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DIAMETER OF COMMUTING GRAPHS OF SYMPLECTIC ALGEBRAS

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Abstract

Let F be an algebraically closed field of characteristic 0 and let $\mathrm{sp}(2l, F)$ be the rank l symplectic algebra of all $2l \times 2l$ matrices $x = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$ over F, where A^t is the transpose of A and B, C are symmetric matrices of order l. The commuting graph $\Gamma(\mathrm{sp}(2l, F))$ of $\mathrm{sp}(2l, F)$ is a graph whose vertex set consists of all nonzero elements in $\mathrm{sp}(2l, F)$ and two distinct vertices x and y are adjacent if and only if $xy = yx$. We prove that the diameter of $\Gamma(\mathrm{sp}(2l, F))$ is 4 when $l > 2$.

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1. Introduction

The diameters of commuting graphs over groups, semigroups, rings and associative algebras and the isomorphisms between commuting graphs are well studied (see, for example, [1, 2]). In particular, let R be a noncommutative ring or an associative algebra and $Z(R)$ be its centre. The commuting graph of R was defined in [3] to be the graph $\Gamma(R)$ whose vertex set is $R \setminus Z(R)$, and two distinct vertices x, y are joined by an edge whenever $xy = yx$, or equivalently, the bracket product $[x, y] = xy - yx$ of x and y is 0. Denote by $M_n(R)$ the full matrix ring of all $n \times n$ matrices over a ring R. Akbari *et al.* [4] proved that if $n \geq 3$ and F is an algebraically closed field, then the diameter of $\Gamma(M_n(F))$ is always 4 and, if F is not algebraically closed, then either the commuting graph is disconnected or the diameter is between 4 and 6. They conjectured that the diameter of $\Gamma(M_n(F))$ is at most 5. When $n = 2$, [5, Remark 8] shows that the commuting graph of $M_n(F)$ is always disconnected. Miguel [12] confirmed the conjecture proposed in [4] by proving that the diameter of the commuting graph of the full matrix ring over the real numbers is at most 5. Dolžan *et al.* [8] determined the diameters of the commuting graphs of the set of all nilpotent matrices over a semiring, the group of all invertible matrices over a semiring and the

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full matrix semiring. Dolžan *et al.* [7] obtained the diameters of commuting graphs of matrices over the binary Boolean semiring, the tropical semiring and an arbitrary nonentire commutative semiring, and found a lower bound for the diameter of the commuting graph of the semigroup of matrices over an arbitrary commutative entire antinegative semiring. For any composite m, Giudici and Pope [9] proved that the diameter of $\Gamma(M_n(Z_m))$ is 3.

Let F be an algebraically closed field of characteristic 0 and let $\mathrm{sp}(2l, F)$ be the symplectic algebra of rank l consisting of all $2l \times 2l$ matrices $x = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$ over F, where A^t is the transpose of A and B, C are symmetric matrices of order l. The commuting graph $\Gamma(\mathrm{sp}(2l, F))$ of $\mathrm{sp}(2l, F)$ is a graph whose vertex set is the set of all nonzero elements in $\mathrm{sp}(2l, F)$, and two distinct vertices x and y are adjacent if and only if $xy = yx$ (or equivalently, the bracket product $[x, y] = xy - yx$ of x and y is zero). The symplectic algebra $\mathrm{sp}(2l, F)$ is important because as a Lie algebra (with respect to the bracket product $[x, y] = xy - yx$), it is one of the nine simple Lie algebras over F. To reveal the commuting relations of elements in $\mathrm{sp}(2l, F)$, we determine the diameter of the commuting graph $\Gamma(\mathrm{sp}(2l, F))$ of $\mathrm{sp}(2l, F)$.

Theorem 1.1. *Let F be an algebraically closed field of characteristic zero. If $l > 2$, then the diameter of the commuting graph $\Gamma(\mathrm{sp}(2l, F))$ of the symplectic algebra $\mathrm{sp}(2l, F)$ is 4.*

Remark 1.2. When $l = 1$, $\mathrm{sp}(2l, F)$ is the Lie algebra of type A_1 consisting of all 2×2 matrices of trace 0. By [5, Remark 8], we easily find that the commuting graph of $\mathrm{sp}(2, F)$ is disconnected. However, the diameter of $\Gamma(\mathrm{sp}(2l, F))$ with $l = 2$ seems quite different from the cases where $l > 2$. We conjecture that the diameter of $\Gamma(\mathrm{sp}(4, F))$ is 5.

2. Proof of Theorem 1.1

Let $M_{2l}(F)$ be the set of all $2l \times 2l$ matrices over F and let $e_{ij} \in M_{2l}(F)$ be the matrix with 1 at the (i, j)th position and 0 elsewhere. Put

$$\begin{aligned} E_{ij} &= e_{ij} - e_{j+l,i+l}, \quad 1 \le i, j \le l, \\ E_{p,-q} &= e_{p,q+l} + e_{q,p+l}, \ 1 \le p < q \le l, \\ E_{-r,s} &= e_{r+l,s} + e_{s+l,r}, \quad 1 \le r < s \le l, \end{aligned}$$

and put

$$\begin{aligned} E_{p,-p} &= e_{p,p+l}, \quad \text{for } 1 \le p \le l, \\ E_{-r,r} &= e_{r+l,r}, \quad \text{for } 1 \le r \le l. \end{aligned}$$

The set

$$\Sigma = \{E_{ij} : 1 \le i, j \le l\} \cup \{E_{p,-q} : 1 \le p \le q \le l\} \cup \{E_{-r,s} : 1 \le r \le s \le l\}$$

forms a basis of $\mathrm{sp}(2l, F)$ and the dimension of $\mathrm{sp}(2l, F)$ is $2l^2 + l$.

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Let

$$J = \sum_{i=1}^{2l-1} e_{i,i+1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

The lemma below follows from the proof of [4, Theorem 3].

Lemma 2.1. *If $l \geq 2$, the distance between J and J^t in $\Gamma(M_{2l}(F))$ is 4, where $\Gamma(M_{2l}(F))$ denotes the commuting graph of $M_{2l}(F)$.*

Let $x_0 = (\sum_{i=1}^{l-1} E_{i,i+1}) + E_{l,-l}$ and let $y_0 = x_0^t = (\sum_{i=1}^{l-1} E_{i+1,i}) + E_{-l,l}$.

Lemma 2.2. *If $l \geq 2$, the distance between x_0 and y_0 in $\Gamma(\mathrm{sp}(2l, F))$ is 4.*

Proof. Let

$$z = \left(\sum_{i=1}^{l} e_{ii}\right) + \left(\sum_{i=1}^{l} (-1)^{l-i} e_{l+i,2l-i+1}\right).$$

By direct calculation, one can verify that

$$z^{-1} x_0 z = J, \quad z^{-1} y_0 z = J^t.$$

Since the distance between J and J^t in $\Gamma(M_{2l}(F))$ is 4, the distance between x_0 and y_0 in $\Gamma(\mathrm{sp}(2l, F))$ is at least 4. Indeed, if $x_0 \sim u \sim v \sim y_0$ is a path in $\Gamma(\mathrm{sp}(2l, F))$, then $J \sim z^{-1}uz \sim z^{-1}vz \sim J^t$ is a path in $\Gamma(M_{2l}(F))$, in contradiction to Lemma 2.1. It is easy to verify that

$$x_0 \sim E_{1,-1} \sim E_{22} \sim E_{-1,1} \sim y_0$$

is a path of length 4 between x_0 and y_0. Consequently, $d(x_0, y_0) = 4$. □

In view of Lemma 2.2, the diameter of $\Gamma(\mathrm{sp}(2l, F))$ is at least 4 when $l \geq 2$. In what follows, we will prove that the distance between any distinct vertices x and y in $\Gamma(\mathrm{sp}(2l, F))$ is at most 4 when $l > 2$.

For $x \in \mathrm{sp}(2l, F)$, denote by $C(x)$ the centraliser of x in $\mathrm{sp}(2l, F)$. That is,

$$C(x) = \{y \in \mathrm{sp}(2l, F) : [x, y] = 0\}.$$

We investigate the dimensions of $C(E_{11}), C(E_{1,-1})$ and $C(E_{1,-2})$.

Lemma 2.3. *Let $l \geq 2$:*

(i) *the dimension of $C(E_{11})$ is $2l^2 - 3l + 2$;*
(ii) *the dimension of $C(E_{1,-1})$ is $2l^2 - l$;*
(iii) *the dimension of $C(E_{1,-2})$ is $2l^2 - 3l + 2$.*

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Proof. Write any $y \in \mathrm{sp}(2l, F)$ as a linear combination of the basis Σ of $\mathrm{sp}(2l, F)$:

$$y = \Big(\sum_{1\le i,j\le l} a_{ij} E_{ij} \Big) + \Big(\sum_{1\le p\le q\le l} b_{p,-q} E_{p,-q} \Big) + \Big(\sum_{1\le r\le s\le l} c_{-r,s} E_{-r,s} \Big)$$

with $a_{ij}, b_{p,-q}, c_{-r,s} \in F$. One easily verifies that y commutes with E_{11} if and only if $a_{1j} = a_{j1} = 0$ for $j = 2, 3, \ldots, l$ and $b_{1,-j} = c_{-j,1} = 0$ for $j = 1, 2, \ldots, l$. As a linear space, $C(E_{11})$ is spanned by a basis

$$\{E_{11}\} \cup \{E_{ij} : 2 \le i, j \le l\} \cup \{E_{p,-q} : 2 \le p \le q \le l\} \cup \{E_{-r,s} : 2 \le r \le s \le l\},$$

which altogether has $2l^2 - 3l + 2$ elements.

Similarly, y commutes with $E_{1,-1}$ if and only if the first column and the $(l+1)$th row of y are zero vectors. As a linear space, $C(E_{1,-1})$ is spanned by a basis

$$\{E_{i,j} : 1 \le i \le l, 2 \le j \le l\} \cup \{E_{p,-q} : 1 \le p \le q \le l\} \cup \{E_{-r,s} : 2 \le r \le s \le l\},$$

which altogether has $2l^2 - l$ elements.

By calculation, we find that y commutes with $E_{1,-2}$ if and only if

$$\begin{aligned} a_{11} &= -a_{22}, \\ a_{j1} &= a_{k2} = 0, \quad \text{for } j = 2, 3, \ldots, l,\ k = 1, 3, 4, \ldots, l, \\ c_{-1,j} &= c_{-2,k} = 0, \quad \text{for } j = 1, 2, \ldots, l,\ k = 2, 3, \ldots, l. \end{aligned}$$

Thus $C(E_{1,-2})$ is a space with basis

$$\{E_{11} - E_{22}\} \cup \{E_{i,j} : 1 \le i \le l, 3 \le j \le l\} \cup \{E_{p,-q} : 1 \le p \le q \le l\} \cup \{E_{-r,s} : 3 \le r \le s \le l\},$$

which altogether has $2l^2 - 3l + 2$ elements. □

The automorphism group of $\mathrm{sp}(2l, F)$ is denoted by $\mathrm{Aut}(\mathrm{sp}(2l, F))$. We now study the action of $\mathrm{Aut}(\mathrm{sp}(2l, F))$ on the basis of $\mathrm{sp}(2l, F)$. Let $\alpha \in M_{2l}(F)$ be invertible. If $\alpha^{-1} x \alpha \in \mathrm{sp}(2l, F)$ for any $x \in \mathrm{sp}(2l, F)$, then the mapping $\overline{\alpha}$ on $\mathrm{sp}(2l, F)$ defined by

$$\overline{\alpha}(x) = \alpha^{-1} x \alpha, \quad \text{for all } x \in \mathrm{sp}(2n, F),$$

is an automorphism of $\mathrm{sp}(2l, F)$ (see [6]).

Lemma 2.4.

(i) *If $1 \le i < j \le l$, there is an invertible $\alpha \in M_{2l}(F)$ such that $\overline{\alpha}(E_{ij}) = E_{1l}$.*
(ii) *There is an invertible $\beta \in M_{2l}(F)$ such that $\overline{\beta}(E_{p,-p}) = E_{1,-1}$, where $1 \le p \le l$.*
(iii) *If $1 \le p < q \le l$, there is an invertible $\gamma \in M_{2l}(F)$ such that $\overline{\gamma}(E_{p,-q}) = E_{1,-2}$.*
(iv) *There is an invertible $\theta \in M_{2l}(F)$ such that $\overline{\theta}(E_{1l}) = E_{1,-l}$.*
(v) *If $1 \le i < j \le l$, there is an invertible $\xi \in M_{2l}(F)$ such that $\overline{\xi}(E_{ij}) = E_{1,-2}$.*

Proof. For $1 \le i \ne j \le l$, let P_{ij} be the permutation matrix obtained by permuting the ith and jth rows of the identity matrix of order l, and put

$$\alpha_{ij} = \begin{pmatrix} P_{ij} & 0 \\ 0 & P_{ij} \end{pmatrix}.$$

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Since $\alpha_{ij}^{-1} x \alpha_{ij} \in \mathrm{sp}(2l, F)$ whenever $x \in \mathrm{sp}(2l, F)$, the mapping

$$\overline{\alpha_{ij}} : x \mapsto \alpha_{ij}^{-1} x \alpha_{ij}, \quad \text{for all } x \in \mathrm{sp}(2l, F),$$

is an automorphism of $\mathrm{sp}(2l, F)$.

If $1 < j < l$, then the automorphism $\overline{\alpha_{jl}}$ sends E_{1j} to E_{1l}. If $1 < i < j$, then the automorphism $\overline{\alpha_{1i}}$ sends E_{ij} to E_{1j}. If $1 < i < j < l$, then the automorphism $\overline{\alpha_{jl}} \cdot \overline{\alpha_{1i}} = \overline{\alpha_{1i} \cdot \alpha_{jl}}$ sends E_{ij} to E_{1l}, which proves (i).

If $p \neq 1$, then the automorphism $\overline{\alpha_{1p}}$ sends $E_{p,-p}$ to $E_{1,-1}$, which proves (ii).

For $3 \leq j \leq l$, the automorphism $\overline{\alpha_{2j}}$ sends $E_{1,-j}$ to $E_{1,-2}$. For $2 \leq i < j \leq l$, the automorphism $\overline{\alpha_{2j}} \cdot \overline{\alpha_{1i}} = \overline{\alpha_{1i} \cdot \alpha_{2j}}$ sends $E_{i,-j}$ to $E_{1,-2}$, which proves (iii).

Let $\theta = I_{2l} - e_{ll} - e_{2l,2l} + e_{l,2l} - e_{2l,l}$. One easily verifies that the mapping $\overline{\theta}$ defined by

$$\overline{\theta} : x \mapsto \theta^{-1} x \theta, \quad \text{for all } x \in \mathrm{sp}(2l, F),$$

stabilises $\mathrm{sp}(2l, F)$, thus is an automorphism of $\mathrm{sp}(2l, F)$. The proof of (iv) is completed by $\overline{\theta}(E_{1l}) = E_{1,-l}$.

Finally, (v) follows immediately from (i), (iii) and (iv). □

Four particular subalgebras of $\mathrm{sp}(2l, F)$ are defined as follows:

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} : A \in M_l(F) \text{ is diagonal} \right\},$$

$$V = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} : B \in M_l(F) \text{ is symmetric} \right\},$$

$$U = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} : A \in M_l(F) \text{ is strictly upper triangular} \right\},$$

$$T = \left\{ \begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix} : A \in M_l(F) \text{ is upper triangular, } B \in M_l(F) \text{ is symmetric} \right\}.$$

Then H, U, V, T are all subalgebras of $\mathrm{sp}(2l, F)$. The following assertions are well known,

- T is a Borel subalgebra (that is, a maximal solvable subalgebra) of $\mathrm{sp}(2l, F)$ (see [10] or [11]),
- The dimension of H is l and the E_{ii}, for $i = 1, 2, \ldots, l$, form a basis of H,
- The dimension of U is $\frac{1}{2}l(l-1)$ and the $E_{i,j}$, for $1 \leq i < j \leq l$, form a basis of U,
- The dimension of V is $\frac{1}{2}l(l+1)$ and the set $\{E_{p,-q} : 1 \leq p \leq q \leq l\}$ forms a basis of V.

Since $T = H \oplus U \oplus V$, any given $t \in T$ has a unique decomposition in the form

$$t = h + u + v, \quad h \in H, u \in U, v \in V,$$

where h, u, v will respectively be called the H-term, the U-term and the V-term of t. We write the V-term v of t as the linear combination of $\{E_{p,-q} : 1 \leq p \leq q \leq l\}$,

$$v = \sum_{1 \leq p \leq q} c_{p,-q} E_{p,-q},$$

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and put

$$\Delta(t) = \{(p, -q) : [h, c_{p,-q}E_{p,-q}] \neq 0\}.$$

If $\Delta(t) \neq \emptyset$, we set

$$d(t) = \max\{p + q : (p, -q) \in \Delta(t)\},$$

and call it the degree of t.

Let

$$\Psi = \left\{ \begin{pmatrix} I_l & C \\ 0 & I_l \end{pmatrix} : C^t = C \right\}.$$

If $\alpha = \left(\begin{smallmatrix} I_l & C \\ 0 & I_l \end{smallmatrix}\right) \in \Psi$, then the mapping $\overline{\alpha} : x \mapsto \alpha^{-1}x\alpha$ for $x \in \mathrm{sp}(2l, F)$ is an automorphism of $\mathrm{sp}(2l, F)$. The set $\{\overline{\alpha} : \alpha \in \Psi\}$ forms a subgroup of $\mathrm{Aut}(\mathrm{sp}(2l, F))$, which will be denoted by G. Direct calculation shows that G stabilises T. In addition, if $\alpha \in \Psi$, then $\overline{\alpha}(t)$ and $t \in T$ have the same H-term and the same U-term. Now we consider how to simplify the V-term of t by applying $\overline{\alpha} \in G$.

Lemma 2.5. *For any given $t \in T$, there exists $\alpha \in \Psi$ such that $\Delta(\overline{\alpha}(t)) = \emptyset$.*

Proof. Suppose to the contrary that $\Delta(\overline{\alpha}(t)) \neq \emptyset$ for any $\alpha \in \Psi$. Choose $\overline{\beta} \in G$ with $\beta \in \Psi$ which minimises $d(\overline{\beta}(t))$ and suppose that $d(\overline{\beta}(t)) = k$. Assume that

$$\overline{\beta}(t) = h + u + v, \quad \text{where } h \in H, u \in U, v \in V,$$

and represent h, u, v as linear combinations of the bases of H, U, V, respectively:

$$h = \sum_{i=1}^{l} a_{ii}E_{ii}, \quad u = \sum_{1 \leq i < j \leq l} b_{ij}E_{ij}, \quad v = \sum_{1 \leq p \leq q \leq l} c_{p,-q}E_{p,-q}.$$

Thus $[h, c_{p,-q}E_{p,-q}] = 0$ when $p + q > k$, and there is $(p', -q')$ such that $p' + q' = k$ and

$$[h, c_{p',-q'}E_{p',-q'}] = (a_{p',p'} + a_{q',q'})c_{p',-q'}E_{p',-q'} \neq 0.$$

Put

$$\gamma = I_{2l} - \sum_{p+q=k,\, a_{pp}+a_{qq} \neq 0} c_{p,-q}(a_{pp} + a_{qq})^{-1}E_{p,-q}.$$

Then $\gamma \in \Psi$. By calculation,

$$\overline{\gamma}(h) = \gamma^{-1}h\gamma = h - \sum_{p+q=k,\, a_{pp}+a_{qq} \neq 0} c_{p,-q}E_{p,-q},$$

$$\overline{\gamma}(v) = \gamma^{-1}v\gamma = v,$$

and

$$\overline{\gamma}(u) = \gamma^{-1}u\gamma = u + v', \quad \text{with } v' \in V_{k-1},$$

where V_{k-1} denotes the subalgebra of V spanned by $\{E_{p,-q} : p + q \leq k - 1\}$. Since

$$\overline{\gamma}(\overline{\beta}(t)) = h + u + \Big(v - \sum_{p+q=k,\, a_{pp}+a_{qq} \neq 0} c_{p,-q}E_{p,-q} + v'\Big)$$

with $v' \in V_{k-1}$, we have $d(\overline{\gamma}(\overline{\beta}(t))) \leq k - 1$, a contradiction to the assumption for $\overline{\beta}$. □

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We need a known result about the Borel subalgebras of an arbitrary Lie algebra to simplify elements in $\mathrm{sp}(2l, F)$.

Lemma 2.6 [10, Theorem 16.4]. *The Borel subalgebras of an arbitrary Lie algebra L are conjugate under $\mathcal{E}(L)$, a subgroup of the automorphism group of L.*

Lemma 2.7. *For a given $x \in \mathrm{sp}(2l, F)$, there exists an automorphism σ of $\mathrm{sp}(2l, F)$, such that $\sigma(x) \in T$ and the H-term of $\sigma(x)$ commutes with both the U-term and the V-term of $\sigma(x)$.*

Proof. Since x lies in a Borel subalgebra of $\mathrm{sp}(2l, F)$ and T is a standard Borel subalgebra of $\mathrm{sp}(2l, F)$, by Lemma 2.6, there is an automorphism τ of $\mathrm{sp}(2l, F)$ such that $\tau(x) \in T$. For convenience, we assume $x \in T$ and that

$$x = \begin{pmatrix} A & C \\ 0 & -A^t \end{pmatrix},$$

where $A \in M_l(F)$ is upper triangular and C is symmetric. By Jordan's theorem, there is an invertible matrix $X \in M_l(F)$ such that $X^{-1}AX = D + W$ and $[D, W] = 0$, where D is diagonal and W is strictly upper triangular. Let $\alpha = \mathrm{diag}(X, (X^t)^{-1})$. Then the mapping $\overline{\alpha} : z \mapsto \alpha^{-1}z\alpha$ on $\mathrm{sp}(2l, F)$ is an automorphism of $\mathrm{sp}(2l, F)$. Denote $\overline{\alpha}(x)$ by y. The H-term and the U-term of y are respectively $\mathrm{diag}(D, -D^t)$ and $\mathrm{diag}(W, -W^t)$, and

$$[\mathrm{diag}(D, -D^t), \mathrm{diag}(W, -W^t)] = 0.$$

By Lemma 2.5, there exists $\beta \in \Psi$ such that $\overline{\beta}(y)$ has the same H-term (respectively, U-term) as y and such that $\Delta(\overline{\beta}(y)) = \emptyset$. The condition $\Delta(\overline{\beta}(y)) = \emptyset$ implies that the H-term of $\overline{\beta}(y)$ commutes with the V-term of $\overline{\beta}(y)$. □

Lemma 2.8. *Let $x \in \mathrm{sp}(2l, F)$, $x \neq 0$. If $l > 2$, there is $y \in C(x)$, $y \neq 0$, such that the dimension of $C(y)$ is greater than half the dimension of $\mathrm{sp}(2l, F)$.*

Proof. By Lemma 2.7, there is an automorphism σ of $\mathrm{sp}(2l, F)$, with $\sigma(x) \in T$ and such that the H-term of $\sigma(x)$ commutes with both the U-term and the V-term of $\sigma(x)$. Assume $\sigma(x) = h + u + v$, where $h \in H$ commutes with $u \in U$ and $v \in V$.

Case 1: $[h, E_{p,-q}] = 0$ for some p, q with $1 \leq p \leq q \leq l$.

Suppose that $E_{p',-q'}$ belongs to $\{E_{p,-q} : [h, E_{p,-q}] = 0, 1 \leq p \leq q \leq l\}$ and minimises $p + q$. We claim that $\sigma(x)$ commutes with $E_{p',-q'}$. For if $[\sigma(x), E_{p',-q'}] \neq 0$, then $[u, E_{p',-q'}] \neq 0$. Write $u = \sum_{1 \leq i < j \leq l} a_{ij}E_{ij}$. Since $[u, E_{p',-q'}] \neq 0$, there are i, j with $1 \leq i' < j' \leq l$ such that $[a_{i'j'}E_{i'j'}, E_{p',-q'}] \neq 0$. The condition $[h, u] = 0$ implies that $[h, E_{i'j'}] = 0$. Note that $[E_{i'j'}, E_{p',-q'}]$ is either $E_{i',-q'}$ (when $j' = p'$) or $E_{i',-p'}$ (when $j' = q'$). From $[h, E_{i'j'}] = [h, E_{p',-q'}] = 0$ we have $[h, [E_{i'j'}, E_{p',-q'}]] = 0$. Thus $[h, E_{i',-q'}] = 0$ or $[h, E_{i',-p'}] = 0$. In either case, we have a contradiction, since $i' + q' < p' + q'$ (when $j' = p'$) and $i' + p' < p' + q'$ (when $j' = q'$), which completes the proof of the claim. If $p' = q'$, then $E_{p',-q'}$ is conjugate to $E_{1,-1}$ under an automorphism of $\mathrm{sp}(2l, F)$ (by Lemma 2.4(ii)), thus $C(E_{p',-q'})$ has the same dimension $2l^2 - l$ as $C(E_{1,-1})$, which is greater than $\frac{1}{2}(2l^2 + l)$. If $p' \neq q'$, then $E_{p',-q'}$ is conjugate to

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$E_{1,-2}$ under an automorphism of $\mathrm{sp}(2l, F)$ (by Lemma 2.4(iii)), thus $C(E_{p',-q'})$ has dimension $2l^2 - 3l + 2$, which is greater than $\frac{1}{2}(2l^2 + l)$ (recalling that $l > 2$). Choose $y = \sigma^{-1}(E_{p',-q'})$ so that $[x, y] = 0$. As $C(y)$ and $C(E_{p',-q'})$ have the same dimension, the dimension of $C(y)$ is greater than $\frac{1}{2}(2l^2 + l)$, that is, half the dimension of $\mathrm{sp}(2l, F)$.

Case 2: $[h, E_{p,-q}] \neq 0$ *for all* p, q *with* $1 \leq p \leq q \leq l$ *and* $[h, E_{ij}] = 0$ *for some* i, j *with* $1 \leq i < j \leq l$.

In this case, the condition $[h, v] = 0$ forces $v = 0$. Thus $\sigma(x) = h + u$. Suppose that $E_{i'j'}$ lies in $\{E_{ij} : [h, E_{ij}] = 0, 1 \leq i < j \leq l\}$ and maximises $j - i$. We claim that $\sigma(x)$ commutes with $E_{i'j'}$. Indeed, if $[\sigma(x), E_{i'j'}] \neq 0$, then $[u, E_{i'j'}] \neq 0$ and there are i_0, j_0 with $1 \leq i_0 < j_0 \leq l$ such that $[a_{i_0,j_0} E_{i_0,j_0}, E_{i'j'}] \neq 0$. The condition $[h, u] = 0$ implies that $[h, E_{i_0,j_0}] = 0$. Note that $[E_{i_0,j_0}, E_{i'j'}]$ is either $E_{i_0,j'}$ (when $j_0 = i'$) or $-E_{i',j_0}$ (when $j' = i_0$). From $[h, E_{i_0,j_0}] = [h, E_{i'j'}] = 0$, we have $[h, [E_{i_0,j_0}, E_{i'j'}]] = 0$. Thus either $[h, E_{i_0,j'}] = 0$ (when $j_0 = i'$) or $[h, E_{i',j_0}] = 0$ (when $j' = i_0$). In either case, we have a contradiction, since $j' - i_0 > j' - i'$ (when $j_0 = i'$) and $j_0 - i' > j' - i'$ (when $j' = i_0$), which completes the proof of the claim. By Lemma 2.4, $E_{i'j'}$ is conjugate to $E_{1,-2}$ under an automorphism of $\mathrm{sp}(2l, F)$, so $C(E_{i'j'})$ has the same dimension $2l^2 - 3l + 2$ as $C(E_{1,-2})$, which is greater than $\frac{1}{2}(2l^2 + l)$ (recalling that $l > 2$). Choose $y = \sigma^{-1}(E_{i'j'})$. Then $[x, y] = 0$. As $C(y)$ and $C(E_{i'j'})$ have the same dimension, the dimension of $C(y)$ is greater than $\frac{1}{2}(2l^2 + l)$.

Case 3: $[h, E_{p,-q}] \neq 0$ *for all* p, q *with* $1 \leq p \leq q \leq l$ *and* $[h, E_{ij}] \neq 0$ *for all* i, j *with* $1 \leq i < j \leq l$.

In this case, the condition $[h, v] = [h, u] = 0$ forces $u = v = 0$. Thus $\sigma(x) = h$ is a diagonal matrix. Let $y = \sigma^{-1}(E_{11})$. Then $[x, y] = 0$ and the dimension $2l^2 - 3l + 2$ of $C(y)$ is the same as that of $C(E_{11})$, which is greater than $\frac{1}{2}(2l^2 + l)$. □

Proof of Theorem 1.1. We have found two distinct vertices in $\Gamma(\mathrm{sp}(2l, F))$ with distance 4. Now it suffices to prove that the distance between any pair of vertices x, y of $\Gamma(\mathrm{sp}(2l, F))$ is at most 4. Let x, y be nonzero elements of $\mathrm{sp}(2l, F)$. By Lemma 2.8, there are nonzero elements x', y' with $x' \in C(x)$ and $y' \in C(y)$ such that the dimensions of $C(x')$ and $C(y')$ are both greater than half the dimension of $\mathrm{sp}(2l, F)$. Thus a nonzero element, say z, lies in $C(x') \cap C(y')$. Consequently, $x \sim x' \sim z \sim y' \sim y$ is a path in $\Gamma(\mathrm{sp}(2l, F))$. Therefore, $d(x, y) \leq 4$. □

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