Canad. Math. Bull. Vol. **33** (3), 1990

ON SUPERSOLVABLE GROUPS AND A THEOREM OF HUPPERT

BY

N. P. MUKHERJEE AND PRABIR BHATTACHARYA

ABSTRACT. We obtain the following generalization of a well known result of Huppert. If p is the largest primer divisor of the order of a finite group G and q is any prime distinct from p, then G is supersolvable if and only if every maximal subgroup whose index is relatively prime to either p or q, has prime index.

1. **Introduction.** It is a well known result of Huppert [1, Hauptsatz 9.5, Kapitel VI] that a finite group G is supersolvable if and only if every maximal subgroup of G is of prime index in G. It would be interesting to investigate whether G is supersolvable if instead of assuming that *every* maximal subgroup of G is of prime index, one assumes this hypothesis only for a certain subclass of maximal subgroups of G. We prove:

THEOREM 1. *Let G be any group. Let p and q be two distinct primes, p being the largest prime dividing the order of G. Then G is supersolvable if and only if the following condition holds:*

($*$) *every maximal subgroup whose index is relatively prime to either p or q, has prime index.*

2. **Preliminaries.** We recall the definition of a particular analog of the *Frattini subgroup* ([2–3]). For any finite group G and any prime q, define

$$S_q(G) = \cap\{M : M <\cdot G,\ [G : M]_q = 1,\ [G : M] \text{is composite}\}$$

where $M <\cdot G$ denotes that M is a maximal subgroup of G. If G has no maximal subgroup M such that both $[G : M]_q = 1$ and $[G : M]$ is composite, then one sets $S_q(G) = G$.

The subgroup $S_q(G)$ is a characteristic subgroup of G containing the *Frattini subgroup*. Various properties of $S_q(G)$ have been investigated in [2-3]. To prove Theorem 1 we shall use the following result.

PROPOSITION 2. ([2, Theorem 8(i)]). *Let p be the largest prime dividing the order of a group G. Then $S_p(G)$ is solvable.*

Received by the editor January 30, 1989.

AMS (1980) Subject Classification: Primary 20D10, 20D25, Secondary 20D99, 20D25.

Key words. solvable, supersolvable, Frattini subgroup.

https://doi.org/10.4153/CMB-1990-051-x Published online by Cambridge University Press

3. **Proof.**

PROOF OF THEOREM 1. Assume the hypothesis (∗). Then, clearly $S_p(G) = S_q(G) = G$. By Proposition 2, it now follows that G is solvable. Therefore each maximal subgroup of G has prime power index and hence must be relatively prime to either p or q. So, condition (∗) now implies that each maximal subgroup of G has prime index. Hence G is supersolvable by Huppert's theorem.

The converse follows trivially by Huppert's theorem. □

COROLLARY 3 (also [2]). *Let G be a group and p be the largest prime dividing the order of G, q be any prime distinct from p. Then G is supersolvable if and only if*

$$G = S_p(G) = S_q(G).$$

Thus some purely set-theoretical conditions may force a group to be supersolvable.

REFERENCES

1. B. Huppert, Endliche Gruppen I. Springer Verlag, Berlin, 1967.
2. N. P. Mukherjee and P. Bhattacharya, *On the intersection of a class of maximal subgroups of a finite group*, Canadian J. Math. **39** (1987), 603–611.
3. ———, *On the intersection of a family of maximal subgroups containing the Sylow subgroups of a finite group*. Canadian J. Math. **40** (1988), 352–359.

Jawaharlal Nehru University,
New Delhi, INDIA

University of Nebraska-Lincoln,
Lincoln, NE, U.S.A.