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A NON-REFLEXIVE SMOOTH SPACE WITH A SMOOTH DUAL

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1. **Introduction.** Let (E, ρ) and (E^*, ρ^*) be a real Banach space and its dual. Restrepo has shown in [4] that, if ρ and ρ^* are both Fréchet differentiable, E is reflexive. The purpose of this note is to show that Fréchet differentiability cannot be replaced by Gateaux differentiability. This answers negatively a question raised by Wulbert [5]. In particular, we will renorm a certain nonreflexive space with a smooth norm whose dual is also smooth.

The author thanks W. J. Davis for suggesting the following approach. For definitions of the notions referred to in this paper, see Day's book [2].

2. **A renorming theorem.** The following theorem is Asplund's renorming theorem [1, Theorem 2] modified so the averaging is done on the dual space.

THEOREM. *If E has equivalent norms α and β such that α^* is rotund and β^{**} is rotund, then E can be renormed with an equivalent norm γ such that γ^* is rotund and γ^{**} is rotund.*

Proof. By applying Asplund's theorem, we can renorm E^* with an equivalent rotund norm σ such that σ^* is a rotund norm on E^{**}. It remains to be shown that σ is a dual norm. This will be done by examining Asplund's averaging process and observing that the resulting norm σ is w^* lower semicontinuous.

For $x^* \in E^*$, let $f_0(x^*)=(1/2)(\alpha^*(x^*))^2$ and $g_0(x^*)=(1/2)(\beta^*(x^*))^2$. Since α^* and β^* are dual norms, f_0 and g_0 are w^* lower semicontinuous, as is $f_1=\frac{1}{2}(f_0+g_0)$.

Let $g_1(x^*)=\inf\{(1/2)(f_0(x^*+y^*)+g_0(x^*-y^*)):y^* \in E^*\}$. Suppose $\{x_n^*\}$ is w^*-convergent to x^*, $g_1(x_n^*)\leq c$ and $g_1(x^*)>c$, for some constant c. Let $\varepsilon>0$. Then, there is an m_0 such that for $n\geq m_0$

$$\frac{1}{2}(f_0(x_n^*+y^*)+g_0(x_n^*-y^*)) \geq \frac{1}{2}(f_0(x^*+y^*)+g_0(x^*-y^*))+\varepsilon \geq g_1(x^*)+\varepsilon. \tag{1}$$

Also there exists $y_0^* \in E^*$ such that

$$\frac{1}{2}(f_0(x_n^*+y_0^*)+g_0(x_n^*-y_0^*)) \leq g(x_{m_0}^*)+\varepsilon. \tag{2}$$

From (1) and (2) we get that $g_1(x_{m_0}^*)>c$ which is a contradiction. Hence g_1 is w^* lower semicontinuous.

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It follows inductively that the iterates

$$f_{n+1}(x^*) = \tfrac{1}{2}(f_n(x^*)+g_n(x^*))$$

and

$$g_{n+1}(x^*) = \inf\{\tfrac{1}{2}(f_n(x^*+y^*)+g_n(x^*-y^*)) : y \in E^*\} \qquad (n \geq 0)$$

are w^* lower semicontinuous.

As Asplund shows, $\{f_n\}$ and $\{g_n\}$ each converge pointwise to a function h and $g_n \leq g_{n+1} \leq h \leq f_{n+1} \leq f_n$. It is easily shown that h must also be w^* lower semicontinuous. Now $\sigma = (2h)^{1/2}$ and, consequently, is also w^* lower semicontinuous. σ is therefore a dual norm and the theorem is proven.

3. **An example.** Consider c_0, the Banach space of null sequences with the usual sup norm. $c_0^{**} = \ell_\infty$ can be renormed by

$$\sigma(u) = \sup |u_n| + \left(\sum \frac{u_n^2}{2^n}\right)^{1/2}$$

where $u = (u_n) \in \ell_\infty$.

Phelps in [3] shows that σ is an equivalent rotund norm on ℓ_∞ which is w^* lower semicontinuous. So σ is the dual of the equivalent norm σ_* on ℓ_1 given by

$$\sigma_*(y) = \sup\{\textstyle\sum u_n y_n : \sigma(u) \leq 1, u \in l_\infty\}.$$

The equivalent norm β on c_0 given by restricting σ to c_0 can be shown to satisfy $\beta^* = \sigma_*$. Consequently, we have renormed c_0 with an equivalent norm β so that β^{**} is rotund.

Since c_0 is separable, c_0 can be renormed with an equivalent rotund norm α [2]. From the renorming theorem, we obtain an equivalent norm γ on c_0 such that γ^* and γ^{**} are both rotund. Thus, (c_0, γ) and (ℓ_1, γ^*) are both smooth and in duality.

References

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