SYMPLECTIC BILINEAR FORMS ON AFFINE REAL ALGEBRAIC SURFACES

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1. Introduction. Given a commutative ring A with identity, let $W^{-1}(A)$ denote the Witt group of skew-symmetric bilinear forms over A (cf. [**1**] or [**7**] for the definition of $W^{-1}(A)$).

For a regular affine 2-dimensional algebra A over the field $\mathbf{R}$ of real numbers, the group $W^{-1}(A)$ has been computed in a quite explicit way in [**1**] under the assumption that the set $X(\mathbf{R})$ of $\mathbf{R}$-rational points of $X = \operatorname{Spec} A$ is compact in the strong topology (induced by the Euclidean topology on $\mathbf{R}$). In fact the calculation of $W^{-1}(A)$ has been reduced to the calculation of the subgroup $H^1_{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2)$ of the cohomology group $H^1(X(\mathbf{R}), \mathbf{Z}/2)$ generated by the cohomology classes of algebraic cycles on X of codimension 1 (cf. Section 2 for the definition of $H^1_{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2)$). This result of [**1**] does not seem directly to extend to the case with $X(\mathbf{R})$ not necessarily compact. Here, without any restrictions on $X(\mathbf{R})$, we show that the group $W^{-1}(A)/2W^{-1}(A)$ is canonically isomorphic to the quotient group of $H^2(X(\mathbf{R}), \mathbf{Z}/2)$ modulo the subgroup

$$\{v^2 = v \cup v \in H^2(X(\mathbf{R}), \mathbf{Z}/2) \mid v \in H^1_{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2)\}.$$

In particular, if $X(\mathbf{R})$ has no compact connected component then $H^2(X(\mathbf{R}), \mathbf{Z}/2) = 0$ and hence $W^{-1}(A)/2W^{-1}(A) = 0$.

For more information about $H^1_{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2)$, we refer to [**2**] and [**8**].

2. Results. First we need some preparation. Let X be a smooth quasi-projective variety over $\mathbf{R}$. The set $X(\mathbf{R})$ of $\mathbf{R}$-rational points of X, equipped with the strong topology, has the natural structure of a C^∞ manifold, $X(\mathbf{R}) = \varnothing$ or $\dim X(\mathbf{R}) = \dim X$. Given a closed d-dimensional subvariety Y of X, let $|Y|$ denote the fundamental class of $Y(\mathbf{R})$ in $H^{BM}_d(Y(\mathbf{R}), \mathbf{Z}/2)$ (cf. [**3**, 5.12]; if $\dim Y(\mathbf{R}) < d$, we set $|Y| = 0$), where $H^{BM}_*(\cdot, \mathbf{Z}/2)$ stands for the Borel–Moore homology with coefficients in $\mathbf{Z}/2$. Let

$$i_Y : H^{BM}_*(Y(\mathbf{R}), \mathbf{Z}/2) \to H^{BM}_*(X(\mathbf{R}), \mathbf{Z}/2)$$

be the homomorphism induced by the inclusion $Y(\mathbf{R}) \subset X(\mathbf{R})$ and let

$$\Delta : H^{BM}_*(X(\mathbf{R}), \mathbf{Z}/2) \to H^*(X(\mathbf{R}), \mathbf{Z}/2)$$

be the Poincaré duality isomorphism, where $H^*(\cdot, \mathbf{Z}/2)$ is the cohomology with coefficients in the constant sheaf $\mathbf{Z}/2$ (which in the case considered here is the same as the singular cohomology). There is a unique homomorphism of graded rings

$$\mathrm{cl}_X : A(X) \to H^*(X(\mathbf{R}), \mathbf{Z}/2)$$

such that $\mathrm{cl}_X([Y]) = \Delta(i_Y(|Y|))$ for all closed subvarieties Y of X, where $A(X)$ is the Chow ring of X and $[Y]$ is the cycle represented by Y (cf. [**3**, 5.13]). We set

$$H^*_{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2) = \bigoplus_{k \geq 0} H^k_{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2) = \mathrm{cl}_X(A(X)).$$

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Note that

$$G(X(\mathbf{R})) = \{v^2 \in H^2(X(\mathbf{R}), \mathbf{Z}/2) \mid v \in H^1_{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2)\}$$

is a subgroup of $H^2(X(\mathbf{R}), \mathbf{Z}/2)$.

THEOREM 1. *Let A be a regular affine $\mathbf{R}$-algebra of dimension 2 and let $X = \operatorname{Spec} A$. Then the groups $W^{-1}(A)/2W^{-1}(A)$ and $H^2(X(\mathbf{R}), \mathbf{Z}/2)/G(X(\mathbf{R}))$ are canonically isomorphic. Moreover, $W^{-1}(A) = 0$ if $X(\mathbf{R})$ is empty.*

In particular, we immediately obtain the following.

COROLLARY 2. *Let A be a regular affine $\mathbf{R}$-algebra of dimension 2 and let $X = \operatorname{Spec} A$. If $X(\mathbf{R})$ is nonempty and the C^∞ surface $X(\mathbf{R})$ has no compact connected nonorientable component, then $W^{-1}(A)/2W^{-1}(A)$ is isomorphic to $(\mathbf{Z}/2)^s$, where s is the number of compact connected components of $X(\mathbf{R})$.*

3. Proof of Theorem 1. Let A be a commutative ring with identity. Denote by $K_0(A)$ the Grothendieck group of finitely generated projective A-modules and by $\operatorname{Pic}(A)$ the Picard group of A, and let $SK_0(A)$ be the kernel of the determinant homomorphism $\det: K_0(A) \to \operatorname{Pic}(A)$. One easily sees that the subgroup $\operatorname{tr} \tilde{K}_0(A)$ of $K_0(A)$ generated by the classes of all elements of the form $P \oplus P^*$, where P is a finitely generated projective A-module and P^* is the dual module, is contained in $SK_0(A)$. We shall be using the main result of [7].

THEOREM 3 [7]. *Let A be a 2-dimensional affine algebra over a real closed field R. Then the groups $W^{-1}(A)/2W^{-1}(A)$ and $SK_0(A)/\operatorname{tr} \tilde{K}_0(A)$ are canonically isomorphic. Moreover, if the set of R-rational points of* Spec A *is contained in a closed subscheme of dimension not exceeding* 1, *then the groups $W^{-1}(A)$ and $SK_0(A)/\operatorname{tr} \tilde{K}_0(A)$ are canonically isomorphic.*

Proof of Theorem 1. Without loss of generality, we may assume that X is irreducible. We shall identify finitely generated projective A-modules with vector bundles on X. Given a vector bundle E on X, let $c_i(E)$ denote its ith Chern class with value in the Chow group $A^i(X)$ of X. We also consider c_i as a mapping from $K_0(A)$ to $A^i(X)$ and denote by $[E]$ the class in $K_0(A)$ represented by E.

The group homomorphism

$$\varphi: SK_0(A) \to H^2(X(R), \mathbf{Z}/2),$$

induced by the mapping which associates to every vector bundle E on X the element $\mathrm{cl}_X(c_2(E))$, satisfies $\varphi(\operatorname{tr} \tilde{K}_0(A)) \subset G(X(R))$ and hence gives rise to the group homomorphism

$$\psi: SK_0(A)/\operatorname{tr} \tilde{K}_0(A) \to H^2(X(R), \mathbf{Z}/2)/G(X(R)).$$

We claim that ψ is a bijection.

Observe that $H^2(X(R), \mathbf{Z}/2) = H^2_{\mathrm{alg}}(X(R), \mathbf{Z}/2)$ since $\dim X = 2$. Thus surjectivity of

ψ follows immediately from the fact that $c_2: SK_0(A) \to A^2(X)$ is an isomorphism (cf. [**5**, Example 15.3.6]).

It remains to show that ψ is injective. First let us observe that

$$\Gamma = \{a \in A^2(X) \mid \mathrm{cl}_X(a) = 0\}$$

is a divisible subgroup of $A^2(X)$. Indeed, by Hironaka's theorem [**6**], we may assume that X is an open subvariety of a projective nonsingular algebraic surface Y over $\mathbf{R}$. Consider the following commutative diagram

$$\begin{tikzcd}
A^2(Y_C) \arrow[r, "\pi_*"] \arrow[d, "\alpha"] & A^2(Y) \arrow[r, "\mathrm{cl}_Y^2"] \arrow[d, "\beta"] & H^2(Y(\mathbf{R}), \mathbf{Z}/2) \arrow[d, "\gamma"] \\
A^2(X_C) \arrow[r, "\rho_*"] & A^2(X) \arrow[r, "\mathrm{cl}_X^2"] & H^2(X(\mathbf{R}), \mathbf{Z}/2)
\end{tikzcd}$$

where $Y_{\mathbf{C}} = Y \times_{\mathbf{R}} \mathbf{C}$, $X_{\mathbf{C}} = X \times_{\mathbf{R}} \mathbf{C}$, the homomorphisms π_*, ρ_* are induced by the canonical projections $\pi: Y_{\mathbf{C}} \to Y$, $\rho = \pi \mid X_{\mathbf{C}}: X_{\mathbf{C}} \to X$, the vertical arrows are the homomorphisms induced by the inclusions $X_{\mathbf{C}} \subset Y_{\mathbf{C}}$, $X \subset Y$, $X(\mathbf{R}) \subset Y(\mathbf{R})$, and the homomorphisms cl_Y^2, cl_X^2 are the restrictions of cl_Y, cl_X. By [**4**], the top row of the diagram is exact. We shall demonstrate that the bottom row is exact too. Obviously, $\mathrm{cl}_X^2 \circ \rho_* = 0$. Let $\mathrm{cl}_X^2(u) = 0$ for some u in $A^2(X)$ and let $u = \beta(v)$ for some v in $A^2(Y)$. Since $\gamma(\mathrm{cl}_Y^2(v)) = 0$, one can find an element w in $A^2(Y)$, represented by a $\mathbf{Z}$-linear combination of points of $Y(\mathbf{R}) \backslash X(\mathbf{R})$, such that $\mathrm{cl}_Y^2(v - w) = 0$. By construction, $\beta(v - w) = 0$ and hence exactness of the top row implies exactness of the bottom row of the diagram.

Let $\tilde{A}^2(Y_{\mathbf{C}})$ be the subgroup of $A^2(Y_{\mathbf{C}})$ of 0-cycles (here $\dim Y_{\mathbf{C}} = 2$) of degree zero. Thus $\tilde{A}^2(Y_{\mathbf{C}})$ is a divisible group (cf. [**5**, Example 1.6.6]). Since $X_{\mathbf{C}}$ is an open affine subvariety of $X_{\mathbf{C}}$, it follows that $\alpha(\tilde{A}^2(Y_{\mathbf{C}})) = A^2(X_{\mathbf{C}})$, and hence $A^2(X_{\mathbf{C}})$ is a divisible group. Therefore, $\mathrm{im}(\rho_*) = \ker(\mathrm{cl}_X^2) = \Gamma$ is a divisible group.

Now we return to the proof of injectivity of ψ. Let ξ be an element of $SK_0(A)$ satisfying $\mathrm{cl}_X(c_2(\xi)) = v^2$ for some v in $H^1_{\mathrm{alg}}(X(R), \mathbf{Z}/2)$. Pick a line bundle L such that $\mathrm{cl}_X(c_1(L)) = v$. Then, for $\eta = \xi + [L \oplus L^*]$, we obtain, $c_1(\eta) = 0$ and $c_2(\eta) = a$, where a is in Γ. Choose an element b in Γ satisfying $2b = -a$. By [**5**, Example 15.3.6], there exists a vector bundle F on X such that $c_1(F) = 0$ and $c_2(F) = b$. Clearly, $c_1(F^*) = 0$ and $c_2(F^*) = b$ and hence setting $\zeta = \eta + [F \oplus F^*]$, we obtain $c_1(\zeta) = 0$ and $c_2(\zeta) = 0$. Applying [**5**, Example 15.3.6] once again, we conclude that $\zeta = 0$ and hence ξ belongs to $\mathrm{tr}\, \tilde{K}_0(A)$. This shows that ψ is injective.

In view of Theorem 3, the proof of Theorem 1 is complete.

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