INEQUALITIES FOR THE MAXIMAL EIGENVALUE OF A NONNEGATIVE MATRIX

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Abstract. Two-sided bounds are obtained for the maximal eigenvalue of a positive matrix by iterating computations of row sums. The result provides an algorithm for approximating the maximal eigenvalue of a nonnegative matrix.

1. Introduction. An $n \times n$ real matrix $A = (a_{ij})$ is called *nonnegative* ($A \geq 0$) if all $a_{ij} \geq 0$; A is *positive* ($A > 0$) if all $a_{ij} > 0$. In 1907 Perron showed that every nonnegative matrix A has an eigenvalue r such that $r \geq |\lambda|$, for every eigenvalue λ of A. The eigenvalue r is called the *maximal eigenvalue* of A. Since then the theory of nonnegative matrices has developed and occurred in various parts of mathematics such as the Theory of Stochastic Process, numerical analysis and dynamic programming. It is well known (see, for example, [**4**]) that the maximal eigenvalue has bounds:

$$\min_i r_i \leq r \leq \max_i r_i, \tag{1}$$

where $r_i = r_i(A) = \sum_{t=1}^n a_{it}$. There have been a number of interesting papers on finding the bounds for the maximal eigenvalue of a nonnegative matrix ([**1**], [**2**], [**5**]). In this paper, we find two-sided bounds for the maximal eigenvalue of a nonnegative matrix by an Iterative algorithm. Furthermore, the algorithm can be used to estimate the maximal eigenvalue. Suppose that A has positive row sums $r_1, \ldots, r_n$, and let $D = \operatorname{diag}(r_1, \ldots, r_n)$. Then the ith row sum of the matrix $D^{-1}AD$ is $\frac{1}{r_i}\sum_{t=1}^n a_{it}r_t$. Since $D^{-1}AD$ and A have the same eigenvalues, we replace A by $D^{-1}AD$ in inequality (1), and obtain

$$\min_i \left(\frac{1}{r_i}\sum_{t=1}^n a_{it}r_t\right) \leq r \leq \max_i \left(\frac{1}{r_i}\sum_{t=1}^n a_{it}r_t\right). \tag{2}$$

For any positive numbers $q_1, \ldots, q_m$ and real numbers $p_1, \ldots, p_m$, we have the inequality (see [**3**, p. 26])

$$\min_i \frac{p_i}{q_i} \leq \frac{p_1 + p_2 + \ldots + p_m}{q_1 + q_2 + \ldots + q_m} \leq \max_i \frac{p_i}{q_i}. \tag{3}$$

Moreover, the equality holds on either side of (3) if and only if p_i/q_i is constant for all $i = 1, 2, \ldots, m$. Suppose A has positive row sums. Applying (3) to the right-hand side of (2), we obtain

$$\max_i \left(\frac{1}{r_i}\sum_{t=1}^n a_{it}r_t\right) = \max_i \frac{\sum_{t=1}^n a_{it}r_t}{\sum_{t=1}^n a_{it}} \leq \max_i r_i. \tag{4}$$

Similarly, we have $\min_i r_i \le \min_i \left(\frac{1}{r_i}\sum_{t=1}^{n} a_{it}r_t\right)$. Thus

$$\min_i r_i \le \min_i \left(\frac{1}{r_i}\sum_{t=1}^{n} a_{it}r_t\right) \le r \le \max_i \left(\frac{1}{r_i}\sum_{t=1}^{n} a_{it}r_t\right) \le \max_i r_i.$$

Hence (2) gives a sharper bound for r than (1).

We shall iterate the process described in (2), and obtain sequences of two-sided bounds for the maximal eigenvalue. Furthermore, we show that the sequences converge to the maximal eigenvalue for a positive matrix in Theorem 4, and nonnegative matrix of the type described in Theorem 5.

2. Preliminaries. Let $A \in M_n$ be a nonnegative matrix with positive row sums. For convenience, we give some symbols and definitions. First, we initialize $A^{(0)} = A = (a_{ij}^{(0)})$, $r_i^{(0)} = \sum_{j=1}^{n} a_{ij}^{(0)}$, and $r^{(0)} = [r_1^{(0)} r_2^{(0)} \dots r_n^{(0)}]^t$. Moreover, set $D^{(0)} = D$. We define, by induction, the diagonal matrices $D^{(k)}$ with the (i,i)th entry being the ith row sum of the matrix

$$(D^{(k-1)})^{-1}(D^{(k-2)})^{-1} \dots (D^{(0)})^{-1}AD^{(0)} \dots D^{(k-2)}D^{(k-1)},$$

for $k = 1, 2, \dots$. Next, for $k = 0, 1, 2, \dots$, set the recurrence

$$A^{(k+1)} = (D^{(k)})^{-1}A^{(k)}D^{(k)} \equiv (a_{ij}^{(k+1)}). \tag{5}$$

Then

$$(D^{(k)})^{-1}A^{(k)} = \left(\frac{a_{ij}^{(k)}}{r_i^{(k)}}\right), \tag{6}$$

and

$$A^{(k+1)} = \left(a_{ij}^{(k)}\frac{r_j^{(k)}}{r_i^{(k)}}\right), \tag{7}$$

where $r_i^{(k)} = \sum_{j=1}^{n} a_{ij}^{(k)}$. Let $r^{(k)} = [r_1^{(k)}\ r_2^{(k)}\ \dots\ r_n^{(k)}]^t$. By (7) we have

$$r_i^{(k+1)} = \sum_{t=1}^{n} a_{it}^{(k+1)} = \sum_{t=1}^{n} a_{it}^{(k)}\frac{r_t^{(k)}}{r_i^{(k)}}. \tag{8}$$

By (6), the vector form of (8) becomes

$$r^{(k+1)} = (D^{(k)})^{-1}A^{(k)}r^{(k)} \qquad (k = 0, 1, 2, \dots). \tag{9}$$

Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size are said to have the *same zero pattern* if $a_{ij} = 0$ whenever $b_{ij} = 0$. As a consequence of (6) and (7), the matrices A, $A^{(k)}$, and $(D^{(k)})^{-1}A^{(k)}$ $(k = 0, 1, 2, \dots)$ all have the same zero pattern.

The next result expresses the row sums of the iterative matrix $A^{(k)}$ in terms of the row sums of A^k.

THEOREM 1. *Let $A \in M_n$ be a nonnegative matrix with nonzero row sums. Then for* *every* $i=1,2,\dots,n$, $r_i^{(k)}=r_i(A^{k+1})/r_i(A^k)$ $(k=1,2,\dots)$ *and the sequence* $\{\max_i r_i^{(k)}\}_k$ *is decreasing, the sequence* $\{\min_i r_i^{(k)}\}_k$ *is increasing and both are convergent, where* $r_i(A^k)$ *is the* ith *row sum of the matrix* A^k.

Proof. The proof is by induction on k. Suppose that $k=1$. Then by (8) we have

$$r_i^{(1)}=\frac{\sum_t a_{it}\sum_j a_{tj}}{r_i^{(0)}}=\frac{r_i(A^2)}{r_i(A)}.$$

Suppose that the assertion is true for $k\le m$. From (5) and (8),

$$r_i^{(m+1)}=r_i(A^{(m+1)})=r_i((D^{(m)})^{-1}A^{(m)}D^{(m)})=\sum_{t=1}^{n}a_{it}^{(m)}\frac{r_t^{(m)}}{r_i^{(m)}},$$

where $D^{(m)}=\operatorname{diag}(r_1^{(m)},r_2^{(m)},\dots,r_n^{(m)})$. By (5) we have

$$A^{(m)}=(D^{(m-1)})^{-1}(D^{(m-2)})^{-1}\dots(D^{(0)})^{-1}AD^{(0)}\dots D^{(m-2)}D^{(m-1)}.$$

Then $a_{it}^{(m)}=(r_i^{(m-1)})^{-1}(r_i^{(m-2)})^{-1}\dots(r_i^{(0)})^{-1}a_{it}r_t^{(0)}\dots r_t^{(m-2)}r_t^{(m-1)}$. By the induction hypothesis, we compute that

$$\begin{aligned}
r_i^{(m+1)}&=\frac{\sum_{t=1}^{n}a_{it}^{(m)}r_t^{(m)}}{r_i^{(m)}}\\
&=\frac{\sum_t (r_i^{(m-1)})^{-1}(r_i^{(m-2)})^{-1}\dots(r_i^{(0)})^{-1}a_{it}r_t^{(0)}\dots r_t^{(m-2)}r_t^{(m-1)}r_t^{(m)}}{r_i^{(m)}}\\
&=\frac{\sum_t \frac{r^i(A^{m-1})}{r_i(A^m)}\frac{r_i(A^{m-2})}{r_i(A^{m-1})}\cdots\frac{1}{r_i(A)}a_{it}r_t(A)\cdots\frac{r_t(A^{m-1})}{r_t(A^{m-2})}\frac{r_t(A^m)}{r_t(A^{m-1})}\frac{r_t(A^{m+1})}{r_t(A^m)}}{\frac{r_i(A^{m+1})}{r_i(A^m)}}\\
&=\frac{\sum_t a_{it}r_t(A^{m+1})}{r_i(A^{m+1})}=\frac{r_i(A^{m+2})}{r_i(A^{m+1})}.
\end{aligned}$$

This completes the induction. Next, by inequality (3), we have

$$r_i^{(k+1)}=\frac{r_i(A^{k+2})}{r_i(A^{k+1})}=\frac{\sum_t a_{it}r_t(A^{k+1})}{\sum_t a_{it}r_t(A^k)}\le\max_t\frac{r_t(A^{k+1})}{r_t(A^k)}=\max_t r_t^{(k)}.$$

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It follows that $\max_i r_i^{(k+1)} \le \max_i r_i^{(k)}$. The sequence $\{\max_i r_i^{(k)}\}_k$ is then decreasing and thus convergent. The proof that the sequence $\{\min_i r_i^{(k)}\}_k$ *is increasing is similar.* □

The iterative matrix $A^{(k+1)}$ in (5) is similar to A; it has the same maximal eigenvalue as A and hence

$$\min_i r_i^{(k+1)} \le r \le \max_i r_i^{(k+1)}. \tag{10}$$

Taking the limit on both sides of (10), we obtain

$$\lim_{k\to\infty} \min_i r_i^{(k+1)} \le r \le \lim_{k\to\infty} \max_i r_i^{(k+1)}. \tag{11}$$

Two matrices A and B are *permutationally equivalent* if there exists a permutation matrix P such that $B = P^tAP$. We show in the following that the inequalities (10) and (11) are invariant under permutation equivalence.

THEOREM 2. *Let $A, B \in M_n$ be permutationally equivalent nonnegative matrices with nonzero row sums. Then, for $k = 0, 1, 2, \ldots$, we have*

$$\max_i r_i^{(k)}(A) = \max_i r_i^{(k)}(B) \quad \textit{and} \quad \min_i r_i^{(k)}(A) = \min_i r_i^{(k)}(B),$$

where $r_i^{(k)}(A)$ and $r_i^{(k)}(B)$ are the i*th row sums of $A^{(k)}$ and $B^{(k)}$ respectively.*

Proof. Suppose $B = P^tAP$, for some permutation matrix P. Then, for every $i = 1, 2, \ldots, n$, there exists j such that $r_i(B) = r_j(A)$. The index j is determined by the permutation P. By Theorem 1 we have

$$\begin{aligned}\max_i r_i^{(k)}(B) = \max_i r_i^{(k)}(P^tAP) &= \max_i \frac{r_i((P^tAP)^{k+1})}{r_i((P^tAP)^k)}\\ &= \max_i \frac{r_i(P^tA^{k+1}P)}{r_i(P^tA^kP)} = \max_j \frac{r_j(A^{k+1})}{r_j(A^k)} = \max_j r_j^{(k)}(A).\end{aligned}$$

The minimum can be proved similarly. □

3. Approximation of the maximal eigenvalue. The inequality (11) gives two-sided bounds for the maximal eigenvalue. the question arises if the equality holds in (11). First we give a positive lower bound for nonzero entries of $(D^{(k)})^{-1}A^{(k)}$.

THEOREM 3. *Let $A \in M_n$ be a nonnegative matrix and let q be a given positive integer. If A has q positive columns and $n-q$ zero columns, then there exists a positive number d with $d \le 1/q$ such that every nonzero entry of $(D^{(k)})^{-1}A^{(k)}$ is not less than d, for all $k = 0, 1, 2, \ldots$.*

Proof. The proof will be by induction on k. Let $r_M^{(k)}$ denote the maximum row sum of $A^{(k)}$, $m^{(0)}$ the minimum of nonzero entries of $A^{(0)}$, $m_l^{(0)}$ the minimum of nonzero entries of the lth column of $A^{(0)}$, and $M_l^{(0)}$ the maximum of nonzero entries of the lth column of

$A^{(0)}$. Set $\alpha = \min_l \frac{m_l^{(0)}}{M_l^{(0)}}$, where l runs over all q positive columns of A. We prove first, by induction, that for every nonzero entry $a_{ij}^{(k)}$,

$$a_{ij}^{(k)} \min_t \frac{a_{jt}^{(k)}}{a_{it}^{(k)}} \geq m^{(0)}\alpha \quad (k = 0, 1, 2, \ldots),$$

where the minimum t is taken over all indices of positive columns of $A^{(k)}$.

It is trivial that the induction hypothesis holds when $k = 0$. Suppose the induction is true for k. If $a_{ij}^{(k+1)}$ is nonzero, then $a_{ij}^{(k)}$ is nonzero. By (7) we have

$$a_{ij}^{(k+1)} \min_t \frac{a_{jt}^{(k+1)}}{a_{it}^{(k+1)}} = a_{ij}^{(k)} \frac{r_j^{(k)}}{r_i^{(k)}} \min_t \frac{a_{jt}^{(k)} \frac{r_t^{(k)}}{r_j^{(k)}}}{a_{it}^{(k)} \frac{r_t^{(k)}}{r_i^{(k)}}} = a_{ij}^{(k)} \min_t \frac{a_{jt}^{(k)}}{a_{it}^{(k)}} \geq m^{(0)}\alpha.$$

This proves the assertion.

Suppose now that $a_{ij}^{(k)}$ is nonzero. From (7) and (8), we deduce that

$$\begin{aligned} a_{ij}^{(k)} = a_{ij}^{(k-1)} \frac{r_j^{(k-1)}}{r_i^{(k-1)}} &= a_{ij}^{(k-2)} \frac{r_j^{(k-2)}}{r_i^{(k-2)}} \frac{\sum_t a_{jt}^{(k-2)} \frac{r_t^{(k-2)}}{r_j^{(k-2)}}}{\sum_t a_{it}^{(k-2)} \frac{r_t^{(k-2)}}{r_i^{(k-2)}}} \\ &= a_{ij}^{(k-2)} \frac{\sum_t a_{jt}^{(k-2)} r_t^{(k-2)}}{\sum_t a_{it}^{(k-2)} r_t^{(k-2)}}. \end{aligned} \tag{12}$$

Since A and $A^{(k-2)}$ have the same zero pattern, by the hypothesis, both the ith row and jth row of $A^{(k-2)}$ have exactly q nonzero entries that locate on the same columns. Thus the sums in the denominator and numerator of (12) have q nonzero terms with the same t indices. Apply those nonzero q terms to the inequality (3). We obtain

$$a_{ij}^{(k)} \geq a_{ij}^{(k-2)} \min_t \frac{a_{jt}^{(k-2)}}{a_{it}^{(k-2)}} \geq m^{(0)}\alpha,$$

where the minimum runs over those q indices. This proves that every nonzero entry of $A^{(k)}$ is not less than $m^{(0)}\alpha$, for all $k = 0, 1, 2, \ldots$.

Finally, for any nonzero entry of $(D^{(k)})^{-1}A^{(k)}$, we obtain

$$(D^{(k)})^{-1}A_{ij}^{(k)} = \frac{a_{ij}^{(k)}}{r_i^{(k)}} \geq \frac{m^{(0)}\alpha}{r_M^{(k)}} \geq \frac{m^{(0)}\alpha}{r_M^{(0)}}.$$

The last inequality follows from the fact that the sequence $\{r_M^{(k)}\}_k$ is decreasing by Theorem 1. Define the positive number $d = \frac{m^{(0)}\alpha}{r_M^{(0)}}$. Since the ith row of A has q nonzero elements, we have

$$1 = \sum_{t=1}^{n} \frac{a_{it}^{(0)}}{r_i^{(0)}} \geq q \frac{m^{(0)}\alpha}{r_M^{(0)}} = qd.$$

Thus $d \leq 1/q$, and the proof is complete. □

Suppose that $A \in M_n$ is positive. Then the hypotheses of Theorem 3 are obviously satisfied. Define the iterative matrix

$$P^{(k+1)} = (D^{(k)})^{-1}A^{(k)}P^{(k)} \quad (k = 1, 2, \ldots), \tag{13}$$

where $P^{(1)} = (D^{(0)})^{-1}A^{(0)}$. We obtain the following approximation.

THEOREM 4. *Let $A \in M_n$ ($n \geq 2$) be a positive matrix. Then $P^{(k)}$ converges to a matrix*

$$S = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_n \end{pmatrix},$$

for some positive numbers $c_1, c_2, \ldots, c_n$, and

$$r = \lim_{k\to\infty} \max_i r_i^{(k)} = \lim_{k\to\infty} \min_i r_i^{(k)} = \lim_{k\to\infty} r_i^{(k)} = \sum_{i=1}^{n} c_i r_i^{(0)}.$$

Proof. For $k = 1, 2, \ldots$ we have from (13)

$$P^{(k+1)} = (D^{(k)})^{-1}A^{(k)}(D^{(k-1)})^{-1}A^{(k-1)} \ldots (D^{(1)})^{-1}A^{(1)}(D^{(0)})^{-1}A^{(0)}. \tag{14}$$

Since $(D^{(k)})^{-1}A^{(k)}$ is positive, it follows from (14), that the matrix $P^{(k+1)}$ is positive. Let $m_1^{(k)}$ and $M_1^{(k)}$ denote respectively the smallest and the largest elements of the first column of $P^{(k)}$. We show first that for $k = 1, 2, \ldots$

$$m_1^{(k+1)} \geq M_1^{(k)}d + m_1^{(k)}(1-d) \quad \text{and} \quad M_1^{(k+1)} \leq m_1^{(k)}d + M_1^{(k)}(1-d). \tag{15}$$

Let $P^{(k+1)} = (p_{ij}^{(k+1)})$ and $(D^{(k)})^{-1}A^{(k)} = (b_{ij}^{(k)})$. By equation (6), we have $\sum_{j=1}^{n} b_{ij}^{(k)} = 1$, and by Theorem 3, we have every entry $b_{ij}^{(k)} \geq d$ for some positive number $d \leq 1/n$. For simplicity, assume that $m_1^{(k)} = p_{11}^{(k)}$ and $M_1^{(k)} = p_{21}^{(k)}$. Computing the first column entry of the matrix (13), we obtain

$$\begin{aligned} p_{i1}^{(k+1)} &= \sum_{j=1}^{n} b_{ij}^{(k)} p_{j1}^{(k)} \\ &= b_{i1}^{(k)}p_{11}^{(k)} + (1 - b_{i1}^{(k)} - b_{i3}^{(k)} - \ldots - b_{in}^{(k)})p_{21}^{(k)} + b_{i3}^{(k)}p_{31}^{(k)} + \ldots + b_{in}^{(k)}p_{n1}^{(k)} \\ &= M_1^{(k)} - b_{i1}^{(k)}(M_1^{(k)} - p_{11}^{(k)}) - b_{i3}^{(k)}(M_1^{(k)} - p_{31}^{(k)}) - \ldots - b_{in}^{(k)}(M_1^{(k)} - p_{n1}^{(k)}) \\ &\leq M_1^{(k)} - b_{i1}^{(k)}(M_1^{(k)} - p_{11}^{(k)}) \\ &= M_1^{(k)} - b_{i1}^{(k)}(M_1^{(k)} - m_1^{(k)}) \\ &\leq M_1^{(k)} - d(M_1^{(k)} - m_1^{(k)}) \\ &= m_1^{(k)}d + (1-d)M_1^{(k)}. \end{aligned}$$

Hence $M_1^{(k+1)} \leq m_1^{(k)}d + (1-d)M_1^{(k)}$, and this proves the second part of (15). On the other hand

$$\begin{aligned} p_{i1}^{(k-1)} &= \sum_{j=1}^{n} b_{ij}^{(k)} p_{j1}^{(k)} \\ &= (1 - b_{i2}^{(k)} - b_{i3}^{(k)} - \ldots - b_{in}^{(k)})p_{11}^{(k)} + b_{i2}^{(k)}p_{21}^{(k)} + b_{i3}^{(k)}p_{31}^{(k)} + \cdots + b_{in}^{(k)}p_{n1}^{(k)} \\ &= m_1^{(k)} + b_{i2}^{(k)}(p_{21}^{(k)} - m_1^{(k)}) + b_{i3}^{(k)}(p_{31}^{(k)} - m_1^{(k)}) + \ldots + b_{in}^{(k)}(p_{n1}^{(k)} - m_1^{(k)}) \\ &\geq m_1^{(k)} + b_{i2}^{(k)}(p_{21}^{(k)} - m_1^{(k)}) \\ &= m_1^{(k)} + b_{i2}^{(k)}(M_1^{(k)} - m_1^{(k)}) \\ &\geq m_1^{(k)} + d(M_1^{(k)} - m_1^{(k)}) \\ &= M_1^{(k)}d + (1-d)m_1^{(k)}. \end{aligned}$$

Thus $m_1^{(k+1)} \geq M_1^{(k)}d + (1-d)m_1^{(k)}$, and the first part of (15) is proved. Now by (15), we have $M_1^{(k-1)} - m_1^{(k+1)} \leq (1-2d)(M_1^{(k)} - m_1^{(k)})$. and thus

$$M_1^{(k+1)} - m_1^{(k+1)} \leq (1-2d)^k(M_1^{(1)} - m_1^{(1)}). \tag{16}$$

Hence $M_1^{(k+1)} - m_1^{(k+1)} \to 0$ as $k \to \infty$. Notice that $0 \leq 1 - 2d < 1$, by Theorem 3. Furthermore, by Theorem 1, the sequence $\{M_1^{(k)}\}_k$ decreases and $\{m_1^{(k)}\}_k$ increases. Therefore both sequences converge to the same limit, and thus the first column of $P^{(k)}$ converges to a column of the form $[c_1 \ c_1 \ \ldots \ c_1]^t$, for some positive number c_1.

Applying the same argument to the remaining columns of $P^{(k)}$, we conclude that $P^{(k)}$ converges to a matrix

$$S = \begin{pmatrix} c_1 & c_2 & \ldots & c_n \\ c_1 & c_2 & \ldots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \vdots & c_n \end{pmatrix},$$

for some positive numbers $c_1, c_2, \ldots, c_n$. From the recurrence relation (9), we obtain

$$r^{(k)} = P^{(k)}r^{(0)}. \tag{17}$$

Then $\{r^{(k)}\}$ converges to $Sr^{(0)}$, and hence for each i, $\{r_i^{(k)}\}_k$ converges to $r = c_1r_1^{(0)} + c_2r_2^{(0)} + \ldots + c_nr_n^{(0)}$. This completes the proof. □

We reduce the positivity in Theorem 4 to certain pattern of matrices, and obtain the following weaker version of Theorem 4.

Theorem 5. *Let $A \in M_n$ be a nonnegative matrix and let q be a given positive integer. If A is permutationally equivalent to a matrix with q positive columns and $n - q$ zero columns, then*

$$r = \lim_{k\to\infty} \max_i r_i^{(k)} = \lim_{k\to\infty} \min_i r_i^{(k)} = \lim_{k\to\infty} r_i^{(k)}.$$

Proof. By Theorem 2 we may assume the positive columns of A appear in the first q columns and zeros elsewhere. Let $P^{(k+1)}$ be the matrix defined in (14). Now A and $(D^{(k)})^{-1}A^{(k)}$ have the same zero pattern; i.e., the first q columns of $(D^{(k)})^{-1}A^{(k)}$ are positive and other columns are zeros. Thus the first q columns of $P^{(k+1)}$ are positive and other columns are zeros. Suppose that $q \geq 2$. Apply the arguments of the proof in

Theorem 4 to the first column of $P^{(k)}$. We obtain the inequality (16) with $0 \le 1-2d < 1$. Repeating this process to the next $q-1$ columns of $P^{(k)}$, we see that the first q columns of $P^{(k)}$ converge to a matrix

$$\begin{pmatrix} c_1 & c_2 & \dots & c_q \\ c_1 & c_2 & \dots & c_q \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \dots & c_q \end{pmatrix} \in M_{n \times q},$$

for some positive numbers $c_1, c_2, \dots, c_q$. Since the last $n-q$ columns of $P^{(k)}$ are zeros, it follows that, as $k \to \infty$, $P^{(k)}$ converges to

$$s = \begin{pmatrix} c_1 & c_2 & \dots & c_q & 0 & \dots & 0 \\ c_1 & c_2 & \dots & c_q & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \dots & c_q & 0 & \dots & 0 \end{pmatrix} \in M_n.$$

By (17) we conclude that $\{r^{(k)}\}$ converges to $Sr^{(0)}$, hence for each i, $\{r_i^{(k)}\}_k$ converges to $r = c_1 r_1^{(0)} + c_2 r_2^{(0)} + \dots + c_q r_q^{(0)}$, and the conclusion follows. When $q = 1$, it is clear that a_{11} and 0 are the eigenvalues of A and so $r = a_{11}$. By (5) we have

$$A^{(1)} = (D^{(0)})^{-1} A^{(0)} D^{(0)} = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{11} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{11} & 0 & \dots & 0 \end{pmatrix}.$$

It follows that $A^{(k)} = A^{(1)}$ for all $k = 2, 3, \dots$. Therefore $r_i^{(k)} = a_{11}$, and thus $r = a_{11} = \lim_{k \to \infty} r_i^{(k)}$ for all i, completing the proof. □

Theorem 4 provides an Iterative algorithm for approximating the maximal eigenvalue of a positive matrix. In general, let $A = (a_{ij}) \in M_n$ be a nonnegative matrix, and r be the maximal eigenvalue of A. Given a positive number ϵ, define a matrix $A_\epsilon \in M_n$ as follows:

$$(A_\epsilon)_{ij} = \begin{cases} a_{ij}, & \text{if } a_{ij} > 0; \\ \epsilon, & \text{if } a_{ij} = 0. \end{cases}$$

Then A_ϵ is positive, and the maximal eigenvalue r_ϵ of A_ϵ is estimated according to Theorem 4. Since eigenvalues are continuous functions of entries of the matrix, it follows that $A_\epsilon \to A$ and $r_\epsilon \to r$ as $\epsilon \to 0$. Hence, if ϵ is small enough then an estimation of r can be achieved by computations of r_ϵ. We summarize the Iterative algorithm.

STEP 1. *Initialize.*

Let $k = 0$, $A^{(0)} = A = (a_{ij}^{(0)})$, $r_i^{(0)} = \sum_{j=1}^{n} a_{ij}^{(0)}$ for $i = 1, 2, \dots, n$ and δ = tolerance.

STEP 2. *Find* min *and* max *row sums:* $r_m^{(k)} = \min_i r_i^{(k)}$, and $r_M^{(k)} = \max_i r_i^{(k)}$.

STEP 3. *Test.*

TABLE 1.

Order.	Power method.	Iterative algorithm.	Matlab.
100	12/4963·6238537076	12/4963·6238537076	4963·6238537076
200	12/9881·5030429984	11/9881·5030429984	9881·5030429984
300	11/14792·7146278852	11/14792·7146278852	14792·7146278852
400	11/19719·9743999372	10/19719·9743999372	19719·9743999372
500	10/24686·1864859857	10/24686·1864859858	24686·1864859857
600	10/29632·9292683024	10/29632·9292683025	29632·9292683024
700	10/34591·5996879745	10/34591·5996879745	34591·5996879745
800	10/39545·8711650438	10/39545·8711650439	39545·8711650439
900	10/44489·6491412335	10/44489·6491412336	44489·6491412336
1000	10/49451·5229094011	9/49451·5229094011	49451·5229094010

If $|r_M^{(k)} - r_m^{(k)}| < \delta$, then we obtain an approximation for the maximal eigenvalue or else $a_{ij}^{(k+1)} = a_{ij}^{(k)} r_j^{(k)} / r_i^{(k)}$ and $r_i^{(k+1)} = \sum_{j=1}^{n} a_{ij}^{(k+1)}$, $i = 1, 2, \ldots, n$. Replace k by $k+1$. Go to step 2.

The test condition Step 3 may be replaced by $|r_M^{(k+1)} - r_M^{(k)}| < \delta$ or $|r_m^{(k+1)} - r_m^{(k)}| < \delta$. We run MATLAB programs on the DEC Alpha Sever-2100, and list in Table 1 the numbers of iterations required to estimate the maximal eigenvalue by the Power method and the Iterative algorithm for nonnegative matrices of order 100 to 1000 with $\delta = 1.0 \times 10^{-10}$. In Table 1, there are two values in the Power method column and Iterative algorithm column. The first value is the number of iterations, the second value is the approximated maximal eigenvalue, and the Matlab software evaluates the maximal eigenvalue in the last column. The entries of a sample run matrix for the Iterative algorithm are generated by the following formula:

$A = \text{rand}(n)$;
if $A(i,j) < 0{\cdot}1$, then $A(i,j) = \text{eps} \simeq 2{\cdot}2 \times 10^{-16}$ ($A(i,j) = 0$ for Power Method)
or else $A(i,j) = A(i,j) \times 100$.

From the experiment, the number of iterations required to approximate the maximal eigenvalue by the Iterative algorithm is close to that of the Power method with initial vector $x = [1]$. This similarity also happens to tridiagonal matrices and matrices whose maximal eigenvalue are relatively dense, such as the tridiagonal matrix $W_{21}^{+} = (w_{ij}) \in M_{21}$ (see [**6**, 5.45]), where $w_{ii} = 11 - i$ ($i = 1, \ldots, 11$), $w_{ii} = i - 11$ ($i = 12, \ldots, 21$) and $w_{i\,i+1} = w_{i+1\,i} = 1$ ($i = 1, \ldots, 20$). The matrix W_{21}^{+} has an eigenvalue which is quite close to the maximal eigenvalue. The Power method takes 136 iterations to approximate the maximal eigenvalue $10{\cdot}7461941835$ of W_{21}^{+}, and 135 iterations for the Iterative algorithm. However, for certain matrices we may obtain fewer iterations by using the test condition $|r_m^{(k+1)} - r_m^{(k)}| < \delta$. For example, consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0{\cdot}01 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

The eigenvalues of A are 3, 3, 1. The Iterative algorithm needs only 3 iterations to approximate the maximal eigenvalue 3·00000000000, but more than 1000 iterations, with estimation 3·00230769231, are needed for the Power method to do the job. This is simply because A has double eigenvalue 3.

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