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UNIQUENESS THEOREMS FOR MAPPINGS OF FINITE DISTORTION

ENRIQUE VILLAMOR

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Abstract

In this paper we prove uniqueness theorems for mappings $F \in W^{1,n}_{\text{loc}}(\mathbb{B}^n; \mathbb{R}^n)$ of finite distortion $1 \le K(x) = \|DF(x)\|^n / J_F(x)$ satisfying some integrability conditions. These types of theorems fundamentally state that if a mapping defined in $\mathbb{B}^n$ has the same boundary limit a on a 'relatively large' set $E \subset \partial\mathbb{B}^n$, then the mapping is constant. Here the size of the set E is measured in terms of its p-capacity or equivalently its Hausdorff dimension.

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1. Introduction

There have been a number of far-reaching generalizations of the famous original F. and M. Riesz's uniqueness theorem that states that if a bounded analytic function in the unit disc of the complex plane has the same radial limit in a set E of positive Lebesgue measure on its boundary, then the function has to be constant.

First Beurling (see [1, 17]), considering the case of nonconstant meromorphic functions mapping the unit disc on a Riemann surface of finite spherical area, was able to prove that if such a function showed an appropriate behavior in the neighborhood of the limit value where the function maps a set on the boundary of the unit disc, then that set has capacity zero. Here the capacity considered is the logarithmic linear capacity.

Carleson, in [3], proved that some condition on the limiting value must be required if we want E to be a set of uniqueness. He constructed a nonconstant function in the unit disc $\mathbb{B}^2$ having the same limiting value in a subset $E \subset \partial\mathbb{B}^2$ of positive capacity.

This led to the following question: what condition must we impose on the limiting value so that if the capacity of E is positive, then the function is constant?

The author of the present note, in [19], was able to weaken Beurling's condition on the limit value. Later Jenkins, in [10], showed that in the presence of such a local condition on the limit value, the global behavior of the image Riemann surface is irrelevant and at the same time he gave an improved and sharper condition.

Those results were quite restrictive in a twofold way. Namely, they were in dimension $n = 2$ and the regularity requirements on the functions under consideration were quite strong; analyticity and meromorphicity. Koskela, in [11], was able to remove those two restrictions by proving a uniqueness result for functions in $W^{1,p}(\mathbb{B}^n)$ (here $\mathbb{B}^n$ is the unit ball of $\mathbb{R}^n$) for values of p in the interval $(1, n]$ and satisfying a condition on the limit value very similar in nature to the one of Jenkins in two dimensions. In particular, Koskela's result recovers Jenkins' in the case $p = n = 2$. He proved that a continuous function u in the Sobolev space $W^{1,p}(\mathbb{B}^n)$, where $1 < p \leq n$, vanishes identically provided

$$I(\epsilon) = \int_{|u(x)-a|<\epsilon} |\nabla u(x)|^p\, dx = O\Big(\epsilon^p \Big(\log\Big(\frac{1}{\epsilon}\Big)\Big)^{p-1}\Big)$$

as $\epsilon \to 0$ and there is a set E on $\partial\mathbb{B}^n$ of positive p-capacity such that for each $x \in E$ and every locally rectifiable curve $\gamma \in \mathbb{B}^n$ ending at x, there is a sequence of points in γ ending at x for which the function u tends to a.

Observe that this is a very strong condition. In our results we are going to require a much weaker condition. Namely, we will require that for each $x \in E$ there are a curve $\gamma \subset \mathbb{B}^n$ ending at x and a sequence of points $\{b_k\}_{k=1}^{\infty} \subset \gamma$ tending to x with

$$\lim_{k\to\infty} u(b_k) = a.$$

Since we are going to be dealing with mappings for which each of their component functions satisfies a Lindelöf-type theorem, see [12, 13], it will follow that if our condition above holds at $x \in E$ with limit a, these Lindelöf-type theorems (see [12, Theorem 2, page 434] and [13, Theorem 2, page 404]) guarantee that, modulo sets of small p-capacity, for any other locally rectifiable curve $\gamma \subset \mathbb{B}^n$ ending at $x \in E$, either the limit exists and is equal to a or the limit does not exist. This allows us to show that our chosen metrics for the module problem $M_\alpha(\Delta(B_{j_0}, E; \mathbb{B}^n))$, where $\Delta(B_{j_0}, E; \mathbb{B}^n)$ consists of the family of curves in $\mathbb{B}^n$ joining B_{j_0}, an open ball in $\mathbb{B}^n$ and $E \subset \partial\mathbb{B}^n$, are admissible and that $M_\alpha(\Delta(B_{j_0}, E; \mathbb{B}^n)) = 0$.

It is well known, see [5], that without loss of generality we can assume that all the curves in $\Delta(B_{j_0}, E; \mathbb{B}^n)$ are locally rectifiable, since the set $\Delta_{NLR}(B_{j_0}, E; \mathbb{B}^n)$ of curves in $\Delta(B_{j_0}, E; \mathbb{B}^n)$ that are not locally rectifiable is an exceptional set for the module problem $M_\alpha(\Delta(B_{j_0}, E; \mathbb{B}^n))$, that is, $M_\alpha(\Delta_{NLR}(B_{j_0}, E; \mathbb{B}^n)) = 0$, and the subadditivity property of the module.

Afterwards, classical symmetrization results [20, Lemmas 4.2 and 4.3] and [12, Lemma 4.7] will allow us to conclude that the same holds for the module $M_\alpha(\Delta(B_{j_0}, E; \mathbb{R}^n)) = 0$ for the family of curves in $\mathbb{R}^n$ joining B_{j_0} and E. Our results will then follow from well-known equivalence results between α-modules and α-capacities; see [8, 22, 23].

In Mizuta [15] and Miklyukov and Vuorinen's [14] papers, they require the existence of fine boundary limits at every point $x \in E \subset \partial\mathbb{B}^n$, which is a much stronger condition than ours; see [2, 4].

Koskela [11] showed that his result is sharp in the sense that $(\log(1/\epsilon))^{p-1}$ cannot be replaced by $(\log(1/\epsilon))^{p-1+\delta}$ for some positive δ (even if u is assumed to be continuous in the closure of $\mathbb{B}^n$).

Mizuta, in [15], showed that under the same hypothesis on the function u as in [11], if

$$I(\epsilon) = O(\epsilon^p \phi(\epsilon))$$

as $\epsilon \to 0$, where ϕ is a positive nonincreasing function on the interval $(0, \infty)$ satisfying the following conditions:

$$A^{-1}\phi(r) \le \phi(r^2) \le A\phi(r)$$

for every $r > 0$ and A a positive constant and

$$\int_0^1 [\phi(r)]^{1/(1-p)} r^{-1}\, dr = \infty,$$

then if there is a set E on $\partial\mathbb{B}^n$ of positive p-capacity such that the same fine boundary limit a exists at each $x \in E$, then the function u is identically equal to a on $\mathbb{B}^n$. It is easy to observe that the function $\phi(\epsilon) = (\log(1/\epsilon))^{p-1}$ satisfies the two conditions in [15].

Lastly, Miklyukov and Vuorinen, in [14], showed that if the integral $I(\epsilon)$ satisfies one of the conditions

$$\int_0 \left(\frac{1}{I'(\epsilon)}\right)^{1/(p-1)} d\epsilon = \infty$$

or

$$\int_0 \left(\frac{\epsilon}{I(\epsilon)}\right)^{1/(p-1)} d\epsilon = \infty,$$

or there exists a nonnegative function $f(\epsilon)$ satisfying the conditions

$$I(\epsilon) \le \epsilon^p (f(\epsilon))^{p-1}$$

for every $0 < \epsilon < \frac{1}{2}$ and

$$\sum_{k=0}^{\infty} \frac{1}{f(2^{-k})} = \infty$$

or

$$\liminf_{\epsilon \to 0} \frac{I(\epsilon)}{\epsilon^p} < \infty,$$

then again the function u is identically equal to a. It is not difficult to show that this result generalizes the one in [15].

In this paper we are going to require a similar type of conditions, but for mappings rather than for functions. Our conditions will be on the integral $\int_{\|F(x)-a\|<\epsilon} J_F(x)^p\, dx$, where J_F is the Jacobian of the mapping F.

Under these conditions, we will be able to show that the mapping F is identically equal to a provided the set $E \subset \partial\mathbb{B}^n$, where for each $x \in E$ there are a curve $\gamma \subset \mathbb{B}^n$ ending at x and a sequence of points $\{b_k\}_{k=1}^{\infty} \subset \gamma$ tending to x with $\lim_{k\to\infty} F(b_k) = a$, has positive p-capacity.

As a consequence of that, we will show that those sets $E \subset \partial\mathbb{B}^n$ have 'small' Hausdorff dimension. Meaning that, if the mappings F are nonconstant, the size of the sets $E \subset \partial\mathbb{B}^n$ where F has the same weak limit a along a curve $\gamma \subset \mathbb{B}^n$ ending at $x \in E$ is 'small'.

Let us remark here that the results of Koskela [11], Mizuta [15] and Miklyukov and Vuorinen [14] are for real-valued functions, while our results are more in the spirit of the initial results of F. and M. Riesz, Beurling and Jenkins, where they considered mappings from the complex plane into the complex plane.

Let us remark also that although our results are established for mappings defined on the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$, they also hold for mappings defined on a sufficiently smooth open connected set Ω for which the corresponding Sobolev and capacity extension results hold.

2. Preliminary definitions and results

In this section we will present several definitions and known results that will be needed in the rest of this paper, as well as the main result of the paper.

Let us start by recalling the definition of monotone function (in this paper we consider only continuous monotone functions).

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set. A continuous function $u : \Omega \to \mathbb{R}$ is monotone (in the sense of Lebesgue) if

$$\max_{\bar{D}} u(x) = \max_{\partial D} u(x)$$

and

$$\min_{\bar{D}} u(x) = \min_{\partial D} u(x)$$

hold whenever D is a domain with compact closure $\bar{D} \subset \Omega$.

The Sobolev space $W^{1,p}(\mathbb{B}^n; \mathbb{R})$ consists of functions $u : \mathbb{B}^n \to \mathbb{R}$ that have first distributional gradient ∇u such that

$$\int_{\mathbb{B}^n} (|u(x)|^p + |\nabla u(x)|^p)\, dx < \infty.$$

For a domain $D \subset \mathbb{R}^n$ and a pair of disjoint, nonempty, compact sets $E, F \subset \bar{D}$, the triple (E, F, D) is called a condenser. Its p-capacity is defined by

$$\mathrm{cap}_p(E, F; D) = \inf_{u \in L} \int_D |\nabla u|^p\, dx,$$

where the infimum is taken over all the functions in the class

$$L = L(E, F; D) = \{u \in L^1_p(D) \cap C(D \cup E \cup F) : u|_E \le 0, u|_F \ge 1\}.$$

Here $L^1_p(D)$ denotes the Sobolev space of measurable functions $u : D \longmapsto \bar{\mathbb{R}}$ satisfying $\int_D |\nabla u|^p\, dx < \infty$. Recall that $W^{1,p}(D) = L^p(D) \cap L^1_p(D)$ and that $L^1_p(D) \subset W^{1,p}(D)$ if ∂D is sufficiently smooth. In particular, this is true for $\mathbb{B}^n$. For a general set

E and a compact F, we define the p-capacity as

$$\operatorname{cap}_p(E, F; D) = \sup_{K\subset E}\{\operatorname{cap}_p(K, F; D)\}.$$

We say that a compact set E is of p-capacity zero if $\operatorname{cap}_p(E, \partial D; D) = 0$ for some bounded (and hence for each bounded) domain D such that $E \subset \bar{D}$. This is the case (see [22, 23]) if the p-modulus of the family of curves in $\mathbb{R}^n$ joining E to the boundary of some open ball not intersecting E is zero.

Next, we will state slight variations of two Lindelöf-type theorems that were proved in [12, 13], respectively, and that we will need in the proofs of Theorems 2.5 and 4.2.

A close examination of the proofs of those theorems in [12, 13] shows that for each locally rectifiable curve $\gamma : [0, 1) \longmapsto \mathbb{B}^n$ ending at E (that is, $\lim_{t\to 1}\gamma(t)$ exists and is equal to b for some $b \in E$), we can assume that our mapping $F(x)$ is absolutely continuous on each closed subcurve of γ. In particular, in the proof of Theorem 2 in [12, 13] (it might be necessary in those proofs to take two subsequences of the two original sequences), there will be a closed subcurve $\gamma_k \subset \gamma \cap \mathbb{B}^n(x_0, |x_0 - b_k|)$ where $|F(x)| < |\alpha| + \eta$ and letting $E = \gamma_k \cap H_k$ the following hold.

Theorem 2.2. *Let u be a continuous monotone function in $W^{1,p}(\mathbb{B}^n)$. Suppose that $n - 1 < p \le n$. Then, for every $\epsilon > 0$, there is an open set U_ϵ in $\mathbb{R}^n$ with $\operatorname{cap}_p(U_\epsilon) < \epsilon$ with the property that if γ is a curve ending at $x_0 \in \partial\mathbb{B}^n\backslash U_\epsilon$ and $\{b_k\}_{k=1}^{\infty} \subset \gamma$ a sequence of points in $\mathbb{B}^n$ tending to x_0 so that*

$$\lim_{k\to\infty} u(b_k) = \alpha,$$

then $u(x)$ has nontangential limit α at x_0.

We say that a nonnegative measurable function w is a Muckenhoupt $A_p(\Omega)$ weight if

$$\sup_{\mathbb{B}\subset\Omega}\left\{\frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} w\,dx\right\}\left\{\frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} w^{1/(1-p)}\,dx\right\}^{p-1} < \infty,$$

where $\mathbb{B}$ is any ball in Ω and $|\mathbb{B}|$ stands for its volume. Then we have the following result.

Theorem 2.3. *Let u be a continuous monotone function in $W^{1,p}(\mathbb{B}^n; w)$, where w is in the class A_q for some q in the range $1 \le q < p/(n - 1)$. Suppose that $n - 1 < p \le n$. Then, for every $\epsilon > 0$, there exists an open set U_ϵ in $\mathbb{R}^n$ with $\operatorname{cap}_p(U_\epsilon) < \epsilon$ with the property that if γ is a curve ending at $x_0 \in \partial\mathbb{B}^n\backslash U_\epsilon$ and $\{b_k\}_{k=1}^{\infty} \subset \gamma$ a sequence of points in $\mathbb{B}^n$ tending to x_0 so that*

$$\lim_{k\to\infty} u(b_k) = \alpha,$$

then $u(x)$ has nontangential limit α at x_0.

The limitation $p > n - 1$ in the two previous theorems appears in a module estimate on $(n-1)$-dimensional spheres.

Let us remark here that it is well known, see [6], that each of the component functions of a mapping $F \in W^{1,n}_{\mathrm{loc}}(\mathbb{B}^n)$ of finite distortion $1 \le K(x) = |DF(x)|^n / J_F(x)$ is continuous and monotone; and thus Theorems 2.2 and 2.3 apply to the component functions of these mappings assuming that the mappings are in the corresponding Sobolev spaces.

We need Theorems 2.2 and 2.3 because in our main uniqueness results we are going to require that our mappings have the same weak limit α along a curve γ in $\mathbb{B}^n$ ending at $x_0 \in E \subset \partial\mathbb{B}^n$. That is, there is a sequence of points $\{b_k\}_{k=1}^{\infty} \subset \gamma$ tending to x_0 with

$$\lim_{k\to\infty} u(b_k) = \alpha,$$

rather than having the same weak boundary limit on the set E, as required in [11]. According to [11], a function u has a weak boundary limit α at a point x_0 in the set $E \subset \partial\mathbb{B}^n$ if for every rectifiable curve $\gamma \in \mathbb{B}^n$ ending at x_0 there is a sequence of points in γ for which u tends to α. This last assumption seems to us a very strong one, and a necessary one, in the absence of a Lindelöf-type theorem as was the case in [11].

Let us continue with some standard notation that will be used throughout the paper. The open ball centered at x_0 with radius r is denoted by $\mathbb{B}^n(x_0, r)$. By $c(\alpha, \beta, \ldots)$, we denote a constant that depends only on the parameters $\alpha, \beta, \ldots$ and that may change value from line to line.

Let Γ be a family of curves in $\mathbb{R}^n$. Denote by $\mathcal{F}(\Gamma)$ the collection of admissible metrics for Γ. These are nonnegative Borel measurable functions $\rho : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ such that

$$\int_\gamma \rho\, ds \ge 1$$

for each locally rectifiable curve $\gamma \in \Gamma$. For $p \ge 1$, the p-module of Γ is defined by

$$M_p(\Gamma) = \inf_{\rho\in\mathcal{F}(\Gamma)} \int_{\mathbb{R}^n} \rho^p\, dx;$$

if $\mathcal{F}(\Gamma) = \emptyset$, we set $M_p(\Gamma) = \infty$. In a similar way, we define the weighted p -module as

$$M_p^w(\Gamma) = \inf_{\rho\in\mathcal{F}(\Gamma)} \int_{\mathbb{R}^n} \rho^p w(x)\, dx,$$

where w is a positive weight defined in $\mathbb{R}^n$: for a more detailed discussion of these topics, see [13, 16, 18, 21]. Upper bounds for moduli are obtained by testing with a particular admissible metric.

Before we state the main result in this paper, let us recall some further definitions.

Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a domain and $F : \Omega \to \mathbb{R}^n$ be a mapping. We will say that the mapping F is in the Sobolev space $W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ if each of its component functions is in the Sobolev space $W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R})$. We can think of F as a deformation of some material whose initial configuration is Ω. The differential of F at a point x is denoted by $DF(x)$, its norm is

$$\|DF(x)\| = \sup\{|DF(x)h| : h \in \mathbb{R}^n, |h| = 1\}$$

and its Jacobian determinant is $J_F(x) = \det DF(x)$. The *distortion* of F at a point x is defined by the ratio

$$K(x) = \frac{\|DF(x)\|^n}{J_F(x)}.$$

If $K(x) \in L^\infty(\Omega; \mathbb{R})$, then F is said to be a quasiregular mapping. We will say that F is a mapping of *finite distortion* if

$$1 \leq K(x) < \infty \quad \text{for a.e. } x \in \Omega,$$

that is, except for a set of measure zero in Ω, $J_F(x) \geq 0$, which implies that F is orientation preserving.

Definition 2.4. Let $F : \mathbb{B}^n \to \mathbb{R}^n$ be a mapping. We define the multiplicity function of F at some point $y \in \mathbb{R}^n$ with respect to some domain $D \subset \mathbb{B}^n$ as

$$N(y, F, D) = \#\{x \in D : F(x) = y\}.$$

Let us remark that although our results are going to be stated for mappings defined on the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$, they also hold for mappings defined in domains $\Omega \subset \mathbb{R}^n$ for which we have both capacity and Sobolev extension theorems. We state our main result.

Theorem 2.5. *Let $F \in W^{1,n}_{\mathrm{loc}}(\mathbb{B}^n; \mathbb{R}^n) \cap W^{1,\alpha}(\mathbb{B}^n; \mathbb{R}^n)$ be a nonconstant mapping for $\alpha = \bar{p} - (1-\eta)^2$ with $0 \leq \eta < 1/2$ small and $\bar{p} = (1-\eta)n$. In addition, let the mapping F have finite distortion $K \in L^{\tilde{p}}$ for $\tilde{p} = n - 1 + \eta \geq n - 1$.*

Let $E \subset \partial\mathbb{B}^n$ be the set where for each $x_0 \in E$ there are a curve $\gamma \subset \mathbb{B}^n$ ending at x_0 and a sequence of points $\{b_k\}_{k=1}^\infty \subset \gamma$ tending to x_0 with

$$\lim_{k\to\infty} F(b_k) = a$$

and $\mathbb{B}_\epsilon = \{x \in \mathbb{B}^n : |F(x) - a| < \epsilon\}$. Suppose, moreover, that for every small $\epsilon_0 \geq \epsilon > 0$,

$$\int_{\mathbb{B}_\epsilon} J_F(x)\, dm(x) = \int_{\{y:|y-a|<\epsilon\}} N(y, \mathbb{B}^n, F)\, dm(y) \leq C\epsilon^n \Big(\log\Big(\frac{1}{\epsilon}\Big)\Big)^{n((\bar{p}-\delta)/\bar{p})},$$

where $\delta = n/(n-1+\eta)$. This condition implies that the $(\bar{p} - (1-\eta)^2)$-capacity of the set $E \subset \partial\Omega$ is zero and thus its Hausdorff dimension is less than or equal to $n - \bar{p} + (1-\eta)^2$.

In other words, if E has positive $(\bar{p} - (1-\eta)^2)$-capacity with $0 \leq \eta < 1/2$, then the mapping F is identically equal to a.

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REMARK 2.6. Our assumption that our mapping F in Theorem 2.5 is in the Sobolev space $W^{1,\alpha}(\mathbb{B}^n;\mathbb{R}^n)$ for some $\alpha < n$ is necessary because we are considering our set $E \subset \partial\mathbb{B}^n$ to be the set on the boundary of $\mathbb{B}^n$ where a 'very weak' limit exists. This assumption would not be necessary if we were to consider radial limits or nontangential limits.

Observe also that by Hölder's inequality, the integrability condition on Theorem 2.5 implies that

$$\int_{\mathbb{B}_\epsilon} J_F(x)^{\bar{p}/n}\, dm(x) \le C\epsilon^{\bar{p}}\Big(\log\Big(\frac{1}{\epsilon}\Big)\Big)^{\bar{p}-\delta}.$$

It is well known that by [6, Theorem 2.3, page 405], the component functions of every mapping in the Sobolev space $W^{1,n}_{\mathrm{loc}}(\mathbb{B}^n;\mathbb{R}^n)$ of finite distortion are monotone. In order to apply the Lindelöf-type Theorem 2.2 in the proof of Theorem 2.5, we need to require that $F \in W^{1,\alpha}(\mathbb{B}^n;\mathbb{R}^n)$ for $\alpha = \bar{p} - (1-\eta)^2$ with $0 \le \eta < 1/2$ small and $\bar{p} = (1-\eta)n$.

Before we pass to the proof of Theorem 2.5, let us examine how it is related to the results of Koskela [11], Mizuta [15] and Miklyukov and Vuorinen [14]. Let us observe that their covering conditions on $\mathbb{B}_\epsilon$ are integral conditions involving the gradient of the real function u to some power p. For example, Koskela's condition requires that

$$I(\epsilon) = \int_{\mathbb{B}_\epsilon} |\nabla u|^p\, dm \le C\epsilon^p\Big(\log\frac{1}{\epsilon}\Big)^{p-1}$$

for $1 < p \le n$. Since the function u is a real function, this covering condition is for an interval $(-\epsilon, \epsilon)$ about zero. This condition should be replaced in the case of mappings from $\mathbb{B}^n$ into $\mathbb{R}^n$ by an integral condition on some power of the norm of the differential matrix $\|DF\|$ or the Jacobian J_F.

The two extra conditions we impose on the mapping F come naturally. First the integrability of the distortion function $K(x)$ and second the monotonicity of the components of the mapping F. Yet, the second condition can be removed if we consider that the limits at the set $E \subset \partial\mathbb{B}^n$ are fine boundary limits; see [2, 4] for the definition of this type of limits.

Observe that the length of the gradient vector of the functions u in the work of Koskela [11], Mizuta [15] and Miklyukov and Vuorinen [14] should be replaced by $J_F^{1/n}$ in the case of a mapping F. Thus, Koskela's condition on $\mathbb{B}_\epsilon$ translates to

$$\int_{\mathbb{B}_\epsilon} J_F(x)^{p/n}\, dm(x) \le C\epsilon^p\Big(\log\frac{1}{\epsilon}\Big)^{p-1}.$$

This is the case under the hypothesis of our theorem on the integrability of the multiplicity function $N(y, \mathbb{B}^n, F)$. We are trying to find an upper bound of the integral $\int_{\mathbb{B}_\epsilon} J_F(x)^{p/n}\, dm(x)$. By Hölder's inequality,

$$\int_{\mathbb{B}_\epsilon} J_F(x)^{p/n}\, dm(x) \le C\Big(\int_{\mathbb{B}_\epsilon} J_F(x)\, dm(x)\Big)^{p/n}.$$

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For the mappings under consideration, the following change of variable formula holds; see [6, Theorem 1.1, page 400]:

$$\int_{\mathbb{B}_\epsilon} J_F(x)\, dm(x) = \int_{\{y:\|y\|<\epsilon\}} N(y, \mathbb{B}^n, F)\, dm(y).$$

Thus, our integrability conditions on the Jacobian $J_F(x)$ or, which is equivalent, on the multiplicity function $N(y, \mathbb{B}^n, F)$ basically say that in order to have uniqueness theorems, the mappings cannot cover 'too much' of a small ϵ-neighborhood of the limiting value a. This is clearly a geometric condition on the mappings, making our results intuitively geometric in nature.

In our proof, it will become clear how conditions similar to the ones in Mizuta [15] and Miklyukov and Vuorinen [14] can be used as the integrability conditions for the multiplicity function $N(y, \mathbb{B}^n, F)$ to obtain similar results to the ones in those two papers.

Our results generalize the previous results related to the original F. and M. Riesz's uniqueness theorem in two directions. Namely:

(1) our results are for mappings;
(2) when restricted to functions, that is, $J_F^{1/n}$ replaced by $|\nabla u|$, we obtain results similar to the ones by Koskela [11], Mizuta [15] and Miklyukov and Vuorinen [14].

Lastly, we examine the role of the monotonicity condition in our results. In our theorems we state that our function 'approaches' the same weak limit a along a curve γ in $\mathbb{B}^n$ ending at $x_0 \in E \subset \partial\mathbb{B}^n$. That is, there is a sequence of points $\{b_k\}_{k=1}^{\infty} \subset \gamma$ tending to x_0 with

$$\lim_{k\to\infty} u(b_k) = a$$

on a set E on the boundary $\partial\mathbb{B}^n$. As will become clear in our proof, we need the monotonicity condition for the Lindelöf-type Theorems 2.2 and 2.3 to hold. Thus, being able to say that for every locally rectifiable curve γ in $\mathbb{B}^n$ ending at a point in E: either the limit of the mapping F along this curve does not exist or else the limit exists and is equal to a.

3. Proof of Theorem 2.5

We now pass to prove Theorem 2.5. Let us point out here that our proof was inspired by the techniques used in the proofs of similar results in [11].

Proof. Without loss of generality, we may assume that $a = 0$. Suppose that F is not identically zero and fix a closed ball $B_{j_0} \subset \mathbb{B}^n$ of positive radius such that $|F(x)| \geq 2^{-j_0}$ for all $x \in B_{j_0}$ and $2^{-j_0} < \epsilon_0$ for some positive integer j_0. Write

$$B_j = \{x \in \mathbb{B}^n : 2^{-j-1} \leq |F(x)| < 2^{-j}\}$$

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for $j = j_0, j_0+1, \dots$, define $\rho(x) = 2^j/(j\log j)\|DF(x)\|$ for $x \in B_j$ and set ρ to be zero elsewhere in $\mathbb{R}^n$. Then ρ is Borel measurable (see, for example, [18, Theorem 26.4]) and

$$\begin{aligned} T(\alpha,\rho) &= \int_{\mathbb{R}^n} \rho^\alpha(x)\,dm(x) \\ &= \sum_{j=j_0}^{\infty} \int_{B_j} \frac{2^{j\alpha}}{j^\alpha(\log j)^\alpha}\|DF(x)\|^\alpha\,dm(x). \end{aligned}$$

Multiplying and dividing inside the above integral by $J_F(x)^\beta$ for some positive $\beta > 0$ to be determined later, we see that

$$T(\alpha,\rho) \le \sum_{j=j_0}^{\infty} \frac{2^{j\alpha}}{j^\alpha(\log j)^\alpha} \int_{B_j} \frac{\|DF(x)\|^\alpha}{J_F(x)^\beta} J_F(x)^\beta\,dm(x).$$

Applying Hölder's inequality to each of the integrals in the series above for the conjugate values p, p' and finding the values of α and β such that $\alpha p = n\tilde{p}$, $\beta p = \tilde{p}$, $\beta p' = 1-\eta$, where $\tilde{p} = n-1+\eta$, and letting $\bar{p} = n(1-\eta)$ yields

$$T(\alpha,\rho) \le \sum_{j=j_0}^{\infty} \frac{2^{j\alpha}}{j^\alpha(\log j)^\alpha}\Big(\int_{B_j}(K(x))^{n-1+\eta}\,dm(x)\Big)^{1/p}\Big(\int_{B_j} J_F(x)^{\bar{p}/n}\,dm(x)\Big)^{1/p'}.$$

Clearly we have that $\alpha = (1-\eta)(n-1+\eta)$, $\beta = (1-\eta)(n-1+\eta)/n$, $p = n/(1-\eta)$ and $p' = n/(n-1+\eta)$. These observations lead to

$$\begin{aligned} T(\alpha,\rho) \le{} & \Big(\int_{\mathbb{B}_\epsilon}(K(x))^{n-1+\eta}\,dm(x)\Big)^{1/p} \\ & \times \sum_{j=j_0}^{\infty} \frac{2^{j\alpha}}{j^\alpha(\log j)^\alpha}\Big(\int_{B_j} J_F(x)^{1-\eta}\,dm(x)\Big)^{(n-1+\eta)/n} \end{aligned}$$

for some $\epsilon < \epsilon_0$. If we now apply the hypothesis of the theorem to the integrals in the series above,

$$\begin{aligned} T(\alpha,\rho) \le{} & \Big(\int_{\mathbb{B}_\epsilon}(K(x))^{n-1+\eta}\,dm(x)\Big)^{1/p} \\ & \times \sum_{j=j_0}^{\infty} \frac{2^{j\alpha}}{j^\alpha(\log j)^\alpha}((2^{-j})^{n(1-\eta)}j^{(n(1-\eta)-\delta)})^{(n-1+\eta)/n}. \end{aligned}$$

The hypothesis $K \in L^{n-1+\eta}$ and $\alpha = (1-\eta)(n-1+\eta)$ now imply that

$$\begin{aligned} T(\alpha,\rho) \le C & \sum_{j=j_0}^{\infty} \frac{2^{j(1-\eta)(n-1+\eta)}}{j^{(1-\eta)(n-1+\eta)}(\log j)^{(1-\eta)(n-1+\eta)}} \\ & \times ((2^{-j})^{n(1-\eta)}j^{(n(1-\eta)-\delta)})^{(n-1+\eta)/n} \end{aligned}$$

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and further

$$T(\alpha,\rho) \le C \sum_{j=j_0}^{\infty} \frac{j^{(n(1-\eta)-\delta)(n-1+\eta)/n}}{j^{(1-\eta)(n-1+\eta)}(\log j)^{(1-\eta)(n-1+\eta)}}.$$

Now, since $\delta = n/(n-1+\eta)$, substituting in the above inequality finally gives us

$$T(\alpha,\rho) \le C \sum_{j=j_0}^{\infty} \frac{1}{j(\log j)^{(1-\eta)(n-1+\eta)}} < \infty$$

since $(1-\eta)(n-1+\eta) > 1$ for small positive η.

Let $\Delta(B_{j_0}, E; \mathbb{B}^n)$ consist of the family of curves in $\mathbb{B}^n$ joining B_{j_0} and E. We will show that $M_\alpha(\Delta(B_{j_0}, E; \mathbb{B}^n)) = 0$. From this, it will follow that $M_\alpha(\Delta(B_{j_0}, E; \mathbb{R}^n)) = 0$ for the family of curves in $\mathbb{R}^n$ joining B_{j_0} and E.

Let us now show that the chosen metric $\rho(x)$ is admissible for the family of curves $\Delta(B_{j_0}, E; \mathbb{B}^n)$ by showing that for each locally rectifiable curve $\gamma \in \Delta(B_{j_0}, E; \mathbb{B}^n)$, $\int_\gamma \rho\, ds \ge 1$.

Fix a locally rectifiable curve $\gamma \in \Delta(B_{j_0}, E; \mathbb{B}^n)$. There are two cases to consider: first, if the limit of $F(x)$ as $x \in \gamma$ approaches E does not exist, we will say in this case that $\gamma \in \Delta(B_{j_0}, E; \mathbb{B}^n)_{NL}$. Second, if the limit of $F(x)$ as $x \in \gamma$ approaches E exists, that is, $\gamma \in \Delta(B_{j_0}, E; \mathbb{B}^n)_L$, then, by our Lindelöf Theorem 2.2, this limit has to be equal to 0.

Remark 3.1. Let us remark here that in order to apply our Lindelöf Theorem 2.2, we need to consider first that for every $\epsilon > 0$, there is an open set U_ϵ in $\mathbb{R}^n$ with $\mathrm{cap}_p(U_\epsilon) < \epsilon$ such that the Lindelöf theorem holds for $x_0 \in \partial\mathbb{B}^n \backslash U_\epsilon$, and then pass to the limit when ϵ goes to 0 to conclude that $M_\alpha(\Delta(B_{j_0}, E; \mathbb{B}^n)_L) = 0$. Thus, in our arguments below we can consider without loss of generality that the Lindelöf condition holds in the whole set E.

We will show next that without loss of generality we can assume from here on that for every $\gamma \in \Delta(B_{j_0}, E; \mathbb{B}^n)$, $\lim_{x\in\gamma,\to x_0\in E} F(x) = 0$. Write

$$\Delta(B_{j_0}, E; \mathbb{B}^n) = \Delta(B_{j_0}, E; \mathbb{B}^n)_{NL} \cup \Delta(B_{j_0}, E; \mathbb{B}^n)_L$$

as the disjoint union of two sets of curves.

By Theorem 2.2, for every $\gamma \in \Delta(B_{j_0}, E; \mathbb{B}^n)_{NL}$, the $\lim_{x\in\gamma,\to x_0\in E} F(x)$ does not exist and for every $\gamma \in \Delta(B_{j_0}, E; \mathbb{B}^n)_L$, the $\lim_{x\in\gamma,\to x_0\in E} F(x) = 0$.

By the subadditivity of the α-modulus,

$$M_\alpha(\Delta(B_{j_0}, E; \mathbb{B}^n)) \le M_\alpha(\Delta(B_{j_0}, E; \mathbb{B}^n)_{NL}) + M_\alpha(\Delta(B_{j_0}, E; \mathbb{B}^n)_L).$$

We pass now to show that $M_\alpha(\Delta(B_{j_0}, E; \mathbb{B}^n)_{NL}) = 0$ by choosing $\tilde{\rho}(x) = \|DF(x)\|$ for each $x \in \mathbb{B}^n$. It follows that $\tilde{\rho}$ is a Borel measurable metric for which

$$\int_{\gamma\in\Delta(B_{j_0},E;\mathbb{B}^n)_{NL}} \tilde{\rho}(x)\, ds(x) = \int_{\gamma\in\Delta(B_{j_0},E;\mathbb{B}^n)_{NL}} \|DF(x)\|\, ds(x) = \infty;$$

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thus, $\tilde{\rho}$ is admissible for the family of curves $\Delta(B_{j_0}, E; \mathbb{B}^n)_{NL}$. Now, since $F \in W^{1,\alpha}(\mathbb{B}^n; \mathbb{R}^n)$,

$$\int_{\mathbb{B}^n} \tilde{\rho}^\alpha(x)\, dm(x) = \int_{\mathbb{B}^n} \|DF(x)\|^\alpha\, dm(x) < \infty.$$

It follows that $M_\alpha(\Delta(B_{j_0}, E; \mathbb{B}^n)_{NL}) = 0$. Hence, we can assume from now on that for every $\gamma \in \Delta(B_{j_0}, E; \mathbb{B}^n)$, $\lim_{x\in\gamma,\to x_0\in E} F(x) = 0$.

Thus,

$$\begin{aligned}\int_\gamma \rho(x)\, ds(x) &\geq \sum_{j=j_0}^{\infty} \int_{\gamma\cap B_j} \frac{2^j}{j\log j}\|DF(x)\|\, ds(x)\\ &\geq \sum_{j=j_0}^{\infty} \frac{2^j}{j\log j} \int_{\gamma\cap B_j} \|DF(x)\|\, ds(x)\\ &\geq C\sum_{j=j_0}^{\infty} \frac{1}{j\log j} = \infty.\end{aligned}$$

It is well known that α-almost every curve in $\mathbb{B}^n$ is rectifiable; by Fuglede's theorem [18, Fuglede's Theorem 28.2], the α-modulus of the curves γ, for which F fails to be absolutely continuous on some closed subcurve, is zero. Hence, the α-modulus

$$\begin{aligned}M_{\alpha=(1-\eta)(n-1+\eta)}(\Delta(B_{j_0}, E; \mathbb{B}^n)) &= \inf_{\tilde{\rho}\in\mathcal{F}(\Delta(B_{j_0},E;\mathbb{B}^n))} \int_{\mathbb{B}^n} \tilde{\rho}^\alpha\, dx\\ &\leq \frac{1}{M^\alpha} T(\alpha, \rho),\end{aligned}$$

for our chosen Borel measurable metric ρ and for every $M > 0$. Letting $M \to \infty$,

$$M_{\alpha=(1-\eta)(n-1+\eta)}(\Delta(B_{j_0}, E; \mathbb{B}^n)) = 0.$$

It follows by [20, Lemmas 4.2 and 4.3] that

$$M_{\alpha=(1-\eta)(n-1+\eta)}(\Delta(B_{j_0}, E; \mathbb{R}^n)) = 0.$$

Thus, see [23, Theorem 3.8, page 122] and [8, Proposition 2.17, page 11], the $(1-\eta)(n-1+\eta)$-capacity of E is equal to 0. Therefore, from [7, Lemma 2.25 and Theorem 2.26], the Hausdorff dimension of E is less than or equal to $n-(1-\eta)(n-1+\eta) > 1$ for every small positive η since $(1-\eta)(n-1+\eta) = \alpha < n-1$. Also observe that for $0 \leq \eta < 1/2$, we have that $1/2(n-1) < \alpha = \bar{p} - (1-\eta)^2 < \bar{p}$. This shows that the set E has Hausdorff dimension less than $(n+1)/2$. □

4. Further results

In this section we are going to state and prove two further results related to the main result of the paper. We will start with the case when the mapping F is a quasiregular mapping.

A mapping F is quasiregular if $K(x)$ is an L^∞ function; thus, $1 \leq K(x) \leq K < \infty$ for almost every x and therefore $(1/C)J_F(x) \leq \|DF(x)\|^n \leq CJ_F(x)$ for almost every x for some constant C independent of x. Then we have the following theorem.

THEOREM 4.1. *Let F be a nonconstant quasiregular mapping in the Sobolev space $W^{1,n}(\mathbb{B}^n;\mathbb{R}^n)$.*

Let $E \subset \partial\mathbb{B}^n$ be the set where for each $x_0 \in E$ there are a curve $\gamma \subset \mathbb{B}^n$ ending at x_0 and a sequence of points $\{b_k\}_{k=1}^{\infty} \subset \gamma$ tending to x_0 with

$$\lim_{k\to\infty} F(b_k) = a$$

and $\mathbb{B}_\epsilon = \{x \in \mathbb{B}^n : |F(x) - a| < \epsilon\}$. If for each small $\epsilon_0 \geq \epsilon > 0$,

$$\int_{\mathbb{B}_\epsilon} J_F(x)\, dm(x) \leq C\epsilon^n \Big(\log \frac{1}{\epsilon}\Big)^{n-1}$$

or, which is equivalent, if

$$\int_{\{y:|y-a|<\epsilon\}} N(y, \mathbb{B}^n, F)\, dm(y) \leq C\epsilon^n \Big(\log \frac{1}{\epsilon}\Big)^{n-1},$$

then the n-capacity of the set E is zero and thus its Hausdorff dimension is zero.

In other words, if E has positive n-capacity then the mapping F is identically equal to a.

PROOF. As in the proof of Theorem 2.5, all we need to show in this theorem is that

$$M_n(\Delta(B_{j_0}, E; \mathbb{B}^n)) = 0.$$

From this, it will follow that $M_n(\Delta(B_{j_0}, E; \mathbb{R}^n)) = 0$ for the family of curves in $\mathbb{R}^n$ joining B_{j_0} and E. The proof of this follows exactly the same steps as the one of Theorem 2.5 for $\alpha = n$ for the same choice of the admissible metric ρ; thus, we omit the details.

It then follows that the n-capacity of the set $E \subset \partial\mathbb{B}^n$ is zero. This implies that its Hausdorff dimension is zero and this finishes the proof of our theorem. □

Finally, we will consider the case when the mappings F under consideration belong to some weighted Sobolev space with a positive weight w in $A_q(\mathbb{R}^n)$. In this case, our theorem is stated as follows.

THEOREM 4.2. *Let $F \in W^{1,n}_{\mathrm{loc}}(\mathbb{B}^n;\mathbb{R}^n)$ be a nonconstant mapping of finite distortion in the weighted Sobolev space $W^{1,\alpha}_{w(x)^{(n-\alpha)/n}}(\mathbb{B}^n;\mathbb{R}^n)$, where $w(x) = |1 - |x||^{\eta}$, and such that $(K(x)^{\alpha/(n-\alpha)}|1 - |x||^{\eta}) \in L^1(\mathbb{B}^n)$ for some $n - 1 < \alpha \leq n$ and η positive such that $\alpha/(n-1) > \eta((n-\alpha)/n) + 1$.*

Let $E \subset \partial\mathbb{B}^n$ be the set where for each $x_0 \in E$ there are a curve $\gamma \subset \mathbb{B}^n$ ending at x_0 and a sequence of points $\{b_k\}_{k=1}^{\infty} \subset \gamma$ tending to x_0 with

$$\lim_{k\to\infty} F(b_k) = a$$

and $\mathbb{B}_\epsilon = \{x \in \mathbb{B}^n : |F(x) - a| < \epsilon\}$. If, for every small $\epsilon_0 \geq \epsilon > 0$,

$$\int_{\mathbb{B}_\epsilon} J_F(x)\, dm(x) \leq C\epsilon^n \Big(\log \frac{1}{\epsilon}\Big)^{\delta}$$

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or, which is equivalent, if

$$\int_{\{y:|y-a|<\epsilon\}} N(y, \mathbb{B}^n, F)\, dm(y) \le C\epsilon^n \Big(\log \frac{1}{\epsilon}\Big)^{\delta},$$

where $\delta = n - n/\alpha$, then the Hausdorff dimension of the set E is less than or equal to $\eta((n-\alpha)/n) + n - \alpha < \alpha/(n-1) - 1 + n - \alpha < 1$ since $n - 1 < \alpha \le n$.

REMARK 4.3. In Theorem 4.2, we need to require that our mapping F is in the weighted Sobolev space $W^{1,\alpha}_{w(x)^{(n-\alpha)/n}}(\mathbb{B}^n; \mathbb{R}^n)$ where $w(x) = |1 - |x||^{\eta}$ in order to use the Lindelöf-type theorem [13, Theorem 3]. This theorem holds whenever $\alpha/(n-1) > \eta((n-\alpha)/n) + 1$.

Considering the case $\alpha = n$ in the above theorem, we obtain the following corollary.

COROLLARY 4.4. *Let $\{\alpha_j\}_{j=1}^{\infty}$, $\{\eta_j\}_{j=1}^{\infty}$ be sequences of positive numbers such that $n - 1 < \alpha_j$, $\lim_{j\to\infty} \alpha_j = n$, $\lim_{j\to\infty} \eta_j = \infty$ and, for every $j = 1, 2, \ldots,$*

$$\frac{\alpha_j}{n-1} > \eta_j\Big(\frac{n-\alpha_j}{n}\Big) + 1.$$

Let F be a nonconstant continuous mapping of finite distortion in the Sobolev space $W^{1,n}(\mathbb{B}^n; \mathbb{R}^n)$ with

$$\lim_{j\to\infty} \int_{\mathbb{B}^n} K(x)^{\alpha_j/(n-\alpha_j)} |1 - |x||^{\eta_j}\, dx < C < \infty,$$

where C is a constant independent of the sequences.

Let $E \subset \partial\mathbb{B}^n$ be the set where for each $x_0 \in E$ there are a curve $\gamma \subset \mathbb{B}^n$ ending at x_0 and a sequence of points $\{b_k\}_{k=1}^{\infty} \subset \gamma$ tending to x_0 with

$$\lim_{k\to\infty} F(b_k) = a$$

and $\mathbb{B}_\epsilon = \{x \in \mathbb{B}^n : |F(x) - a| < \epsilon\}$. If we have that for every small $\epsilon_0 \ge \epsilon > 0$,

$$\int_{\mathbb{B}_\epsilon} J_F(x)\, dm(x) \le C\epsilon^n \Big(\log \frac{1}{\epsilon}\Big)^{n-1}$$

or, which is equivalent, if

$$\int_{\{y:|y-a|<\epsilon\}} N(y, \mathbb{B}^n, F)\, dm(y) \le C\epsilon^n \Big(\log \frac{1}{\epsilon}\Big)^{n-1},$$

then the Hausdorff dimension of the set E is zero.

REMARK 4.5. In [11], Koskela showed that his result is sharp. It will be worth studying whether his sharpness result could be used to show whether the results in this paper are in some sense sharp too. It will also be worth pursuing to study the connections between uniqueness results and results concerning the size of the inverse image $F^{-1}(a)$ of points a in the image of mappings in our Sobolev spaces with finite distortion $K(x)$ in some $L^p(\mathbb{B}^n)$.

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Proof. As in the proof of Theorem 2.5, all we need to show in this theorem is that

$$M_\alpha^{w(x)^{(n-\alpha)/n}}(\Delta(B_{j_0}, E; \mathbb{B}^n)) = 0,$$

where $w(x) = |1 - |x||^\eta$ is an A_p-weight in $\mathbb{R}^n$ for every $p > n - 1$; see [13]. From this, it will follow that $M_\alpha^{w(x)^{(n-\alpha)/n}}(\Delta(B_{j_0}, E; \mathbb{R}^n)) = 0$ for the family of curves in $\mathbb{R}^n$ joining B_{j_0} and E.

Without loss of generality, we may assume that $a = 0$. Suppose that F is not identically zero and fix a closed ball $B_{j_0} \subset \mathbb{B}^n$ of positive radius such that $|F(x)| \geq 2^{-j_0}$ for all $x \in B_{j_0}$ and $2^{-j_0} < \epsilon_0$ for some positive integer j_0. Write

$$B_j = \{x \in \mathbb{B}^n : 2^{-j-1} \leq |F(x)| < 2^{-j}\}$$

for $j = j_0, j_0 + 1, \ldots$, define $\rho(x) = 2^j/(j \log j)\|DF(x)\|$ for $x \in B_j$ and set ρ to be zero elsewhere in $\mathbb{R}^n$. Then ρ is Borel measurable (see, for example, [18, Theorem 26.4]) and

$$\begin{aligned} T(\alpha,\rho) &= \int_{\mathbb{R}^n} \rho^\alpha(x)[w(x)]^{(n-\alpha)/n}\, dm(x) \\ &= \sum_{j=j_0}^{\infty} \int_j \frac{2^{j\alpha}}{j^\alpha(\log j)^\alpha} \|DF(x)\|^\alpha [w(x)]^{(n-\alpha)/n}\, dm(x). \end{aligned}$$

Multiplying and dividing inside the above integral by $K(x)^\beta$ for some positive $\beta > 0$ to be determined later,

$$T(\alpha,\rho) \leq \sum_{j=j_0}^{\infty} \int_{B_j} \frac{2^{j\alpha}}{j^\alpha(\log j)^\alpha} \frac{\|DF(x)\|^\alpha}{K(x)^\beta} K(x)^\beta [w(x)]^{(n-\alpha)/n}\, dm(x).$$

Applying now Hölder's inequality to each of the integral terms of this sum for the conjugate values $p = n/\alpha$ and $p' = n/(n - \alpha)$ and letting $\beta = \alpha/n$,

$$\begin{aligned} T(\alpha,\rho) \leq \sum_{j=j_0}^{\infty} \frac{2^{j\alpha}}{j^\alpha(\log j)^\alpha} &\Big(\int_{B_j} (K(x))^{\alpha/(n-\alpha)} w(x)\, dm(x)\Big)^{(n-\alpha)/n} \\ &\times \Big(\int_{B_j} J_F(x)\, dm(x)\Big)^{\alpha/n}, \end{aligned}$$

using now the hypothesis of the theorem, for some $\epsilon < \epsilon_0$,

$$\begin{aligned} T(\alpha,\rho) \leq &\Big(\int_{B_\epsilon} (K(x))^{\alpha/(n-\alpha)} w(x)\, dm(x)\Big)^{(n-\alpha)n} \\ &\times \Big(\sum_{j=j_0}^{\infty} \frac{2^{j\alpha}}{j^\alpha(\log j)^\alpha} (2^{-jn} j^{n-(n/\alpha)})^{\alpha/n}\Big) < \infty, \end{aligned}$$

since the first integral above is bounded by hypothesis.

Let $\Delta(B_{j_0}, E; \mathbb{B}^n)$ consist of the family of curves in $\mathbb{B}^n$ joining B_{j_0} and E. We will show that $M_\alpha^{w(x)^{(n-\alpha)/n}}(\Delta(B_{j_0}, E; \mathbb{B}^n)) = 0$. From this, it will follow that $M_\alpha^{w(x)^{(n-\alpha)/n}}(\Delta(B_{j_0}, E; \mathbb{R}^n)) = 0$ for the family of curves in $\mathbb{R}^n$ joining B_{j_0} and E.

Fix $\gamma \in \Delta(B_{j_0}, E; \mathbb{B}^n)$ locally rectifiable. It follows as in the proof of Theorem 2.5 (using now Theorem 2.3) that without loss of generality we can assume that the limit of $F(x)$ as $x \in \gamma$ approaches E exists and is equal to 0, and thus we conclude that

$$\begin{aligned}\int_\gamma \rho(x)\,ds(x) &\geq \sum_{j=j_0}^{\infty} \int_{\gamma\cap B_j} \frac{2^j}{j\log j}\|DF(x)\|\,ds(x)\\ &\geq \sum_{j=j_0}^{\infty} \frac{2^j}{j\log j} \int_{\gamma\cap B_j} \|DF(x)\|\,ds(x)\\ &\geq C\sum_{j=j_0}^{\infty} \frac{1}{j\log j} = \infty.\end{aligned}$$

Hence, the α-weighted modulus

$$\begin{aligned}M^{w(x)^{(n-\alpha)/n}}_{\alpha=(1-\eta)(n-1+\eta)}(\Delta(B_{j_0}, E; \mathbb{B}^n)) &= \inf_{\tilde\rho\in\mathcal{F}(\Delta(B_{j_0},E;\mathbb{B}^n))} \int_{\mathbb{B}^n} \tilde\rho^\alpha\,dx\\ &\leq \frac{1}{M^\alpha} T(\alpha, \rho)\end{aligned}$$

for our chosen Borel measurable metric ρ and for each $M > 0$. Letting $M \to \infty$,

$$M^{w(x)^{(n-\alpha)/n}}_{\alpha=(1-\eta)(n-1+\eta)}(\Delta(B_{j_0}, E; \mathbb{B}^n)) = 0.$$

It follows by [13, Lemma 4.7] that

$$M^{w(x)^{(n-\alpha)/n}}_{\alpha=(1-\eta)(n-1+\eta)}(\Delta(B_{j_0}, E; \mathbb{R}^n)) = 0.$$

Let us choose $w(x) = |1 - |x||^\eta$, which is an $A_p(\mathbb{R}^n)$-weight for $p > \eta + 1$; see [13]. Thus, $[w(x)]^{(n-\alpha)/n} = |1 - |x||^{\eta((n-\alpha)/n)}$ is an $A_q(\mathbb{R}^n)$-weight where we choose η to be a positive number such that $\alpha/(n-1) > q > \eta((n-\alpha)/n) + 1$ whenever $n - 1 < \alpha \leq n$.

It is well known, see [13], that the Hausdorff dimension of the set E is less than or equal to $\eta((n-\alpha)/n) + n - \alpha < \alpha/(n-1) - 1 + n - \alpha < 1$ for $n - 1 < \alpha \leq n$.

This shows that the set $E \subset \partial\mathbb{B}^n$ where the nonconstant mapping F can have the same nontangential boundary limit is very small in the sense that its Hausdorff dimension is strictly less than 1. □

In the quasiregular case (Theorem 4.1), one could ask whether the assumption

$$\int_{\mathbb{B}_\epsilon} J_F(x)\,dm(x) \leq C\epsilon^n\Big(\log\frac{1}{\epsilon}\Big)^{n-1}$$

is necessary. In other words, could the quasiregularity of the mapping F somehow imply the above inequality? Some results in this direction can be found in Hencl and Koskela [9], where they estimate the integral

$$\int_{\mathbb{B}_\epsilon} \frac{J_F(x)}{|F(x)|^n \log^n \frac{1}{|F(x)|}}\,dm(x)$$

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for various mappings of finite distortion. Clearly the assumption

$$\int_{\mathbb{B}_\epsilon} J_F(x)\, dm(x) \le C\epsilon^n \Big(\log \frac{1}{\epsilon}\Big)^{n-1}$$

implies the finiteness of the integral

$$\int_{\mathbb{B}_\epsilon} \frac{J_F(x)}{|F(x)|^n \log^{n+\delta} \frac{1}{|F(x)|}}\, dm(x)$$

for all $\delta > 0$. It would also be worth checking whether in our assumptions the exponent $n-1$ in the logarithmic term is sharp for quasiregular mappings.

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References

[1] A. Beurling, 'Ensembles exceptionnels', *Acta Math.* **72** (1940), 1–13.
[2] M. Brelot and J. L. Doob, 'Limites angulaires et limites fines', *Ann. Inst. Fourier (Grenoble)* **13** (1963), 395–415.
[3] L. Carleson, 'Sets of uniqueness for functions analytic in the unit disc', *Acta Math.* **87** (1952), 325–345.
[4] J. L. Doob, *Classical Potential Theory and its Probabilistic Counterpart* (Springer, New York, 1984).
[5] B. Fuglede, 'Extremal length and functional completion', *Acta Math.* **98** (1957), 171–219.
[6] V. Goldstein and S. Vodopyanov, 'Quasiconformal mappings and spaces of functions with generalized first derivatives', *Sibirsk. Mat. Zh.* **17** (1976), 515–531.
[7] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory* (Oxford University Press, New York, 1993).
[8] J. Heinonen and P. Koskela, 'Quasiconformal maps in metric spaces with controlled geometry', *Acta Math.* **181** (1998), 1–61.
[9] S. Hencl and P. Koskela, *Mappings of Finite Distortion* (Springer, New York, 2013).
[10] J. A. Jenkins, 'On a result of Beurling', *Indiana Univ. Math. J.* **41** (1992), 1077–1080.
[11] P. Koskela, 'A radial uniqueness theorem for Sobolev functions', *Bull. Lond. Math. Soc.* **27** (1995), 460–466.
[12] J. Manfredi and E. Villamor, 'Traces of monotone Sobolev functions', *J. Geom. Anal.* **6**(3) (1996), 433–444.
[13] J. J. Manfredi and E. Villamor, 'Traces of monotone functions in weighted Sobolev spaces', *Illinois J. Math.* **45** (2001), 403–422.
[14] V. M. Miklyukov and M. Vuorinen, 'A boundary uniqueness theorem for Sobolev functions', *Tohoku Math. J.* **50** (1998), 503–511.
[15] Y. Mizuta, 'Remarks on the results by Koskela concerning the radial uniqueness for Sobolev functions', *Proc. Amer. Math. Soc.* **126** (1998), 1043–1047.
[16] M. Ohtsuka, 'On weighted extremal length of families of curves', in: *Complex Analysis* (Birkhauser, Basel, 1988), 191–200.
[17] M. Tsuji, 'Beurling's theorem on exceptional sets', *Tohoku Math. J.* **2** (1950), 113–125.

[18] J. Väisälä, *Lectures in n-Dimensional Quasiconformal Mappings*, Lecture Notes, 229 (Springer, Berlin, 1971).
[19] E. Villamor, 'Boundary values for some classes of functions', *J. Lond. Math. Soc.* **45** (1992), 508–518.
[20] M. Vuorinen, 'On the existence of angular limits of n-dimensional quasiregular mappings', *Ark. Mat.* **18** (1980), 157–180.
[21] H. Yamamoto, 'On weighted extremal length and p-capacity', *Hiroshima Math. J.* **20** (1990), 633–642.
[22] W. Ziemer, 'Extremal length and p-capacity', *Michigan Math. J.* **16** (1969), 43–51.
[23] W. Ziemer, 'Extremal length as capacity', *Michigan Math. J.* **17** (1969), 117–128.

ENRIQUE VILLAMOR, Department of Mathematics and Statistics,
Florida International University, Miami, FL 33199, USA
e-mail: villamor@fiu.edu

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