ contains a non-zero element.

We shall do this by constructing an entire function $K(z)$ of exponential type, which satisfies (6), such that

$$\int |K(t)E(b, t)|^{-2}\, dt < \infty. \tag{28}$$

Then, by a theorem of Beurling and Malliavin **(1)**, there is an entire function $F(z)$ of exponential type a, which is bounded on the real axis and does not vanish identically, such that $K(z)F(z)$ is bounded on the real axis. In fact, the construction can always be made so that $F(z)$ and $K(z)F(z)$ are the Fourier transforms of square integrable functions which vanish outside of some finite interval. Because of (28), such a function $F(z)$ belongs to $\mathfrak{M}(a)$.

To construct $K(z)$, let $c > 0$ be chosen so that $\phi'(b, x) < c$ for all real x. Let

$$W(z) = e^{icz}/E(b, z)$$

and

$$\psi(x) = cx - \phi(b, x).$$

Then, $\psi(x)$ is a continuous, increasing function of real x which has value 0 at the origin, and $|W(x + iy)|$ is a non-decreasing function of $y > 0$ for each fixed x. Since

$$\log |W(iy)| \leqslant cy,$$

Formula (3) holds for $y > 0$ by the Poisson representation of a function positive and harmonic in a half-plane. Since $\psi'(x)$ is bounded, (2) holds. Let $E(z)$ be chosen for $\psi(x)$ as in Theorem II so that $E(z)/W(z)$ is never negative for $y > 0$. Let $f(z)$ be the choice of a function, analytic for $y > 0$ and continuous in the closed half-plane, such that $\operatorname{Re} f(z) \geqslant 0$ and $f(z)^4 = W(z)/E(z)$. By the Poisson representation of a function positive and harmonic in a half-plane, there is a number $p \geqslant 0$ such that

$$\operatorname{Re} f(z) = py + \pi^{-1} y \int |t - z|^{-2} \operatorname{Re} f(t)\, dt$$

for $y > 0$, and hence

$$f(x + iy) = O(x^2)$$

as $|x| \to \infty$ for each fixed $y > I$. Since $|W(x + iy)|$ is a non-decreasing function of $y > 0$ for each fixed x,

$$|E(x + iy)E(b, x)|^{-1} = O(x^4)$$

as $|x| \to \infty$ for each fixed $y > 0$. Therefore, (28) holds if we choose

$$K(z) = (z + ih)^5\, E(z + ih)$$

for some number $h > 0$. The function $K(z)$ is of exponential type $\tau = c - \tau(b)$ and satisfies (6) because (26) holds for $y > 0$ with absolute convergence of the integral. This completes the proof of the theorem.

Proof of Theorem VIII. Let $\mathfrak{C}(X)$ be the space of continuous complex-valued functions $f(x)$ of x in X such that $\lim f(x) = 0$ as $|x| \to \infty$, in the

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non-compact case. We consider $\mathfrak{C}(X)$ as a Banach space in the uniform norm. Let E be the space of entire functions $F(z)$ of exponential type at most c such that $\lim F(x) = 0$ as $|x| \to \infty$. If μ is a measure of finite total variation on the Borel sets of the real line, whose Fourier transform $\int e^{ixt}\, d\mu(t)$ vanishes in $[-c, c]$, then $\int F(t)\, d\mu(t) = 0$ by **(5**, Theorem I**)**, since $e^{-a|y|} F(iy)$ is bounded by Boas **(2**, p. 82**)**. Conversely, if μ is a measure of finite total variation on the Borel sets of the real line such that $\int F(t)\, d\mu(t) = 0$ for every $F(z)$ in E, then $\int e^{ixt}\, d\mu(t) = 0$ in $[-c, c]$, for $[e^{ihz} - e^{ihw}]/(z - w)$ belongs to E as a function of z for every complex number w when $-a \leqslant h \leqslant a$. Since we have

$$\int [e^{iht} - e^{ihw}]/(t - w)\, d\mu(t) = 0$$

by the choice of μ,

$$\begin{aligned}\int e^{iht}\, d\mu(t) &= \lim w \int (w - t)^{-1} e^{iht}\, d\mu(t)\\ &= \lim w e^{ihw} \int (w - t)^{-1}\, d\mu(t) = 0\end{aligned}$$

by the Lebesgue dominated convergence theorem if we let $|w| \to \infty$ with w on the upper or lower half of the imaginary axis, depending on the sign of h.

Let $U(E)$ be the convex set of all real-valued measures of total variation at most 1, supported in X, such that $\int e^{ixt}\, d\mu(t) = 0$ in $[-c, c]$, or equivalently, $\int F(t)\, d\mu(t) = 0$ for every $F(z)$ in E. By the Krein-Milman convexity theorem, $U(E)$ is the closed convex span of its extreme points in the weak topology induced by $\mathfrak{C}(X)$. A useful property of such measures is given in **(6)** as a lemma to the Stone-Weierstrass theorem. If $g(x)$ is a Borel measurable function of x in X, which is essentially bounded with respect to $|\mu|$, such that $\int F(t)\, g(t)\, d\mu(t) = 0$ for every $F(z)$ in E, then g is equal to a constant a.e. with respect to $|\mu|$. Equivalently, if $f(x)$ is in $L^1(|\mu|)$ and if $\int f(t)\, d\mu(t) = 0$, there is a sequence $(F_n(z))$ in E such that $f(x) = \lim F_n(x)$ in the metric of $L^1(|\mu|)$. These conclusions depend essentially on the fact that $F^*(z)$ belongs to E whenever $F(z)$ belongs to E.

If μ is any fixed element of $U(E)$ with $\int |d\mu(t)| = 1$, let

$$M(z) = \sup |F(z)|,$$

where $F(z)$ ranges in the elements of E with $\int |F(t)\, d\mu(t)| \leqslant 1$. Since

$$[F(z) - F(w)]/(z - w)$$

belongs to E whenever $F(z)$ belongs to E,

$$\int [F(t) - F(z)]/(t - z)\, d\mu(t) = 0$$

for all complex z. When z is not real,

$$|F(z) \int (t - z)^{-1}\, d\mu(t)| \leqslant |y|^{-1} \int |F(t)\, d\mu(t)|$$

and hence

$$M(z)\, |\int (t - z)^{-1}\, d\mu(t)| \leqslant |y|^{-1} \tag{29}$$

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by the arbitrariness of $F(z)$. Since μ is real and does not vanish identically, $\int (t-z)^{-1}\,d\mu(t)$ does not vanish identically for $y > 0$. Since this function is bounded by 1 for $y \geqslant 1$,

$$\int (1+x^2)^{-1} \log^- \left|\int (t-x-i)^{-1}\,d\mu(t)\right| dx > -\infty \tag{30}$$

by Boas (**2**, p. 85). Since 1 belongs to E and μ has total variation 1, $M(z) \geqslant 1$ for all complex z. It follows from (29) and (30) that

$$\int (1+t^2)^{-1} \log M(t+i)\,dt < \infty.$$

By Boas (**2**, p. 93), the inequality

$$\log |F(x+iy_2)| \leqslant c|y_2 - y_1| + \pi^{-1}|y_2 - y_1| \int |t - x + iy_2 - iy_1|^{-2} \times \log^+ |F(t+iy_1)|\,dt$$

holds for every $F(z)$ in E when $y_1 \neq y_2$. If $\int |F(t)\,d\mu(t)| \leqslant 1$, then

$$\log |F(x+iy_2)| \leqslant c|y_2 - y_1| + \pi^{-1}|y_2 - y_1| \int |t - x + iy_2 - iy_1|^{-2} \times \log M(t+iy_1)\,dt.$$

By the arbitrariness of $F(z)$,

$$\log M(x+iy_2) \leqslant c|y_2 - y_1| + \pi^{-1}|y_2 - y_1| \int |t - x + iy_2 - iy_1|^{-2} \log M(t+iy_1)\,dt \tag{31}$$

when $y_1 \neq y_2$. Since the integral has been shown to be finite when $y_1 = 1$, it converges for all values of y_1 by the semi-group properties of the Poisson kernels under convolution. Therefore, $M(z)$ is finite and locally bounded in the complex plane.

Let us show that an entire function $G(z)$ which vanishes a.e. with respect to $|\mu|$ must vanish identically if $G(z)/M(z)$ is bounded in the complex plane. To see this, observe that

$$g(z) = G(z) \int (z-t)^{-1}\,d\mu(t) = \int [G(t) - G(z)]/(t-z)\,d\mu(t)$$

is an entire function. For if $[a, b]$ is any finite interval,

$$\begin{aligned} g(z) = &\int_{-\infty}^{a} G(t)/(t-z)\,d\mu(t) - G(z) \int_{-\infty}^{a} 1/(t-z)\,d\mu(t) \\ &+ \int_a^b [G(t) - G(z)]/(t-z)\,d\mu(t) \\ &+ \int_b^{\infty} G(t)/(t-z)\,d\mu(t) - G(z) \int_b^{\infty} 1/(t-z)\,d\mu(t), \end{aligned}$$

where the first two terms are analytic in the complement of the real segment $(-\infty, a)$ and the last two terms are analytic in the complement of (b, ∞). Since the middle term is a limit of Riemann-Stieltjes partial sums, boundedly for z in any bounded set, it is an entire function. We have shown that $g(z)$

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is analytic for $y > 0$, for $y < 0$, and across the real segment (a, b). By the arbitrariness of a and b, $g(z)$ is an entire function. If $|G(z)/M(z)| \leqslant k$, we have

$$|g(z)| \leqslant kM(z)|\int (t-z)^{-1} \, d\mu(t)| \leqslant k\,|y|^{-1}$$

for all complex z. Since

$$\log |g(z)| \leqslant (2\pi)^{-1}\int_0^{2\pi} \log |g(z+e^{i\theta})| \, d\theta$$
$$\leqslant \log k + (2\pi)^{-1}\int_0^{2\pi} \log |\sin \theta|^{-1} \, d\theta$$

for all complex z, $g(z)$ is a constant by Liouville's theorem and vanishes identically by estimates at the far end of the imaginary axis. Since $\int (t-z)^{-1} \, d\mu(t)$ does not vanish identically, $G(z)$ vanishes identically.

With μ still held fixed, let $\mathfrak{M}$ be the space of all entire functions $G(z)$ such that $G(z)/M(z)$ is bounded in the complex plane, and $G(x) = \lim F_n(x)$ in the metric of $L^1(|\mu|)$, where $(F_n(z))$ is a sequence of elements of E. Since

$$|F(z)| \leqslant M(z) \int |F(t) \, d\mu(t)| \tag{32}$$

for every $F(z)$ in E, by the definition of $M(z)$, every $F(z)$ in E belongs to $\mathfrak{M}$. If $F_n(x) \to G(x)$ in the metric of $L^1(|\mu|)$,

$$|F_m(z) - F_n(z)| \leqslant M(z) \int |F_m(t) - F_n(t)| \, d\mu(t)$$

for all complex z and hence $H(z) = \lim F_n(z)$ exists uniformly on bounded sets. Since $H(z)$ belongs to $\mathfrak{M}$ and since $H(x) = G(x)$ a.e. with respect to $|\mu|$, $H(z) = G(z)$ for all complex z. Therefore, $\mathfrak{M}$ determines a closed subspace of $L^1(|\mu|)$ and (32) holds for every $F(z)$ in $\mathfrak{M}$.

The argument depends on showing that certain special functions belong to $\mathfrak{M}$. Let us show that if $G(z)$ is an entire function which belongs to $L^1(|\mu|)$ and if $G(z)/M(z)$ is bounded in the complex plane, then $[G(z) - G(w)]/(z-w)$ belongs to $\mathfrak{M}$ as a function of z for every complex number w. To see this, we must show that if $h(x)$ is any Borel measurable function such that $|h(x)| \leqslant 1$ and $\int F(t)\,h(t) \, d\mu(t) = 0$ for every $F(z)$ in E, then

$$L(z) = \int [G(t) - G(z)]/(t-z)\, h(t) \, d\mu(t)$$

vanishes identically. Since

$$F(z)\,L(z) = \int [F(z)\,G(t) - F(t)\,G(z)]/(t-z)\, h(t) \, d\mu(t)$$

for every $F(z)$ in E, we have

$$|F(z)\,L(z)| \leqslant |y|^{-1}\,|F(z)| \int |G(t) \, d\mu(t)| + |y|^{-1}\,|G(z)|$$

if $\int |F(t) \, d\mu(t)| \leqslant 1$. By the arbitrariness of $F(z)$,

$$|L(z)| \leqslant |y|^{-1} \int |G(t) \, d\mu(t)| + |y|^{-1}\,|G(z)/M(z)|.$$

Since $G(z)/M(z)$ is bounded, $L(z)$ vanishes identically by an argument earlier in the proof. Since $h(x)$ is arbitrary, $[G(z) - G(w)]/(z-w)$ does belong to $\mathfrak{M}$.

If μ is actually a non-zero extreme point of $U(E)$, we obviously have $\int |d\mu(t)| \leqslant 1$. Every element $f(x)$ of $L^1(|\mu|)$ such that $\int f(t)\, d\mu(t) = 0$ may be identified with a unique element $F(z)$ of $\mathfrak{M}$. Since $\int d\mu(t) = 0$, the support of μ contains more than one point. If (a, b) is any finite interval which contains at least two points of the support of μ, there is certainly an element $f(x)$ of $L^1(|\mu|)$ such that $\int |f(t)\, d\mu(t)| \leqslant 1$ and $\int f(t)\, d\mu(t) = 0$, and $f(x) = 0$ a.e. with respect to $|\mu|$ outside of (a, b). Since $f(x)$ is equivalent to an entire function $F(z)$ in $\mathfrak{M}$ and since

$$\int_{-\infty}^{a} |F(t)\, d\mu(t)| = 0 = \int_{b}^{\infty} |F(t)\, d\mu(t)|,$$

the support of μ is contained in the union of (a, b) with a discrete set. By the arbitrariness of a and b, the support of μ has no finite limit points.

If t_0 and t_1 are distinct points in the support of μ, consider the function $f(x)$ which vanishes at all other points of the support of μ and has

$$f(t_0)\, \mu(\{t_0\}) = t_0 - t_1,$$
$$f(t_1)\, \mu(\{t_1\}) = t_1 - t_0.$$

Since $\int f(t)\, d\mu(t) = 0$, $f(x) = F(x)$ a.e. with respect to μ, where $F(z)$ is in $\mathfrak{M}$. Therefore, $S(z) = (z_0 - t_0)(z - t_1)F(z)$ is an entire function which vanishes on the support of μ. Let w be any other zero of $S(z)$. Since $(z-t_0)^{-1}(z-t_1)^{-1}S(z)$ belongs to $\mathfrak{M}$ and since this space contains difference quotients,

$$(z - t_0)^{-1}\, (z - t_1)^{-1}\, (z - w)^{-1} S(z)$$

belongs to $\mathfrak{M}$. It follows that

$$(z - t_0)^{-1}\, (z - w)^{-1}\, S(z) = (z - t_0)^{-1}\, (z - t_1)^{-1} S(z) + (w - t_1)(z - t_0)^{-1}(z - t_1)^{-1}(z - w)^{-1} S(z)$$

belongs to $\mathfrak{M}$. This function vanishes at all points of the support of μ except for t_0 and w if w is in the support of μ. Since this function must have mean zero with respect to μ and since it has a non-zero value at t_0, $w = t_n$ is a real number in the support of μ and

$$\begin{aligned} 0 &= \int (t - t_0)^{-1}(t - t_n)^{-1}\, S(t)\, d\mu(t) \\ &= S'(t_n)\, (t_n - t_0)^{-1}\, \mu(\{t_n\}) - (t_n - t_0)^{-1}. \end{aligned}$$

It follows that $S'(t_n) \neq 0$ and that

$$\mu(\{t_n\}) = S'(t_n)^{-1}.$$

The entire function

$$g(z) = S(z) \int (z - t)^{-1}\, d\mu(t)$$

has value 1 at the zeros of $S(z)$. If

$$|(z - t_0)^{-1}\, (z - t_1)^{-1}\, S(x)| \leqslant kM(z)$$

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for all complex z, then

$$|g(z)| \leqslant k\,|(z-t_0)(z-t_1)|\,M(z)\,|\int (z-t)^{-1}\,d\mu(t)| \leqslant k\,|y|^{-1}\,|(z-t_0)(z-t_1)|.$$

An argument above will show that $g(z)$ is a linear function, which is identically 1 by its value at the zeros of $S(z)$. Since

$$\int F(t)\;(z-t)^{-1}\,d\mu(t) = F(z)\int (z-t)^{-1}\,d\mu(t)$$

for every $F(z)$ in $\mathfrak{M}$ when $y \neq 0$,

$$\sum_{S(t)\neq 0} F(t)\;S'(t)^{-1}\,(z-t)^{-1} = F(z)\;S(z)^{-1} \tag{33}$$

whenever $S(z) \neq 0$. When $F(z) = 1$ an argument similar to the proof of Theorem III will show that

$$\log|S(x+iy)|^{-1} \leqslant \pi^{-1}\,|y|\int |t-z|^{-2}\log|S(t)|^{-1}\,dt$$

with absolute convergence of the integral for $y > 0$. By Boas (**2**, p. 92), there is a number $a \geqslant 0$ such that

$$\log|S(x+iy)| = a|y| + \pi^{-1}\,|y|\int|t-z|^{-2}\log|S(t)|\,dt$$

for $y \neq 0$, with absolute convergence of the integral. In particular, $S(z)$ satisfies (6), and it has exponential type by the Krein estimate (cf. Boas, **2**, p. 132). Since $S(z)\;(z-t_1)^{-1}\;(z-t_2)^{-1}$ is in $\mathfrak{M}$ and since (31) and (32) hold, $S(z)$ has type at most c. By the Lebesgue dominated convergence theorem in (33),

$$F(iy) = o(S(iy))$$

as $|y| \to \infty$ for every $F(z)$ in $\mathfrak{M}$, and hence every $F(z)$ in E. By the arbitrariness of $F(z)$, $S(z)$ is not of smaller type than c. Since we may choose $F(z) = 1$ in (33), $S(z)$ is real for real z. We have seen that $S(z)$ has real simple zeros and that (8) holds with a sum of 1.

The necessity is now clear since $\mathfrak{M}(E)$ must contain a non-zero element. Although the argument is essentially the same as that of (**7**), we have taken care to give the proof in detail because our previous argument contains gaps which are not easily filled. The sufficiency is quite clear from previous work and need not be reproduced so carefully. By (**7**, Lemma 2) Formula (33) must hold for every $F(z)$ in E. If μ is the measure defined in the statement of the theorem, we have $\int F(t)\;d\mu(t) = 0$ for every $F(z)$ in E by (**7**, Lemma 1) and hence $\int e^{ixt}\,d\mu(t)$ vanishes in $[-c, c]$.

Proof of Theorem IX. If (B) holds and if a is given, choose b so that $a < b < c$. By (**13**, Theorem VIII), $e^{ihz}\,F(z)$ belongs to $\mathfrak{H}(E_{b-a})$ as a function of z when $-a \leqslant h \leqslant a$. Because of (**13**, Theorem X), the orthogonal complement of $\mathfrak{H}(E_{b-a})$ in $\mathfrak{H}(E)$ has dimension greater than 1. Therefore, there is an element $G(z)$ of $\mathfrak{H}(E)$ which is orthogonal to $\mathfrak{H}(E_{b-a})$ and does not vanish identically. By **(13**, Theorem V**)**, we may choose $G(z)$ in the closure of the domain of multiplication by z in $\mathfrak{H}(E)$. By the formula of **(8)**, which is justified by **(10**, Theorem VB**)**,

$$\pi \sum_{B(t)=0} F(t)\, e^{iht}\, \bar{G}(t)\, B'(t)^{-1} A(t)^{-1} = \langle F(t)\, e^{iht}, G(t)\rangle = 0$$

for $-a \leqslant h \leqslant a$, with absolute convergence of the integral by the Schwarz inequality. The inner product is of course taken in $\mathfrak{H}(E)$. Let μ be the measure supported in the zeros of $B(z)$ with mass

$$[F(t)\, \bar{G}(t)]/[B'(t)\, A(t)]$$

at each such zero t. Then μ has finite total variation, is supported at points t where $\phi(t) \equiv 0 \pmod{\pi}$, and $\int e^{ixt}\, d\mu(t)$ vanishes in $[-a, a]$. We must show that this Fourier transform does not vanish identically.

Let us argue by contradiction, supposing that μ were identically zero. Then, every zero of $B(z)$ is a zero of $F(z)$ or $G(z)$. But $G(z)$ cannot vanish at all the zeros of $B(z)$ since

$$||G||^2 = \pi \sum_{B(t)=0} |G(t)|^2/[B'(t)\, A(t)]$$

and $G(z)$ cannot vanish identically by construction. Therefore, $B(z)$ has a zero w which is a zero of $F(z)$ but not of $G(z)$. Since $E_b(z)$ has no real zeros by hypothesis, $F(z)/(z-w)$ is in $\mathfrak{H}(E_b)$, and so is orthogonal to $G(z)$. Since all the zeros of $B(z)$ are zeros of $G^*(z)\, F(z)/(z-w)$, except possibly for the zero at w, w must be a zero of $F(z)/(z-w)$. Continuing inductively, we may show that $F(z)/(z-w)^n$ belongs to $\mathfrak{H}(E_b)$ for every $n = 0, 1, 2, \ldots,$ and vanishes at w. By analyticity, $F(z)$ vanishes identically contrary to construction. We must therefore grant that μ does not vanish identically. This completes the proof that (B) implies (A).

Conversely, suppose that (A) holds. We shall show that (B) must hold, at least in the case that $\phi(x)$ satisfies (9). If b is given, choose $a = b$. By hypothesis, there is a measure μ of finite total variation, supported in the zeros of $B(z)$, whose Fourier transform vanishes in $[-a, a]$ and does not vanish identically. By Theorem VIII, there is an entire function $S(z)$ of exponential type a, and not of smaller type, which is real for real z, has only real simple zeros contained in the zeros of $B(z)$, and satisfies (6) and (8). By the arbitrariness of $b < c$, a similar function could also be constructed of slightly larger type, from which it follows that the zeros of $S(z)$ are not all the zeros of $B(z)$. Therefore, $G(z) = B(z)/S(z)$ is an entire function which is real for real z, has only real simple zeros, and has at least one zero. The values of $G(z)$ at a zero t of $B(z)$ are either 0 if $S(t) \neq 0$, or $B'(t)/S'(t)$ if $S(t) = 0$. Since (9) holds by hypothesis and since $\phi'(t) = B'(t)/A(t)$ when $B(t) = 0$,

$$\sum_{B(t)=0} (1+t^2)^{-1} B'(t)/A(t) < \infty,$$

and

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$$\sum_{B(t)=0} (1 + t^2)^{-1} |G(t)|^2/[B'(t)\, A(t)] < \infty. \tag{34}$$

Let $F(z) = G(z)/(z - w_0)$, where w_0 is the choice of a zero of $G(z)$. Since $S(z)$ has exponential type a and satisfies (6),

$$\log|F(z)/B(z)| = -a\,|y| + \pi^{-1}\,|y| \int |t - z|^{-2} \log|F(t)/B(t)|\, dt \tag{35}$$

with absolute convergence when $y \neq 0$. Because of (34),

$$\sum_{B(t)=0} |F(t)|^2/[B'(t)\, A(t)] < \infty.$$

Under these conditions, the proof of (**11**, Lemma 11) can be applied, and $F(z)$ belongs to $\mathfrak{H}(E)$. Let $\mathfrak{M}(a)$ be the closed span in $\mathfrak{H}(E)$ of the functions $F(z)/(z - w)$, where w ranges in the zeros of $F(z)$. Since $\mathfrak{M}(a)$ satisfies (H1), (H2), and (H3) in the metric of $\mathfrak{H}(E)$, it is equal isometrically to $\mathfrak{H}(E_a)$, where $E_a(z)$ is an entire function which satisfies (1). Since $L(z)/(z - w)$ belongs to $\mathfrak{M}(a)$ whenever $L(z)$ belongs to $\mathfrak{M}(a)$ and $L(w) = 0$, $E_a(z)$ has no real zeros. Since (35) holds when $F(z)$ is replaced by any non-zero element of $\mathfrak{M}(a)$, $E_a(z)$ satisfies (10) with $b = a$. This completes the proof that (A) implies (B) when (9) holds.

Proof of Theorem X. Since (7) holds, $\psi(x)$ is unbounded above and below, and this continuous function must take the value zero. By a translation, we may suppose that $\psi(0) = 0$ to agree with previous normalizations. Since we assume that $\psi(x)$ is uniformly continuous, we may choose $E(z)$, as in Theorem V, so that $\phi'(x)$ is bounded. Since (7) holds by hypothesis, we may take $E(z)$ of exponential type τ so as to satisfy (6). If b is a given number with $0 < b < \tau$, then by Theorem VII, there is an entire function, which we shall now call $E_b(z)$, which is of exponential type $\tau - b$, satisfies (1), and has no real zeros, such that $\mathfrak{H}(E_b)$ is contained isometrically in $\mathfrak{H}(E)$. Since $E(z)$ satisfies (6), so does $E_b(z)$, and by Boas (**2**, p. 116),

$$\lim y^{-1} \log|E(iy)| = \tau,$$

$$\lim y^{-1} \log|E_b(iy) = \tau - b$$

as $y \to -\infty$. Therefore, the condition (B) of Theorem IX is satisfied by $E(z)$ with $c = \tau$. The theorem now follows from condition (A) of Theorem IX.

Proof of Theorem XI. An entire function of minimal exponential type determines a local operator on Fourier transforms. By the lemmas of (**4**) and (**5**), we must have

$$\int [K^n(t) - K^n(z)]/(t - z)\, d\mu(t) = 0$$

for every $n = 1, 2, 3, \ldots$, since $K^n(z)$ has minimal type and $\int e^{ixt}\, d\mu(t)$ is a function in the domain of $K^n(H)$ which vanishes in an interval $[-a, a]$. When z is not real,

$$K^n(z) \int (z - t)^{-1}\, d\mu(t) = \int K^n(t)\ (z - t)^{-1}\, d\mu(t)$$

and so
$$|K^n(z) \int (z - t)^{-1}\, d\mu(t)| \leqslant M^n\, |y|^{-1} \int |d\mu(t)|,$$

where M is a bound for $K(z)$ on the support of μ. By the arbitrariness of n, $|K(z)| \leqslant M$ whenever z is not real and $\int (z - t)^{-1}\, d\mu(t) \neq 0$. Since the Fourier transform of μ does not vanish identically by hypothesis, $\int (z - t)^{-1}\, d\mu(t)$ does not vanish identically in either the half-plane $y > 0$, or the half-plane $y < 0$. Since the Fourier transform of $\bar{\mu}$ also vanishes in $[-a, a]$, $|K(z)| \leqslant M$ in both half-planes, and on the real axis by continuity. This entire function is a constant by Liouville's theorem.

Proof of Theorem XII. Our proof will suppose that $b_n - a_n \geqslant 4$ for every n. This situation can always be obtained by omitting any of the given intervals which are smaller. These omissions cannot affect the divergence of (12) since (11) is assumed to hold. In fact, if we let $c_n = a_n + 1$, $d_n = b_n - 1$, the sequence of intervals (c_n, d_n) satisfies (12). Let μ be the absolutely continuous measure which has density 0 in the union of the intervals (c_n, d_n), and which has density 1 otherwise. It is clear that

$$\int (1 + t^2)^{-1}\, d\mu(t) < \infty$$

and that this non-negative measure does not vanish identically. Because of (**11**, Theorem XII), there is an entire function $E(z)$ of exponential type, which satisfies (1) and (6) and has no real zeros, such that $\mathfrak{H}(E)$ is contained isometrically in $L^2(\mu)$. Let us now show that $E(z)$ cannot have positive type τ.

If this were the case, and if b were a given number $0 < b < \tau$, then by the proof of Theorem IX, there would exist entire functions $F(z)$ and $G(z)$ in $\mathfrak{H}(E)$, which do not vanish identically, such that $e^{ihz} F(z)$ is orthogonal to $G(z)$ for $-a \leqslant h \leqslant a$. Since $\mathfrak{H}(E)$ is contained isometrically in $L^2(\mu)$, it follows that

$$\int e^{ixt}\, F(t)\, \bar{G}(t)\, d\mu(t) = 0$$

for $-a \leqslant x \leqslant a$. This cannot be, because of a theorem of Levinson (cf. **5**). Let $K(x)$ be the continuous function of real x defined by

$$\log K(x) = \min(x - c_n, d_n - x)$$

in each interval $[c_n, d_n]$, and by $\log K(x) = 0$ otherwise. Then, $\log K(x)$ is uniformly continuous, and a quick calculation from (12) will show that

$$\int (1 + t^2)^{-1} \log K(t)\, dt = \infty.$$

Since
$$\int K(t)\, |F(t)\, G(t)\, d\mu(t)| < \infty,$$

$F(z)\, G^*(z)$ must vanish a.e. on the support of μ by (**5**, Theorem V). By analyticity, $F(z)\, G^*(z)$ vanishes identically, contrary to construction. Therefore, we must grant that $E(z)$ has minimal exponential type.

If $g(z)$ is the choice of an element of $\mathfrak{H}(E)$ which does not vanish identically, it has minimal exponential type and

$$f(z) = \int_{-1}^{1} g(z - t)\, g^*(z - t)\, dt$$

is an entire function of minimal exponential type which remains bounded on the real complement of $\cup(a_n, b_n)$ and is strictly positive on the real axis. Since $\lim f(x) = 0$ as $x \to -\infty$, this function is not a constant.

ADDED IN PROOF: Theorem VII was conjectured as a result of work with non-self-adjoint transformations, applied to $-id/dx$ in $\mathfrak{H}(E)$. Although a proof has yet to be obtained by these methods, much progress has recently been made with the theory (see *Some Hilbert spaces of analytic functions I*, Trans. Amer. Math. Soc. ,*106* (1963), 445–468). An announcement of results from Part II will appear in the *Bulletin of the American Mathematical Society*. The present proof of Theorem VII depends on an equivalent theorme of A. Beurling and P. Malliavin announced at the Stanford Conference on Functional Analysis, August 1961, and later published in **(1)**. I wish to thank Professor Malliavin for discussing his work with me. It seems to be closely related to mine in spite of a great difference in method.

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