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A uniqueness theorem for the Chaplygin-Frankl problem

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In a paper dealing with trans-sonic jet flows Frankl (*Bull. Acad. Sci. URSS Sér. Math.* [*Izv. Akad. Nauk SSSR*] 9 (1945), 121-143) considered the following problem (T) by applying the condition

(1) $$F(y) = 1 + 2(k/k')' > 0 \quad \text{for} \quad y < 0 \ ,$$

where $k = k(y)$ is a monotone increasing function with a continuous second derivative, $k(0) = 0$, $F(0) > 0$, $k'(y) \neq 0$ for $y < 0$. Consider an equation of the form

(2) $$\bar{L}[u] = k(y)\cdot u_{xx} + u_{yy} = 0 \ ,$$

which is elliptic for $y > 0$, hyperbolic for $y < 0$, and parabolic for $y = 0$. Consider equation (2) in a bounded simply connected region $D \subset R^2$ which is bounded by the following three curves: a piecewise smooth curve Γ_0 lying in the half-plane $y > 0$ which intersects the line $y = 0$ at the points $A(0, 0)$ and $B(1, 0)$; for $y < 0$ by a smooth curve Γ_2 through B which meets the characteristic of (2) issuing from $A(0, 0)$ at the point P; and the curve Γ_1 which consists of the portion PA of the characteristic through A. The problem (T) (or problem of Tricomi-Frankl) consists of finding a solution $u = u(x, y) \in C^2(D)$ assuming prescribed values on $\Gamma_0 \cup \Gamma_2$. In the present paper we generalize Frankl's uniqueness theorem; our uniqueness theorem includes cases where $F(y)$ *may be negative*.

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The Chaplygin-Frankl problem

Consider the equation

(3) $$L[u] = k(y)\cdot u_{xx} + u_{yy} + \lambda(x, y)\cdot u = f(x, y)$$

in a bounded simply connected region $G \subset R^2$, where $k = k(y)$ is a monotone increasing function with a continuous second derivative, $k(0) = 0$, $k'(y) \neq 0$ for $y < 0$, and the region G is bounded by the following curves: a piecewise smooth curve Γ_0 lying in the half-plane $y > 0$ which intersects the line $y = 0$ at the points $A(0, 0)$ and $B(1, 0)$; for $y < 0$ by a smooth curve Γ_2 through B which meets the characteristic of (3) issuing from $A(0, 0)$ at the point P ; and the curve Γ_1 which consists of the portion PA of the characteristic through A ; $\lambda(x, y) \in C^1(\overline{G})$, $f(x, y) \in C^0(\overline{G})$.

The Chaplygin-Frankl Problem, or *problem* (F), consists in finding a solution $u = u(x, y) \in C^2(\overline{G})$ assuming prescribed values on $\Gamma_0 \cup \Gamma_2$; that is,

(4) $$u = 0 \quad \text{on} \quad \Gamma_0 \cup \Gamma_2 .$$

DEFINITION ([2], p. 234, [3], [4]). A function $u = u(x, y)$ is called *a quasi-regular solution* of problem (F) if u satisfies equation (3) in $G \subset R^2$ and in addition the following conditions:

(i) the integral $\int_A^B u(x, 0)\cdot u_y(x, 0)dx$ exists;

(ii) if $G_+ = G \cap \{y > 0\}$, $G_- = G \cap \{y < 0\}$, and if $G_{\pm(\varepsilon)}$ are regions with boundaries $\partial G_{\pm(\varepsilon)}$ lying entirely in $G_\pm$, then the line integrals along $\partial G_{\pm(\varepsilon)}$ which result from the application of Green's Theorem to the integrals

$$\iint_{G_{\pm(\varepsilon)}} u\cdot L[u]\cdot dxdy \, , \quad \iint_{G_{\pm(\varepsilon)}} u_x\cdot L[u]\cdot dxdy \, , \quad \iint_{G_{\pm(\varepsilon)}} u_y\cdot L[u]\cdot dxdy$$

have a limit when $\partial G_{\pm(\varepsilon)}$ approaches the boundary of $G_\pm$.

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THEOREM. *Let* $k(y)$ *be a monotone increasing function with a continuous second derivative,* $k(0) = 0$, $F(0) > 0$, $k'(y) \neq 0$ *for* $y < 0$, $\lambda(x, y) \in C^1(\overline{G})$, $f(x, y) \in C^0(\overline{G})$ *, where* G *is the domain described above. Moreover, assume the conditions* $\lambda|_{\Gamma_1} \leq 0$ *and*

$k \cdot a_{xx} + a'' + 2\lambda \cdot a \geq d_1 > 0$ *in* G_+ *, and*

$$R(x, y) = a'' - 4\lambda \cdot (k/k') \cdot \left[a' + a_x \cdot (-k)^{\frac{1}{2}} \cdot e^{\beta x}\right] + k \cdot a_{xx}$$
$$+ 2a \cdot \left\{(-2) \cdot (k/k') \cdot \left[\lambda' + (\lambda_x + \beta \cdot \lambda) \cdot (-k)^{\frac{1}{2}} \cdot e^{\beta x}\right] + \lambda \cdot R(y)\right\} \geq d_2 > 0 \quad in \quad G_- ,$$

where $R(y) = 1 - 2 \cdot (k/k')'$ *, and* $a = a(x, y) \in C^2(\overline{G})$ *is a given negative function of the independent variables* $x, y \in R$ *, such that* $\lim_{y \to 0-} (k/k') = 0$ *, and* β *is a given positive constant* $R(x) = e^{\beta x} - 1 \geq 0$ *in* G *. In addition, we assume* $R^*(x, y) = \left[a' - (\beta \cdot a + a_x) \cdot e^{\beta \cdot x} \cdot (-k)^{\frac{1}{2}}\right] \geq 0$ *in* G_- *, and if* $R^*(x) = e^{2\beta \cdot x} - 1$ *, then* $V(x, y) = A \cdot F^2 + B \cdot F + C \leq 0$ *in* G_- *, where*

$$A = a^2 \cdot R^*(x) ,$$

$$B = 4 \cdot \left(R^*(x) \cdot a' + \beta \cdot e^{\beta \cdot x} \cdot a \cdot (-k)^{\frac{1}{2}}\right) \cdot a \cdot (k/k') ,$$

and

$$C = 4\left\{-\left[\beta \cdot a(\beta \cdot a + 2 \cdot a_x) \cdot e^{2\beta \cdot x} + R^*(x) \cdot (a_x)^2\right] \cdot k \right.$$
$$\left. + 2\beta \cdot e^{\beta \cdot x} \cdot a \cdot a' \cdot (-k)^{\frac{1}{2}} + R^*(x) \cdot (a')^2\right\} \cdot (k/k')^2 .$$

Finally, we assume

$$R_1(x, y) = ak \cdot F(y) + 2 \cdot R^*(x, y) \cdot (k^2/k') \geq d_3 > 0 ,$$

$$R_2(x, y) = (-a) \cdot F(y) - 2 \cdot R^*(x, y) \cdot (k/k') \geq d_4 > 0 \quad in \quad G_- ,$$

and

$$-a_x(-k)^{\frac{1}{2}} + \left((a \cdot k')/4k\right)|_{\Gamma_1} = R_3(x, y)|_{\Gamma_1} \geq 0 ,$$

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$$\left[(-k)^{\frac{1}{2}}\cdot e^{\beta\cdot x}\cdot \nu_1+\nu_2\right]\cdot\left[k\cdot \nu_1^2+\nu_2^2\right]\Big|_{\Gamma_2} = R_4(x, y)\big|_{\Gamma_2} \geq 0 ,$$

where $\nu = (\nu_1, \nu_2)$ *is the outer normal unit vector on* Γ_2 .

The prime (') differentiation is meant with respect to the variable y .

If the above hypotheses hold, then there exist a constant $d_0 < 0$ *, and another constant* $d^0 > 0$ *such that if* $d_0 \leq F(y) \leq d^0$ *in* G_- *, and* $u(x, y)$ *is a quasi-regular solution of* (3) *which vanishes on* $\Gamma_0 \cup \Gamma_2$ *, then* $u = 0$ *in* G *.*

Proof. We investigate the expression

(5) $$2(l[u], L[u]) = 2 \cdot \iint_G l[u]\cdot L[u]\cdot dxdy ,$$

where

(6) $$l[u] = a(x, y)\cdot u \quad \text{in} \quad G_+ ,$$

and

$$l[u] = a(x, y)\cdot\left[u+4\left((-k)^{\frac{1}{2}}\cdot e^{\beta x}\cdot u_x+u_y\right)\cdot(k/k\ )\right] \quad \text{in} \quad G_- ,$$

where $a = a(x, y) \in C^2(\overline{G})$ is a given negative function of the independent variables $x, y \in R$, and β is a given positive constant.

If $u(x, y)$ is a solution of (3), then (5) will vanish.

We note the following identities:

$$2ak\cdot uu_{xx} = 2\left(ak\cdot uu_x\right)_x - 2ak\cdot u_x^2 - \left(a_x k\cdot u^2\right)_x + ka_{xx}\cdot u^2 ,$$

$$2a\cdot uu_{yy} = 2\left(a\cdot uu_y\right)_y - 2a\cdot u_y^2 - \left(a_y\cdot u^2\right)_y + a_{yy}\cdot u^2 ,$$

$$2bk\cdot u_x u_{xx} = \left(bk\cdot u_x^2\right)_x - kb_x\cdot u_x^2 ,$$

$$2b\cdot u_x u_{yy} = 2\left(b\cdot u_x u_y\right)_y - 2b_y\cdot u_x u_y - \left(b\cdot u_y^2\right)_x + b_x\cdot u_y^2 ,$$

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$$2ck\cdot u_y u_{xx} = 2\left(ck\cdot u_x u_y\right)_x - \left[ck\cdot u_x^2\right]_y + (ck)_y\cdot u_x^2 - 2kc_x\cdot u_x u_y ,$$

$$2c\cdot u_y u_{yy} = \left[c\cdot u_y^2\right] - c_y\cdot u_y^2 ,$$

$$2\lambda\cdot b\cdot uu_x = \left(b\lambda u^2\right)_x - (b\lambda)_x\cdot u^2 ,$$

$$2c\cdot uu_y = \left(b\lambda\cdot u^2\right)_y - (b\cdot\lambda)_y\cdot u^2 ,$$

where $b = b(x, y) \in C^1(\overline{G})$ and $c = c(x, y) \in C^1(\overline{G})$ are defined from (6) as the coefficients of u_x and u_y, respectively, in $\overline{G}$.

Substitution of these identities into (5) and an application of Green's Theorem yield

$$
\begin{aligned}
(7)\quad 0 &= 2(l[u], L[u]) \\
&= 2\iint_G l[u]\cdot L[u]\cdot dxdy \\
&= \iint_G \{ka_{xx}+a_{yy}+2\lambda\cdot a-[(\lambda\cdot b)_x+(\lambda\cdot c)_y]\}\cdot u^2\cdot dxdy \\
&\quad + \iint_G \left[\left(-2ak-k\cdot b_x+(ck)_y\right)\cdot u_x^2-2\cdot\left(kc_x+b_y\right)\cdot u_x u_y+\left(-2a+b_x-c_y\right)\cdot u_y^2\right]\cdot dxdy \\
&\quad + \int_{\partial G} \lambda\cdot\left(b\cdot v_1+c\cdot v_2\right)\cdot u^2\cdot ds \\
&\quad + \int_{\partial G} \left[\left(2ak\cdot uu_x\cdot v_1+2a\cdot uu_y v_2\right)-\left(ka_x\cdot v_1+a_y v_2\right)\cdot u^2\right]\cdot ds \\
&\quad + \int_{\partial G} \left[\left(kb\cdot v_1-kc\cdot v_2\right)\cdot u_x^2+\left(-b\cdot v_1+c\cdot v_2\right)\cdot u_y^2+2\left(b\cdot v_2+kc\cdot v_1\right)\cdot u_x u_y\right]\cdot ds \\
&= J_1 + J_2 + J_3 + J_4 + J_5 ,
\end{aligned}
$$

where $b = c = 0$ in G_+, and

$$(8)\qquad b = c(-k)^{\frac{1}{2}}\cdot e^{\beta x} , \quad c = (4ak/k') \text{ in } G_- .$$

LEMMA 1.

$$(9)\qquad J_5^{(1)} = \int_{\Gamma_1} Q\left(u_x, u_y\right)\cdot ds \geq 0 ,$$

where $Q = Q\left(u_x, u_y\right) = \alpha_1\cdot u_x^2 + 2\beta_1\cdot u_x u_y + \gamma_1\cdot u_y^2$ *is a quadratic form with* *respect to* u_x, u_y , *where* $\alpha_1 = kb\cdot v_1 - kc\cdot v_2$, $\beta_1 = b\cdot v_2 + kc\cdot v_1$, $\gamma_1 = -b\cdot v_1 + c\cdot v_2$, *and where* $v = (v_1, v_2)$ *is the outer normal unit vector on* Γ_2 .

Proof.

$$\alpha_1\cdot dx = kb\cdot dy + kcdx = \left(kb\left(-1/(-k)^{\frac{1}{2}}\right)+kc\right)\cdot dx = \left(b(-k)^{\frac{1}{2}}+kc\right)\cdot dx$$

$$= kc\left(-e^{\beta x}+1\right)\cdot dx = (-k)\cdot c\cdot R(x)\cdot dx \geq 0 \quad (ds > 0) ;$$

$$\beta_1\cdot ds = -b\cdot dx + kc\cdot dy = \left(-b+kc\cdot\left(-1/(-k)^{\frac{1}{2}}\right)\right)\cdot dx = \left(-b+c(-k)^{\frac{1}{2}}\right)\cdot dx$$

$$= -c(-k)^{\frac{1}{2}}\cdot R(x)\cdot dx ;$$

$$\gamma_1\cdot ds = \left(-b\cdot v_1+c\cdot v_2\right)\cdot ds = -b\cdot dy - c\cdot dx = -\left(b\left(-1/(-k)^{\frac{1}{2}}\right)+c\right)\cdot dx$$

$$= -\left(-e^{\beta\cdot x}+1\right)\cdot c\cdot dx = c\cdot R(x)\cdot dx \geq 0 ;$$

$$\beta_1^2 - \alpha_1\cdot\gamma_1 = 0 \quad \text{since} \quad k\cdot v_1^2 + v_2^2 = 0 \quad \text{on} \quad \Gamma_1 .$$

Therefore

$$(10) \qquad J_5^{(1)} = \int_{\Gamma_1} \left((-k)^{\frac{1}{2}}\cdot u_x - u_y\right)^2\cdot c\cdot R(x)\cdot dx \geq 0 . \qquad \square$$

LEMMA 2.

$$(11) \qquad J_5^{(2)} = \int_{\Gamma_0\cup\Gamma_2} Q\left(u_x, u_y\right)\cdot ds \geq 0 ,$$

where $Q = Q\left(u_x, u_y\right)$ *is defined as in Lemma* 1*, such that* $R_4(x, y)|_{\Gamma_2} \geq 0$.

Proof. By (4) we get $du = u_x dx + u_y dy$, and $u_x = N\cdot v_1$, $u_y = N\cdot v_2$, where N is a normalizing factor. By substituting these expressions in $Q\left(u_x, u_y\right)$ we obtain

$$(12) \quad Q = \left[\left(kb\cdot v_1 - kc\cdot v_2\right)\cdot v_1^2 + 2\left(b\cdot v_2 + kc\cdot v_1\right)\cdot v_1 v_2 + \left(-b\cdot v_1 + c\cdot v_2\right)\cdot v_2^2\right]\cdot N^2$$

$$= N^2\cdot\left(b\cdot v_1 + c\cdot v_2\right)\cdot\left\{k\cdot v_1^2 + v_2^2\right\} \geq 0$$

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on $\Gamma_0 \cup \Gamma_2$, because $b = c = 0$ on Γ_0 , and

$$(b\cdot v_1+c\cdot v_2)\cdot\left\{k\cdot v_1^2+v_2^2\right\}\Big|_{\Gamma_2}$$

$$= c\cdot\left[(-k)^{\frac{1}{2}}\cdot e^{\beta x}\cdot v_1+v_2\right]\cdot\left[k\cdot v_1^2+v_2^2\right]\Big|_{\Gamma_2} = c\cdot R_4(x,\ y)\big|_{\Gamma_2} \geq 0 ,$$

by hypothesis. □

LEMMA 3.

$$(13) \qquad J_4 = \int_{\partial G}\left[2a\left(k\cdot u_x\cdot v_1+u_y\cdot v_2\right)\cdot u-\left(ka_x\cdot v_1+a_y\cdot v_2\right)\cdot u^2\right]\cdot ds \geq 0 ,$$

where $a = a(x,\ y) \in C^2(\overline{G})$ *is a given negative function of the independent variables* $x,\ y \in R$ *, such that* $R_3(x,\ y)\big|_{\Gamma_1} \geq 0$.

Proof. Condition (4) implies

$$J_4 = \int_{\Gamma_1}\left[2a\left(k\cdot u_x\cdot v_1+u_y\cdot v_2\right)\cdot u-\left(ka_x\cdot v_1+a_y\cdot v_2\right)\cdot u^2\right]\cdot ds$$

$$= \int_{\Gamma_1}\left[2a(-k)^{\frac{1}{2}}udu-(-k)^{\frac{1}{2}}\cdot u^2\cdot da\right] = -\int_{\Gamma_1}\left\{a_x(-k)^{\frac{1}{2}}-a_y+(ak'/-4k)\right\}\cdot u^2\cdot dx$$

$$= \int_{\Gamma_1} R_3(x,\ y)\cdot u^2\cdot dx \geq 0 ,$$

by hypothesis. □

LEMMA 4.

$$(14) \qquad J_3 = \int_{\partial G}\lambda\cdot\left(b\cdot v_1+c\cdot v_2\right)\cdot u^2\cdot ds \geq 0 ,$$

where $\lambda\big|_{\Gamma_1} \leq 0$.

Proof. Condition (4) implies

$$J_3 = \int_{\Gamma_1}\lambda\cdot\left(b\cdot v_1+c\cdot v_2\right)\cdot u^2\cdot ds = \int_{\Gamma_1}\lambda\cdot(bdy-cdx)\cdot u^2$$

$$= \int_{\Gamma_1}\lambda\cdot u^2\cdot\left(b\left(-1/(-k)^{\frac{1}{2}}\right)-c\right)\cdot dx = \int_{\Gamma_1}(-\lambda)\cdot c\cdot\left(e^{\beta x}+1\right)\cdot u^2\cdot dx \geq 0 ,$$

because $\lambda|_{\Gamma_1} \le 0$, by hypothesis. □

LEMMA 5.

(15) $$J_1 = \iint_G \{k\cdot a_{xx}+a_{yy}+2\lambda\cdot a-[(\lambda\cdot b)_x+(\lambda\cdot c)_y]\}\cdot u^2\cdot dxdy \ge 0 ,$$

if $k\cdot a_{xx} + a_{yy} + 2\lambda\cdot a \ge d_1 > 0$ *in* G_+ , *and* $R(x, y) \ge d_2 > 0$ *in* G_- . □

LEMMA 6.

(16) $$J_2^{(1)} = \iint_{G_+} \tilde{Q}(u_x, u_y)\cdot dxdy \ge 0 ,$$

where $\tilde{Q}(u_x, u_y) = \alpha_2\cdot u_x^2 + 2\beta_2\cdot u_x u_y + \gamma_2\cdot u_y^2$ *is a quadratic form with respect to* u_x, u_y , *where* $\alpha_2 = -2ak - k\cdot b_x + (ck)_y$, $\beta_2 = -(kc_x+b_y)$, $\gamma_2 = -2a + b_x - c_y$.

Proof. Condition (6) implies that $\tilde{Q} = 2(-a)\cdot\{k\cdot u_x^2+u_y^2\} \ge 0$ in G_+ .

LEMMA 7.

(17) $$J_2^2 = \iint_{G_-} \tilde{Q}(u_x, u_y)\cdot dxdy \ge 0 ,$$

if $\tilde{Q}$ *is defined as in Lemma* 6, *and if conditions* $R^*(x, y) \ge 0$, $V(x, y) \le 0$, $R(x) > 0$, $R_1(x, y) \ge d_3 > 0$, $R_2(x, y) \ge d_4 > 0$, *and* $\lim_{y\to 0-} (k/k) = 0$, *hold in* G_- .

Proof. From (6), and by differentiation with respect to x and y , we find

$$c_x = 4(k/k')\cdot a_x ,$$

$$c_y = 4[a_y\cdot(k/k')+a\cdot(k/k')'] ,$$

$$b_x = (-k)^{\frac{1}{2}}(c_x+\beta\cdot c)\cdot e^{\beta x} = 4(-k)^{\frac{1}{2}}\cdot(k/k')\cdot(a_x+\beta\cdot a)\cdot e^{\beta x} ,$$

$$b_y = \{c_y\cdot(-k)^{-\frac{1}{2}}+c\cdot(-k'/2\cdot(-k)^{\frac{1}{2}})\}\cdot e^{\beta x}$$
$$= 4\cdot(-k)^{\frac{1}{2}}(a_y\cdot(k/k')+a\cdot(k/k')'+(\tfrac{1}{2})\cdot a)\cdot e^{\beta x} ,$$

$$\alpha_2 = 2ak \cdot F(y) - 4 \cdot \left[\left(\beta \cdot a + a_x\right) \cdot (-k)^{\frac{1}{2}} \cdot e^{\beta x} - a_y\right] \cdot \left(k^2/k'\right)$$
$$= 2\left[ak \cdot F(y) + 2 \cdot R^*(x,\ y) \cdot \left(k^2/k'\right)\right] \geq 0 \ ,$$

by hypothesis $\left(R_1(x,\ y) \geq d_3 > 0\right)$,

$$\beta_2 = -2\left[2a_x \cdot \left(k^2/k'\right) + \left(a \cdot F(y) + 2a_y \cdot (k/k')\right) \cdot (-k)^{\frac{1}{2}} \cdot e^{\beta x}\right] \ ,$$

$$\gamma_2 = 2\left[(-a) \cdot F(y) + 2R^*(x,\ y) \cdot (-k/k')\right] \geq 0 \ ,$$

by hypothesis $\left(R_2(x,\ y) \geq d_4 > 0\right)$,

$$\Delta = \left\{\beta_2^2 - \alpha_2\gamma_2\right\}/4 = \left(1-e^{2\beta \cdot x}\right) \cdot a^2 \cdot k \cdot F^2(y) + 4\left[\left(1-e^{2\beta \cdot x}\right) \cdot a_y - \beta \cdot a \cdot (-k)^{\frac{1}{2}} \cdot e^{\beta \cdot x}\right]$$
$$\cdot a \cdot \left(k^2/k'\right) \cdot F(y) + 4\left\{\left[\left[\left(\beta \cdot a + a_x\right) \cdot e^{\beta x}\right]^2 - \left(a_x\right)^2\right] \cdot k + \left(1-e^{2\beta \cdot x}\right) \cdot \left(a_y\right)^2\right.$$
$$\left. -2\beta \cdot e^{\beta x} \cdot a \cdot a_y \cdot (-k)^{\frac{1}{2}}\right\} \cdot \left(k^3/(k')^2\right) \leq 0 \ ,$$

because $V(x,\ y) \leq 0$ in G_- , and $B^2 - 4AC \geq 0$ always in G_- .

From hypotheses, $V(x,\ y) \leq 0$, $R_1(x,\ y) \geq d_3 > 0$, $R_2(x,\ y) \geq d_4 > 0$, implies that there exist two constants $d_0 < 0$ and $d^0 > 0$, such that $d_0 \leq F(y) \leq d^0$ in G_- .

Lemmas 1 to 7 imply the required result.

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