INVARIANT MEASURES ON COSET SPACES

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1. Notation and statement of results. In this note we consider measures on a left coset space G/H, where G is a locally compact group and H is a closed subgroup. We assume the natural topology in G/H and we denote the generic element of this space by xH $(x \in G)$. Every element $t \in G$ defines a homeomorphism of G/H given by $t(xH) = (tx)H$. A. Weil showed that a Baire measure on G/H invariant under all these homeomorphisms can exist only if

$$\Delta(\xi) = \delta(\xi) \quad \text{for each} \quad \xi \in H, \tag{1}$$

where $\Delta(x)$, $\delta(\xi)$ denote the modular functions in G, H [**6**, pp. 42–45]. We shall devote our investigations to *inherited* measures on G/H (cf. [**3**] and the definition below) invariant under homeomorphisms belonging to a normal and closed subgroup $T \subset G$.

In the sequel we shall use Baire measures only, and it is part of our definition that such measures are finite on compact sets and positive on open sets. We denote by $L(X)$ the class of continuous functions with compact supports (i.e. vanishing outside compact sets) defined on a topological space X. We say that h is an LB-function on X (locally Baire) if fh is a Baire function whenever $f \in L(X)$. We use dx, $d\xi$ to indicate integration with respect to the left Haar measures in G, H.

As in [**3**], we call a measure μ on G/H *inherited* if there is an LB-function h on G such that, for every $f \in L(G)$,

$$\int_G f(x)h(x)\,dx = \int_{G/H} d\mu(xH) \int_H f(x\xi)\,d\xi. \tag{2}$$

(We note that $\int_H f(x\xi)\,d\xi$ is constant on the cosets xH and thus it may be regarded as a member of $L(G/H)$ [**2**, § 33A].) A function h satisfying the above condition will be called a *μ-factor function.*

Let T be a subgroup of G. We say that a measure μ on G/H is *T-invariant* if, for every $t \in T$, $\mu(E) = \mu(tE)$, where $tE = \{txH : xH \in E\}$. In the sequel we assume that T is a closed normal subgroup of G. We denote by D the closure of the group $TH = HT \subset G$.

THEOREM 1. *There exists a T-invariant inherited measure on the coset space G/H if and only if either of the following conditions holds:*

C_1. *The function $\delta(\xi)/\Delta(\xi)$ (which is continuous and multiplicative on H) admits a continuous and multiplicative extension q to the group D such that q is constant on the T-cosets.*

C_2. *For every compact set $C \subset G$, the function $\delta(\xi)/\Delta(\xi)$ is bounded on $H \cap CT$.*

We observe that if $T = G$, then each of the conditions C_1, C_2 is equivalent to (1).

To state our next theorem briefly, we adopt the following convention. If M is a group and S is a subgroup, we shall identify a function defined on M and constant on the left S-cosets with the corresponding function on the left coset space M/S. It is known that if M is locally compact and S is closed, then LB-functions correspond in this way to LB-functions [**3**, proof of Theorem 6]. In particular, every LB-function on the left coset space G/D is also an LB-function on G/H. Given a measure μ on G/H and a function on this space, we shall call the function locally μ-integrable if its μ-integrals over compact sets are finite. We shall say that μ is stable if $\mu(E) = 0$ implies that $\mu(tE) = 0$, for every $t \in G$.

THEOREM 2. *Given a T-invariant stable measure μ on G/H, all other such measures μ^* are given by the formula*

$$\mu^*(E) = \int_E r(xH)\, d\mu\,(xH), \tag{3}$$

where r runs over the class of all positive LB-*functions on the left coset-space G/D which are locally μ-integrable on G/H.*

Remarks. It is known that a measure on G/H is stable if and only if it is an inherited measure that has a positive factor function [**3**, Theorem 2]. Any two such measures are equivalent, i.e. each is absolutely continuous with respect to the other. (The proof is the same as for Theorem 1 in [**4**].)

Assuming either of the conditions C_1, C_2, we can construct a T-invariant inherited measure μ that has a continuous and positive factor function (cf. § 4, (III) below). Such a measure is pseudo-invariant in the sense defined in [**4**]; in particular it is stable.

2. Properties of factor functions. Let h be a μ-factor function, where μ is an inherited measure on G/H. Since μ is a Baire measure, we have, by (2),

(*a*) $\displaystyle\int_U h(x)\, dx > 0,$ *for every open set* $U \subset G,\ U \neq \emptyset,$

(*b*) $\displaystyle\int_C h(x)\, dx < \infty,$ *for every compact set* $C \subset G.$

Replacing in (2) the function $f(x)$ by $g(x) = f(x\xi_0)$, where ξ_0 is an arbitrary fixed element of H, and comparing the resulting equation with (2), we deduce without difficulty that

(*c*) *for every* $\xi \in H$, $h(x\xi) = h(x)\,\delta(\xi)/\Delta(\xi)$ *holds for almost all* $x \in G$ (*in the Haar measure*).

Conversely, it follows from Theorem 2 in [**4**] that an LB-function h which satisfies (*a*), (*b*) and (*c*) is the factor function for an inherited measure.

It is obvious that if μ is T-invariant, then, for every $t \in T$, $f \in L(G)$,

$$\int_{G/H} d\mu(xH) \int_H f(tx\xi)\, d\xi = \int_{G/H} d\mu(xH) \int_H f(x\xi)\, d\xi.$$

Conversely, the above condition is sufficient for the T-invariance of μ, because every function in $L(G/H)$ is of the form $\displaystyle\int_H f(x\xi)\, d\xi$. where $f \in L(G)$ [**2**, § 33B]. It follows, by (2), that an inherited measure is T-invariant if and only if its factor function h satisfies

(*d*) *for every* $t \in T$, $h(tx) = h(x)$ *for almost all* $x \in G$.

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From these remarks we conclude that Theorems 1 and 2 are equivalent to theorems about the existence of LB-functions h on G which satisfy the conditions (a), (b), (c) and (d).

3. Two lemmas. Let P be a closed subgroup of G and let $s(p)$ be a continuous and multiplicative function on P. In our proofs we shall need the following two lemmas.

LEMMA 1. *If h_1 is an* LB-*function on G, such that for every $p \in P$, we have $h_1(xp) = h_1(x)s(p)$ for almost all $x \in G$, then there exists an* LB-*function h_0 such that $h_0 = h_1$ almost everywhere, and $h_0(xp) = h_0(x)s(p)$ holds identically for $x \in G$, $p \in P$. Moreover, if h_1 is everywhere positive, then a positive function h_0 with the above property can be found* [**3**, Theorem 1].

LEMMA 2. *There exists a positive and continuous function h on G such that $h(xp) = h(x)s(p)$ holds identically for $x \in G$, $p \in P$* [**4**, last remark].

The proofs given in [**3**] and [**4**] refer to the case when $P = H$ and $s = \delta/\Delta$ but it is easily seen that this restriction is inessential.

4. Proof of Theorem 1. (I) *The necessity of* C_2. We assume that μ is a T-invariant inherited measure and we show that C_2 follows. Let h be a μ-factor function; hence h satisfies (a), (b), (c) and (d). Applying Lemma 1 with $P = T$, $s = 1$ and $h_1(x) = h(x^{-1})$, we obtain, by (d), that there is a μ-factor function h_0 such that $h_0(tx) = h_0(x)$ holds identically for $t \in T$, $x \in G$. This means that h_0 is constant on the cosets Tx; but since $Tx = xT$, we have

(d') $h_0(xt) = h_0(x)$ *for every* $x \in G$, $t \in T$.

We define now a measure ν on G by the formula

$$\nu(E) = \int_E h_0(x)\, d^*x, \tag{4}$$

where d^*x indicates integration with respect to the right Haar measure on G. By (a) and (b), ν is a Baire measure. From (c) and (d'), we have

(i) $\nu(E\xi) = \nu(E)\,\delta(\xi)/\Delta(\xi)$, *for every* $\xi \in H$,
(ii) $\nu(Et) = \nu(E)$, *for every* $t \in T$.

Let, in particular, E be a compact set such that $\nu(E) > 0$. Then, for every compact set $C \subset G$ and $t \in T$, we have $\nu(ECt) = \nu(EC) < \infty$, by (ii). Using (i), we see that if $\xi \in CT \cap H$, then $\delta(\xi)/\Delta(\xi) \leqq \nu(EC)/\nu(E) < \infty$; i.e. we have C_2.

(II) *The implication* $C_2 \Rightarrow C_1$. We assume C_2. Taking $C = \{e\}$, we have that $\delta(\xi)/\Delta(\xi)$ is bounded on the group $H \cap T$, and, by the multiplicative property of this function, we must have $\delta(\xi)/\Delta(\xi) = 1$ on $H \cap T$. This shows that the function δ/Δ admits a multiplicative extension q_1 to the group HT such that q_1 is constant on the T-cosets.

Let V be a symmetric neighbourhood of unity in G which has a compact closure $\bar{V} = C$. We prove first that q_1 is bounded on $C \cap HT$. Indeed, the values taken by q_1 on this set are the same as those taken on $CT \cap H$, and on the latter set q_1 is equal to δ/Δ, where, by C_2, this function is bounded. From the multiplicative property of q_1 and since V is symmetric, it follows that the values taken by q_1 on $C \cap HT$ are contained between two positive constants. Consequently, there is a finite number m such that $|\log q_1| < m$ holds on $C \cap HT$.

Taking a neighbourhood U of unity such that $U^n \subset C$, we find, by the additive property of $\log q_1$, that $|\log q_1| < m/n$ holds on U. Thus, since $\log q_1(e) = 0$, we have that $\log q_1$ is continuous at e with respect to the topology induced by G. Since this is an additive function, it is uniformly continuous on HT and hence it admits a continuous extension l to the closure D of HT. It is clear that l is also constant on T-cosets and additive. It suffices now to define $q = \exp l$.

(III) *The sufficiency of* C_1. We assume C_1 and show that there exists a T-invariant inherited measure on G/H. We consider the group G/T, its closed subgroup D/T and the continuous multiplicative function q defined on D/T (cf. our convention preceding Theorem 2, § 1). Applying Lemma 2, we obtain that there is a positive continuous function $h(\kappa)$ on G/T such that $h(\kappa p) = h(\kappa)q(p)$ holds identically for $\kappa \in G/T$, $p \in D/T$. Considering h as a function on G we have that h is constant on T-cosets and that identically $h(x\xi) = h(x)\,\delta(\xi)/\Delta(\xi)$ holds for $x \in G$, $\xi \in H$. Therefore conditions (*a*), (*b*), (*c*) and (*d*) are satisfied and this proves that h is the factor function for a certain inherited T-invariant measure on G/H.

5. Further lemmas. To prove Theorem 2 we need the result stated in Lemma 4 below. Lemma 3 finds application in the proof of Lemma 4.

LEMMA 3. *If ν is a measure on G which is equivalent to the Haar measure, then there exists an open basis $\mathfrak{B}$ in G (i.e. a family $\mathfrak{B}$ of open sets such that every open set in G is a union of members of $\mathfrak{B}$) with the property that, for every $U \in \mathfrak{B}$, the function $g(x) = \nu(Ux)$ is a continuous function of x.*

Proof. Let $\mathfrak{B}$ consist of all open sets U which have compact closures $\overline{U}$ satisfying $\nu(\overline{U}) = \nu(U)$. This is a basis by Lemma 2.1 in [**5**]. Since $\mathfrak{B}$ is invariant under the right translations by elements of G, it is enough to prove the continuity of every function $g(x)$ at $x = e$. Let $U \in \mathfrak{B}$ and let $\varepsilon > 0$ be arbitrary. Since ν is equivalent to the Haar measure, it is regular (cf. [**1**]); hence there is an open set Q such that $\overline{U} \subset Q$ and $\nu(Q - \overline{U}) < \varepsilon$. From the assumption that $\overline{U}$ is compact, there is a neighbourhood V of e such that $\overline{U}V \subset Q$. It follows that $g(x) - g(e) < \varepsilon$ for $x \in V$. Using again the regularity of ν, we deduce that there is a compact set $C \subset U$ such that $\nu(U - C) < \varepsilon$. Again, there is a neighbourhood V_0 of e such that $CV_0 \subset U$; hence $(G - U)V_0^{-1}$ is disjoint to C. Therefore, if $x \in V_0^{-1}$, $(G - U)x \cap C = \emptyset$, and consequently $C \subset Ux$. This proves that $g(x) \geqq \nu(C) > g(e) - \varepsilon$. We have $|g(x) - g(e)| < \varepsilon$ on $V \cap V_\epsilon^{-1}$ and this completes our proof.

LEMMA 4. *If μ is a T-invariant stable measure on G/H, then μ is inherited and there exists a μ-factor function $h > 0$ such that (in the notation of condition C_1)*

$$h(x\gamma) = h(x)q(\gamma) \text{ holds identically for } x \in G,\ \gamma \in D.$$

Proof. We assume that μ is T-invariant and stable. It is known [**3**, Theorem 2] that μ must be an inherited measure having a positive factor function. Let $h_0 > 0$ be a μ-factor

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function which satisfies (d') (cf. § 4, (I)). We denote by ν the measure defined by formula (4). It is clear, by (i) and (ii), that

(e) $\nu(E\gamma) = q_1(\gamma)\nu(E)$ *holds for each* $\gamma \in HT$,

where q_1 is the multiplicative extension of δ/Δ defined in § 4, (II). By C_1, we have that q_1 admits a continuous extension q to the closure $D = \overline{HT}$. Let $\mathfrak{B}$ satisfy the assertion of Lemma 3. If $U \in \mathfrak{B}$, then the function $\nu(Ux)$ is continuous and hence, by (e)

(e′) $\nu(U\gamma) = q(\gamma)\nu(U)$ *holds for each* $\gamma \in D$.

Since $\mathfrak{B}$ is a basis, it is clear that (e′) holds if U is replaced by an arbitrary Baire set E. This implies, by (2), that for every $\gamma \in D$, $h_0(x\gamma) = h_0(x)q(\gamma)$ holds for almost all $x \in G$. Hence, by Lemma 1, there is a μ-factor function $h > 0$ which satisfies this equality identically for $x \in G$, $\gamma \in D$. This we wished to show.

6. Proof of Theorem 2. Assume μ, μ^* both inherited, T-invariant and stable. Let h, h^* be the factor functions which satisfy the assertion of Lemma 4. We define $r = h^*/h$. This function is positive and constant on every coset xD, hence it is an LB-function on the spaces G/D, G/H and G.

If $f \in L(G)$ is positive and F is a continuous function on G/H, then, by (2),

$$\int_G F(x)f(x)h(x)\,dx = \int_{G/H} F(xH)\,d\mu(xH)\int_H f(x\xi)\,d\xi. \tag{5}$$

We consider (5) with fixed f and h but varying F. Clearly, if $\{F_i\}$ is a monotone sequence of non-negative Baire functions and (5) holds for each F_i, then it holds also for $\lim F_i$. It follows that (5) is true for any non-negative LB-function in place of F, e.g. for r. Then, since $r = h^*/h$,

$$\int_G r(x)f(x)h(x)\,dx = \int_{G/H} r(xH)\,d\mu(xH)\int_H f(x\xi)\,d\xi$$

$$= \int_G f(x)h^*(x)\,dx = \int_{G/H} d\mu^*(xH)\int_H f(x\xi)\,d\xi. \tag{6}$$

Since $f \in L(G)$ is arbitrary positive, the above equalities mean that μ^* satisfies (3). By assumption, μ^* is finite on compact sets; hence, by (3), the function r is μ-integrable.

Now suppose that a T-invariant and stable measure μ on G/H is given and that μ^* is defined by (3), where r is a positive LB-function on G/D which is locally μ-integrable. Then it is clear that μ^* is a stable measure on G/H. By the remark following Theorem 2 (§ 1) both these measures are inherited. It is easy to see (cf. (6)) that if h is a μ-factor function, then $h^* = rh$ is a μ^*-factor function. The function h^* satisfies (d), because h satisfies this condition by assumption, and r is constant on T-cosets. Therefore μ^* is T-invariant. This completes our proof.

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