SERIAL RINGS WITH KRULL DIMENSION

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1. Introduction. A module is said to be serial if it has a unique chain of submodules, and a ring is serial if it is a direct sum of serial right ideals and a direct sum of serial left ideals. The serial rings of Krull dimension 0 are the Artinian serial (or generalised uniserial) rings studied by Nakayama and for which there is an extensive theory (see for example [**4**]). Warfield in [**10**] extended the theory to the non-Artinian case. In particular he showed that a Noetherian serial ring is a direct sum of Artinian serial rings and prime Noetherian serial rings, and he gave a structure theorem in the prime Noetherian case. A Noetherian non-Artinian serial ring has Krull dimension 1. Serial rings of arbitrary Krull dimension have been studied by Wright ([**9**], [**12**], [**13**], [**14**]) with special results being proved when the Krull dimension is 1 or 2.

In this paper, we extend some of these results to serial rings of arbitrary Krull dimension. The methods used rely heavily on making use of weak chain conditions. In Section 3, it is shown that certain indecomposable serial rings with Krull dimension are prime or can be written as triangular matrices, extending results of Singh, Warfield and Wright. A structure theorem for prime serial rings of finite Krull dimension is given in Section 4, which generalises results of Warfield and Wright when the Krull dimension is 1 or 2.

2. Background material and conventions. All rings considered here are associative with identity element. We shall use N or $N(R)$ to denote the nil radical of a ring R, and J or $J(R)$ to denote the Jacobson radical of R. The set of elements regular modulo an ideal I will be denoted by $C(I)$. We refer to [**3**] and [**5**] for general material on ring theory and Krull dimension.

Let R be a ring. As in [**10**], an R-module M is said to be serial if for any submodules A and B of M we have $A \subseteq B$ or $B \subseteq A$. The ring R is said to be serial if it is the direct sum of serial right ideals and is also the direct sum of serial left ideals. Such rings are discussed in [**10**] and Chapter 6 of [**3**]. Serial rings with Krull dimension have been studied in [**9**], [**12**], [**13**], [**14**]. It is shown in Theorem 6 of [**14**] that the left and right Krull dimensions of a serial ring are equal. A serial ring R is Noetherian if and only if $\bigcap J^n = 0$, and under these conditions $\mathrm{Kdim}(R) \leqslant 1$ [**10**, Theorem 5.11].

Let R be a serial ring with Krull dimension. Then N is nilpotent by Theorem 5 of [**6**]. Also R/N is a serial semi-prime Goldie ring [**5**, Theorem 5.1] in which every finitely-generated one-sided ideal is projective [**10**, Theorem 4.1]. Therefore R/N is a direct sum of prime rings by Theorem 4.3 of [**7**]. Also R has an Artinian quotient ring if and only if R satisfies the ascending chain condition for right annihilators [**11**, Theorem 5].

Let R be an indecomposable non-singular serial ring. Then R has a two-sided quotient ring which is a blocked triangular matrix ring over a division ring D [**10**, Theorem 4.1]. The corresponding full matrix ring over D is both the maximal left and right quotient ring of R. Hence every complement one-sided ideal of R is generated by an idempotent [**1**, Theorems 5.2 and 2.4]. The structure of R can now be described

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completely in terms of blocked triangular matrix rings over prime rings as in Theorem 6.10 of [1]. We shall use this in Corollary 4.3 to give the structure of non-singular serial rings of finite Krull dimension.

3. Certain serial rings are prime. We shall establish some sufficient conditions for a serial ring with Krull dimension to be prime. The first result of this type was Warfield's theorem that an indecomposable Noetherian non-Artinian serial ring R is prime (in this case $\mathrm{Kdim}(R)=1$). A result of this type when $\mathrm{Kdim}(R)=2$ was proved by Wright in Proposition 2.7 of [12].

Lemma 3.1. *Let R be a serial ring with Krull dimension and let K be a non-zero serial right R-module with $KN=0$. Then there is a unique minimal prime ideal P of R such that $KP=0$. Also $KQ=K$ for every minimal prime ideal Q of R with $Q\neq P$.*

Proof. We know that $KN=0$ and that N contains a product of minimal primes (because R/N is a Goldie ring). Also $K\neq 0$. Therefore there is a minimal prime ideal P with $KP\neq K$. Let Q be a minimal prime ideal of R with $Q\neq P$. Because R/N is a direct sum of prime rings, we have $P+Q=R$. Hence $K=KP+KQ$. But $KP\neq K$ and K is serial. Therefore $K=KQ$. Let $Q_1,\ldots,Q_n$, P be the distinct minimal primes of R. Then $KQ_i=K$ for all i and $Q_1Q_2\ldots Q_nP\subseteq N$. Therefore $0=KN=KQ_1\ldots Q_nP=KP$.

Lemma 3.2. *Let R be a serial ring and let K be a semi-prime ideal of R such that R/K is a Goldie ring and K contains no non-zero idempotent elements of R. Then $K=Kc=cK$ for all $c\in C(K)$.*

Proof. This is based on Remark (6) at the end of Chapter 6 of [3]. Let $c\in C(K)$. By Theorem 3.3 of [10], there are non-zero orthogonal idempotents $e_1,\ldots,e_n$ of R adding to 1 such that each e_iR is a serial right R-module and $cR=\sum_i(e_iR\cap cR)$. We fix an integer j with $1\leq j\leq n$. We have $e_j\notin K$. Also $(cR+K)/K$ is an essential right ideal of R/K. Hence $e_jR\cap(cR+K)$ is not contained in K. Because e_jR is serial, it follows that $e_jR\cap(cR+K)\supsetneqq e_jK$. But

$$e_jR\cap(cR+K)=e_jR\cap\left(\sum_i(e_iR\cap cR)+\sum_i e_iK\right)=e_jR\cap cR+e_jK.$$

Hence $e_jR\cap cR+e_jK\supsetneqq e_jK$. Because e_jR is serial, it follows that $e_jK\subseteq e_jR\cap cR$. Hence $e_jK\subseteq cR$ for all j and $e_1+\ldots+e_n=1$. Therefore $K\subseteq cR$. Because $c\in C(K)$, we have $cR\cap K=cK$. Therefore $K=cK$.

Theorem 3.3. *Let R be an indecomposable serial ring with Krull dimension and suppose that R has a non-nilpotent ideal X such that $\bigcap_{n=1}^{\infty}X^n=0$. Then R is a prime ring.*

Proof. We fix a minimal prime ideal P of R such that P does not contain X. We wish to show that $N=NP$. Let e be an idempotent element of R such that eR is serial. It is enough to show that $eN=eNP$, and this is trivial if $eN=0$. Suppose that $eN\neq 0$ and that $eN\neq eNP$. By taking $K=eN/eN^2$ in Lemma 3.1, we see that $eNP=eN^2$. Because X is not contained in P, there is no minimal prime of R which contains $X+P$. Therefore

$c \in X + P$ for some $c \in C(N)$. But $Nc = N$ by Lemma 3.2. Hence

$$eN = eNc = eN(X + P) = eNX + eNP = eNX + eN^2.$$

Therefore $eN = eNX = eNX^2 = \ldots$. This is a contradiction because $eN \neq 0$ and $\cap X^n = 0$.

Thus $N = NP$. Because R/N is a direct sum of prime rings, we know that P/N is generated by a central idempotent of R/N. Hence there is an idempotent element f of R such that $P = fR + N = Rf + N$. Thus

$$N = NP = Nf + N^2 = Nf + N(Nf + N^2) = Nf + N^3 = \ldots$$

Hence $N = Nf$ and $P = Rf + N = Rf + Nf = Rf$. By symmetry, we have $P = Rf = fR$. It follows that f is a central idempotent element of R. Therefore $P = 0$ because R is indecomposable.

Corollary 3.4 (Warfield). *Let R be a serial ring. Then $R\Big/\left(\bigcap_{n=1}^{\infty} J^n\right)$ is a direct sum of Artinian rings and prime rings.*

Proof. Without loss of generality, we may suppose that $\bigcap J^n = 0$ and that R is indecomposable. Either J is nilpotent and R is Artinian, or J is not nilpotent and R is prime by Theorem 3.3.

Corollary 3.5 (Wright [**12**, Proposition 2.7]). *Let R be an indecomposable serial ring with Krull dimension. Set $J_1 = \bigcap_{n=1}^{\infty} J^n$. Suppose that J_1 is not nilpotent and that $\bigcap_{n=1}^{\infty} J_1^n = 0$. Then R is a prime ring.*

Theorem 3.6. *Let R be an indecomposable serial ring with Krull dimension. Suppose that R has an Artinian quotient ring and that* $\mathrm{Kdim}(N) < \mathrm{Kdim}(R)$ *as right R-modules. Then there is a unique minimal prime ideal P of R such that* $\mathrm{Kdim}(R/P) = \mathrm{Kdim}(R)$. *Also $P = Rf$ for some idempotent element f of R.*

Proof. Because $\mathrm{Kdim}(R/N) = \mathrm{Kdim}(R)$, there is a minimal prime P of R such that $\mathrm{Kdim}(R/P) = \mathrm{Kdim}(R)$. We shall show that $N = NP$. Suppose that e is an idempotent element of R such that eN is a non-zero serial right R-module and $eNP = eN^2$; we shall obtain a contradiction. We have $\mathrm{Kdim}(eN/eNP) \leqslant \mathrm{Kdim}(N) < \mathrm{Kdim}(R) = \mathrm{Kdim}(R/P)$. Let x be a non-zero element of eN/eNP. Then $\mathrm{Kdim}(xR) < \mathrm{Kdim}(R/P)$. Therefore xR is torsion as a right R/P-module. Hence $xc = 0$ for some $c \in C(P)$. Let $y \in eN$ with $y + eNP = x$. Then $y(cR + P) \subseteq eN^2$. But $d \in cR + P$ for some $d \in C(N)$. Thus $yd \in eN^2$. But $N = Nd$ by Lemma 3.2 or [**11**, Theorem 5]. Thus $yd \in eN^2d$. Because R has an Artinian quotient ring, we know that d is a regular element of R. Therefore $y \in eN^2$, i.e. $x = 0$. This is a contradiction. It follows, as in the proof of Theorem 3.3, that $N = NP$ and that $P = Rf$ for some idempotent f.

Suppose that Q is a minimal prime of R with $Q \neq P$ and $\mathrm{Kdim}(R/Q) = \mathrm{Kdim}(R)$. Then $N = NQ$ and $Q = Rg$ for some idempotent g. We have $g \notin P$, i.e. $gR \neq gP$. By Lemma 3.1 with $K = gR/gN$, we have $gN = gP$. Thus $QP = RgP = RgN \subseteq N$. Hence $N^2 \supseteq NQP = NP = N$. Therefore $N = 0$. But R is indecomposable. Therefore R is prime. This is a contradiction because $P \neq Q$.

Corollary 3.7. *Let R be an indecomposable serial ring with Krull dimension. Suppose that R has an Artinian quotient ring and that* $\mathrm{Kdim}(N) < \mathrm{Kdim}(R)$ *as both left R-modules and right R-modules. Then R is a prime ring.*

Corollary 3.8 (Singh [8, Theorem 2.11]). *Let R be an indecomposable serial right Noetherian ring which is neither prime nor right Artinian. Then*

$$R \cong \begin{bmatrix} S & M \\ 0 & T \end{bmatrix},$$

where S is a prime Noetherian serial ring, T is an Artinian serial ring, and M is an S-T-bimodule.

Proof. We know that R has an Artinian quotient ring ([11, Theorem 5] or [3, Theorem 6.10]). Also $Nc = N$ for every regular element c of R (Lemma 3.2 or [11, Theorem 5]). But N is finitely-generated as a right ideal. Therefore N is Artinian as a right R-module (see for example the proofs of [2, Lemma (A)] or [3, Lemma 5.2]). Thus $\mathrm{Kdim}(N) = 0$ and $\mathrm{Kdim}(R) = 1$ as right R-modules. Let $P = Rf$ as in Theorem 3.6. Then

$$R \cong \begin{bmatrix} S & M \\ 0 & T \end{bmatrix},$$

where $S \cong R/P$ and $T \cong fRf$. The minimal primes of T correspond to minimal primes Q of R with $Q \neq P$. But R/Q is Artinian for every such Q by Theorem 3.6. It follows that T is Artinian.

4. Prime serial rings of finite Krull dimension. We shall determine the structure of prime serial rings with finite Krull dimension in terms of blocked matrices over integral domains. The cases in which the Krull dimension is 1 or 2 were done by Warfield [10, Theorem 5.14] and Wright [12, Theorem 2.11]. The proof will be module-theoretic and will require that the domain of definition of certain homomorphisms can be extended. The following lemma gives enough injectivity for this purpose and was suggested by Theorem 1.4 of [4].

Lemma 4.1. *Let R be a serial ring with primitive idempotents e and f. Let $x \in eR$, and let $a: xR \to fR$ be a homomorphism of right R-modules. Then one at least of the following statements is true.*

(1) *There is a homomorphism $b: eR \to fR$ with $b(x) = a(x)$, or*

(2) *there is a homomorphism $b: fR \to eR$ with $ba(x) = x$.*

Proof. There are orthogonal primitive idempotents $g_1, \ldots, g_n$ of R adding to 1 such that each Rg_i is a serial left R-module. We have $xR = xg_1R + \ldots + xg_nR$. Because eR is serial, we have $xR = xg_iR$ for some i. Thus $xR = xgR$ for some primitive idempotent g. Set $w = a(xg) = a(x)g$. Because Rg is serial, we have $Rxg \subseteq Rw$ or $Rw \subseteq Rxg$.

Case (a). Suppose that $Rxg \subseteq Rw$. Then $xg = rw$ for some $r \in R$. But $x = ex$ and $w = fw$. Hence, without loss of generality, we may suppose that $r = erf$. Define $b: fR \to eR$ by $b(fs) = rs$ for all $s \in R$. Then $ba(xg) = b(w) = rw = xg$. But $xR = xgR$. Hence $x = xgt$ for some $t \in R$. Therefore $ba(x) = ba(xgt) = (ba(xg))t = xgt = x$.

Case (b). Suppose that $Rw \subseteq Rxg$. Then $w = rxg$ for some $r \in R$ with $r = fre$. Define $b: eR \to fR$ by $b(es) = rs$ for all $s \in R$. Then $b(xg) = rxg = w = a(xg)$. As in (a), it follows that $b(x) = a(x)$.

Theorem 4.2. *Let R be a prime serial ring with finite Krull dimension. Then there is a serial integral domain T such that either*

(1) *$R \cong M_n(T)$ for some positive integer n, or*

(2) *there is a positive integer $k \neq 1$ and sets H_{ij} for $1 \leq i, j \leq k$ such that R is isomorphic to the ring of k by k matrices with (i, j)-entries in H_{ij}, where*

(i) *if $i < j$ then H_{ij} is the set of all n_i by n_j matrices with entries in T for some positive integers n_i and n_j,*

(ii) *if $i > j$ then H_{ij} is the set of all n_i by n_j matrices over $J(T)$, and*

(iii) *each H_{ii} is a prime serial ring of finite Krull dimension and with smaller Goldie rank than that of R.*

Thus each H_{ii} can in turn be written as a matrix ring in the same way that R is, and so on.

Proof. We shall frequently use the well-known fact that a nonzero homomorphism between uniform right ideals of R is injective. Set $J = J(R)$. Suppose firstly that J is a prime ideal of R. Then all simple right R-modules are isomorphic. There are orthogonal idempotents $e_1, \ldots, e_n$ of R adding to 1 such that each e_iR is a serial right R-module. Each e_iR/e_iJ is a simple module. Therefore $e_iR/e_iJ \cong e_jR/e_jJ$ for all i and j. It is now routine to show that $e_iR \cong e_jR$ and that (1) holds with $T = \mathrm{End}_R(e_1R)$.

For the remainder of the proof, we suppose that J is not a prime ideal of R. Set $J_0 = J$ and $J_n = \bigcap_{i=1}^{\infty} J_{n-1}^i$ for every positive integer n. Because R has finite Krull dimension, we have $J_h = 0$ for some h [**14**, Theorem 6]. For each i, let K_i be the ideal of R containing J_i such that $K_i/J_i = N(R/J_i)$. Each K_i contains no non-zero idempotent element of R. We have $K_0 = J$ and $K_h = 0$. Thus K_0 is not prime and K_h is prime. Let p be the largest integer such that K_p is not a prime ideal. Set $K = K_p$ and $L = K_{p+1}$. These meanings for K and L will be retained for the rest of the proof.

Set $I = \bigcap_{i=1}^{\infty} K^i$ and let e be a primitive idempotent of R. We shall show that $I = L$. Because K is nilpotent modulo J_p we have $I = \bigcap_{i=1}^{\infty} K_p^i = \bigcap J_p^i = J_{p+1} \subseteq L$. Also $L^r \subseteq J_{p+1}$ for some non-negative integer r. Hence $L^r \subseteq J_p \subseteq K$. Therefore $L \subseteq K$. But L is prime and K is not. Therefore $L \subsetneq K$. Let i be a positive integer. Then eK^i is not contained in the prime ideal L. Because eR is serial it follows that $eK^i \supsetneq eL$. Because 1 is the sum of such idempotents e, we have $L \subsetneq K^i$. Hence $L \subseteq I$. Therefore $L = I = J_{p+1}$. From "$eK^i \supsetneq eL$", it also follows that $eK^i \neq eK^j$ if $i \neq j$.

The reason for choosing K to be semi-prime but not prime is that we take k in (2) to be the number of prime ideals of R minimal over K.

Set $C = C(K)$. We shall show that C is an Ore set in R. Let $a \in R$ and $c \in C$. Because R/K is a semi-prime Goldie ring we have $ad = cb + u$ for some $d \in C$, $b \in R$, $u \in K$. By Lemma 3.2, we have $u = cv$ for some $v \in K$. Thus $ad = c(b + v)$. Therefore C is an Ore set in R, and the elements of C are regular in R. Let S be the partial quotient ring of R with respect to C. Then $K = J(S)$ and L is a prime ideal of S. Because $\cap K^i = L$ it follows that S/L is a prime Noetherian serial ring.

Let e be a primitive idempotent of R. Then eR is a serial R-module. Hence eS is serial as a right S-module. For each i, set $M_i = \mathrm{ann}_S(eK^i/eK^{i+1})$. Because S/L is a prime Noetherian serial ring and $eK^i \supseteq eL$ for all i, it follows, as in the proof of [**10**, Theorem 5.14], that the sequence $M_0, M_1, M_2, \ldots$ starts by running through the distinct maximal ideals of S and then repeats itself. The only way in which e influences this sequence is in determining its starting point. In R, this means that: the primes of R minimal over K are precisely the ideals of the form $\mathrm{ann}_R(eK^i/eK^{i+1})$ for some non-negative integer i; $\mathrm{ann}_R(eK^i/eK^{i+1}) = \mathrm{ann}_R(eK^j/eK^{j+1})$ if and only if $i \equiv j \bmod(k)$, where k is the number of primes of R minimal over K; $\mathrm{ann}(eK^i/eK^{i+1})$ is determined by $\mathrm{ann}(eK^{i-1}/eK^i)$ and not by e.

Notation. From now on, g denotes a fixed primitive idempotent element of R; P is a fixed prime ideal of R minimal over K such that $gR \neq gP$, i.e. $gK = gP$ (Lemma 3.1); $T = \mathrm{End}_R(gK)$.

We shall show that $T \cong \mathrm{End}_S(gS)$. Recall that $K = J(S)$. Let $a \in T$. For $x \in gK$ and $c \in C(K)$, we have $a(xc^{-1})c = a(xc^{-1}c) = a(x)$, i.e. $a(xc^{-1}) = a(x)c^{-1}$. Thus $a : gK \to gK$ is a right S-module homomorphism. Because gS/gL is a cyclic serial module over the Noetherian ring S/L, we have $gK = yS$ for some $y \in S$. We shall use Lemma 4.1 to show that a can be extended to an element of $\mathrm{End}_S(gS)$. Suppose that there is a right S-module homomorphism $b : gS \to gS$ such that $b(y) = y$. Then $b(gK) \supseteq b(a(gK)) = ba(yS) = yS = gK$. Hence $b(gK) = gK$ and $b(gS) = gS$. Thus b is an automorphism of gS and its inverse is an extension of a. If no such b exists then a can be extended to an element of $\mathrm{End}_S(gS)$ by Lemma 4.1. It follows that $T \cong \mathrm{End}_S(gK) \cong \mathrm{End}_S(gS)$.

From now on, we identify T with $\mathrm{End}_S(gS)$. We have $J(T) = \{a \in T : a(gS) \subseteq gK\}$. Let $a \in T$ with $a \neq 0$. Then $a(gS)/a(gK) \cong gS/gK$, and $gS/gK \cong gK^i/gK^{i+1}$ if and only if i is divisible by k. Hence $J(T) = \{a \in T : a(gS) \subseteq gK^k\} = \{a \in T : a(gK) \subseteq gK^{k+1}\}$.

Let e be a primitive idempotent element of R. Because L is prime, we have $eRg \nsubseteq L = \bigcap_{i=1}^{\infty} K^i$. Let i be the smallest positive integer such that $eRg \nsubseteq K^i$, i.e. such that there is a right R-module homomorphism $u : gR \to eR$ with $u(gR) \supsetneqq eK^i$. These meanings for i and u will be fixed until further notice. We have $eK^{i-1} \supseteq u(gR) \supsetneqq eK^i$. Hence $u(gP) = u(gK) \subseteq eK^i$. Let Q be any prime of R minimal over K with $Q \neq P$. Then $P + Q = R$ and $eK^i \subsetneqq u(gR)P + u(gR)Q \subseteq eK^i + u(gR)Q$. Hence $eK^i \subsetneqq u(gR)Q \subseteq eK^{i-1}Q$. It follows from Lemma 3.1 that $eK^{i-1}P = eK^i$. Thus $P = \mathrm{ann}_R(eK^{i-1}/eK^i)$. Now set $Q = \mathrm{ann}_R(eK^i/eK^{i+1})$. Then $Q \neq P$ and $eK^i = eK^iP + eK^iQ \subseteq u(gR)P + eK^{i+1} \subseteq u(gP) + u(gR)K \subseteq eK^i$. Therefore $u(gK) = eK^i$. Let s be a positive integer such that $\mathrm{ann}_R(eK^{s-1}/eK^s) = P$. Then $eK^{s-1}g$ is not contained in eK^s because $g \notin P$. The minimality of i gives $s \geqslant i$. It follows from the periodicity of the sequence of ideals $\mathrm{ann}(eK^j/eK^{j+1})$ that $i \leqslant k$.

Let f also be a primitive idempotent of R and let j and v be determined by f in the same way that i and u were determined by e. Set $H = \mathrm{Hom}_R(eR, fR)$. The aim is to identify H as either T or $J(T)$ according to the relative sizes of i and j. Let $h \in H$. Because $hu(gR) \subseteq fR$, we have $hu(gR) \subseteq fK^{j-1}$ by definition of j. Hence $hu(gK) \subseteq fK^j = v(gK)$. Thus h induces an element $v^{-1}hu$ of $\mathrm{End}_R(gK)$. We shall now investigate the function from H to T which sends h to $v^{-1}hu$.

Case (a). Suppose that $i > j$. Let $h \in H$. Then $h(eK^{i-1}) = h(eR)K^{i-1} \subseteq fK^{i-1}$. Set $Q = \mathrm{ann}_R(fK^{i-1}/fK^i)$. We have $1 \leqslant j < i \leqslant k$. Hence $j < i < k + j$. Therefore $i \not\equiv j \bmod(k)$, i.e. $\mathrm{ann}_R(fK^{i-1}/fK^i) \neq \mathrm{ann}_R(fK^{j-1}/fK^j)$, i.e. $Q \neq P$. Hence $P + Q = R$ and $h(eK^{i-1}) = h(eK^{i-1}P) + h(eK^{i-1}Q) = h(eK^i) + h(eK^{i-1})Q \subseteq fK^i + fK^{i-1}Q = fK^i$. If $i + 1 < k + j$, we can repeat the argument and obtain $h(eK^{i-1}) \subseteq fK^{i+1}$. After a finite number of steps, we have $h(eK^{i-1}) \subseteq fK^{k+j-1}$. Hence $h(eK^i) \subseteq fK^{k+j}$, i.e. $hu(gK) \subseteq v(gK)K^k$, i.e. $v^{-1}hu(gK) \subseteq gK^{k+1}$, i.e. $v^{-1}hu \in J(T)$. Now let $t \in J(T)$. We shall show that $t = v^{-1}hu$ for some $h \in H$. We have $t(gK) \subseteq gK^{k+1}$. Because $eK^i = u(gK)$, it makes sense to set

$a(x)=vtu^{-1}(x)$ for all $x \in eK^i$. We have $a(eK^i)=vt(gK)\subseteq v(gK^{k+1})=v(gK)K^k=fK^{k+j}$. Thus $a:eK^i \to fK^{k+j}$ is a right R-module homomorphism. We fix $x \in eK^i$ with $x \notin eK^{i+1}$. Let w be the restriction of a to xR. Because R is a non-singular ring, any extension of w to an element of H will also be an extension of a. Suppose that there is a homomorphism $b:fR\to eR$ with $bw(x)=x$. We have $eK^{i+1}\subsetneq xR=bw(xR)=ba(xR)\subseteq ba(eK^i)\subseteq b(fK^{k+j})=b(fR)K^{k+j}\subseteq eK^{k+j}$. Hence $eK^{i+1}\subsetneq eK^{k+j}$, i.e. $i+1>k+j$. This is a contradiction because $k\geqslant i$ and $j\geqslant 1$. Hence there is no such b. Therefore w can be extended to an element h of H, by Lemma 4.1. It follows that h is an extension of a and that $t=v^{-1}au=v^{-1}hu$. Thus $h\to v^{-1}hu$ gives a bijective function between H and $J(T)$. We used w rather than a above because the domain of a may not be a cyclic submodule of eR.

Case (b). Suppose that $i<j$. We shall identify H with T. Let $t\in T$. As above set $a=vtu^{-1}:eK^i\to fK^j$, fix $x\in eK^i$ with $x\notin eK^{i+1}$, and let w be the restriction of a to xR. Suppose that there is a homomorphism $b:fR\to eR$ with $bw(x)=x$. Then $eK^{i+1}\subsetneq xR=bw(xR)=ba(xR)\subseteq b(fK^j)\subseteq eK^j$. Hence $i+1>j$, which is a contradiction. Therefore no such b exists, and w can be extended to an element h of H by Lemma 4.1. We have $t=v^{-1}hu$ and we can identify H with T via $h\to v^{-1}hu$.

We do not wish to consider the case $i=j$.

We drop the special meanings for e, i, u, f, j, v.

Let $e_1,\ldots,e_n$ be orthogonal primitive idempotents of R adding to 1. For each j, let $i(j)$ be the smallest positive integer s such that $P=\operatorname{ann}_R(e_jK^{s-1}/e_jK^s)$. From above, we have $i(j)\leqslant k$. Suppose that there is a non-negative integer i with $K^iP=K^i$. Then for every primitive idempotent e and every integer $j\geqslant i$ we have $K^jP=K^j$ and $P\neq \operatorname{ann}(eK^j/eK^{j+1})$; this is a contradiction. Hence, for each integer s with $1\leqslant s\leqslant k$, there is an integer j with $1\leqslant j\leqslant n$ such that $e_jK^{s-1}P\neq e_jK^{s-1}$, i.e. $P=\operatorname{ann}(e_jK^{s-1}/e_jK^s)$, i.e. $i(j)=s$. We can arrange the numbering so that $i(1)=i(2)=\ldots=i(j_k)=k$, $i(j_k+1)=\ldots=i(j_k+j_{k-1})=k-1,\ldots,\ldots=i(n)=1$. For $1\leqslant s$, $t\leqslant n$, set $H_{st}=\operatorname{Hom}_R(e_tR, e_sR)$. Then R is isomorphic to the ring of n by n matrices with (s,t)-entries in H_{st}. If $i(s)<i(t)$ then we can identify H_{st} with $J(T)$ as in case (a). If $i(s)>i(t)$ then we can take $H_{st}=T$ by case (b).

COROLLARY 4.3. *Let R be an indecomposable non-singular serial ring with finite Krull dimension. Then there is a division ring D such that R is isomorphic to a blocked upper triangular matrix ring in which the above-diagonal blocks are full sets of matrices over D, and each diagonal block is a ring as described in Theorem* 4.2 *which has a full matrix ring over D as its quotient ring.*

Proof. This follows from Theorem 4.2 and the last paragraph of Section 2 combined with [**1**, Theorem 6.10 and Corollary 3.3].

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