



RESEARCH ARTICLE

Estimates for evolutionary partial differential equations in classical function spaces

Alejandro J. Castro¹, Anders Israelsson², Wolfgang Staubach³ and Madi Yerlanov⁴

¹Nazarbayev University, Department of Mathematics, Qabanbay Batyr Ave 53, 010000 Astana, Kazakhstan; E-mail: alejandro.castilla@nu.edu.kz.

²Uppsala University, Department of Mathematics, Lägerhyddsvägen 1, 752 37 Uppsala, Sweden; E-mail: anders.Israelsson@math.uu.se.

³Uppsala University, Department of Mathematics, Lägerhyddsvägen 1, 752 37 Uppsala, Sweden; E-mail: wulf@math.uu.se.

⁴University of Colorado Boulder, Department of Applied Mathematics, Engineering Center, ECOT 225 526 UCB Boulder, CO 80309-0526, USA; E-mail: Madi.Yerlanov@colorado.edu.

Received: 26 March 2023; Revised: 20 July 2023; Accepted: 3 August 2023

2020 Mathematics Subject Classification: *Primary* – 42B20, 42B35, 42B37, 47D06, 47D08, 35G10; *Secondary* – 35S30, 37L50

Abstract

We establish new local and global estimates for evolutionary partial differential equations in classical Banach and quasi-Banach spaces that appear most frequently in the theory of partial differential equations. More specifically, we obtain optimal (local in time) estimates for the solution to the Cauchy problem for variable-coefficient evolutionary partial differential equations. The estimates are achieved by introducing the notions of Schrödinger and general oscillatory integral operators with inhomogeneous phase functions and prove sharp local and global regularity results for these in Besov–Lipschitz and Triebel–Lizorkin spaces.

Contents

1	Introduction	2
1.1	Some relevant results regarding boundedness of nondegenerate oscillatory integral operators	3
1.2	Some relevant results in the theory of linear partial differential equations	4
1.3	Synopsis of the results of this paper	5
2	Definitions and preliminaries	7
2.1	Notations	7
2.2	Function spaces	7
2.3	Oscillatory integral operators	10
2.4	Schrödinger integral operators	12
3	Main regularity results and applications	13
3.1	Local regularity results	13
3.1.1	Local boundedness of oscillatory integrals	13
3.1.2	Local boundedness of Schrödinger integrals	15
3.2	Global regularity results	15
3.2.1	Global boundedness of oscillatory integrals	15
3.2.2	Global boundedness of Schrödinger integrals	16

3.3	A parameter-dependent composition formula	18
3.4	Global $L^p - L^q$ estimates	19
3.5	Applications to harmonic analysis and PDEs	20
4	Estimates for phases and kernels	22
5	L^2 -boundedness	28
6	Boundedness of low-frequency portion	29
7	Boundedness of middle-frequency portion	33
8	Local $h^p - L^p$ boundedness	34
9	Boundedness of high-frequency portion	35
10	The $h^p - L^p$ boundedness of Schrödinger integral operators	38
11	Action of parameter-dependent pseudodifferential operators on oscillatory integrals	42
12	Regularity on Besov–Lipschitz spaces	48
13	Regularity on Triebel–Lizorkin spaces	53
14	Sharpness of the results	55

1. Introduction

In this paper, we prove sharp estimates for the solutions to initial value problems for a large class of linear evolutionary partial differential equations (PDEs) in classical function spaces. Examples include linear water-wave and capillary-wave equations, the Klein–Gordon equation, the Schrödinger-type equations, the Airy equation and higher-order dispersive equations. To achieve this, we develop a fairly general framework for the investigation of the regularity of a wide range of oscillatory integral operators that appear in the theory of partial differential equations and mathematical physics. Our results are obtained in the Besov–Lipschitz and Triebel–Lizorkin spaces which contain most classical function spaces such as the L^p , Sobolev, Hölder (or Lipschitz), Hardy and BMO spaces, just to mention some well-known examples.

In the context of well-posedness of the initial value problems for linear evolutionary PDEs, one has a variety of optimal regularity results for;

- i*) Local and global regularity of variable-coefficient strictly hyperbolic linear PDEs in various function spaces; see, for example, [20] and [29].
- ii*) Regularity of variable-coefficient linear Schrödinger-type PDEs in modulation spaces as well as certain Sobolev spaces (however, not L^p -spaces with $p \neq 2$); see, for example, [7] and [8].
- iii*) Regularity of constant coefficient linear dispersive PDEs of the form

$$-i\partial_t u + \Delta^{k/2} u = 0 \quad (1)$$

in Hardy and Hölder spaces; see, for example, [12] and [23].

The main contributions of this paper are twofold.

Firstly, from the point of view of PDE theory, we prove regularity results for variable coefficient Klein–Gordon equations which are hyperbolic equations that are solved with oscillatory integral ansatz with inhomogeneous phase functions (see further improvements of our results in the Klein–Gordon setting in [21]). This is in contrast to the strictly hyperbolic equations in item *i*) above, where the corresponding phase functions are positively homogeneous of degree one. We also obtain optimal regularity results for variable coefficient Schrödinger equations, in L^p , Hölder, Hardy and BMO spaces, which widely extends the results mentioned in item *ii*). For the dispersive equations of item *iii*), we not only handle the case of variable coefficient dispersive equations but also extend the regularity results to the realm of more general function spaces than just Hardy and Hölder spaces.

Secondly, from the point of view of Fourier analysis, we extend the local L^p -regularity theory of Fourier integral operators as was done in [29], to the case of general oscillatory integral operators with inhomogeneous phase functions. Furthermore, we investigate the action of pseudodifferential operators

on the general oscillatory integrals in the same spirit as in [18], which can be used in developing a calculus for general oscillatory integral operators.

1.1. Some relevant results regarding boundedness of nondegenerate oscillatory integral operators

We start by giving an overview of the previously known regularity results for oscillatory integral operators, which is to a large extent biased by their relevance to our current paper.

For simplicity, we confine ourselves to oscillatory integral operators of the form

$$T_a^\varphi f(x) := \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi) \widehat{f}(\xi) \, d\xi, \tag{2}$$

with amplitude $a(x, \xi)$ and phase function $\varphi(x, \xi)$. Here, following L. Hörmander [18] one can assume that the $a(x, \xi)$ belongs to the class $S_{\rho,\delta}^m(\mathbb{R}^n)$, which means that the amplitude is in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies

$$\left| \partial_\xi^\alpha \partial_x^\beta a(x, \xi) \right| \leq C_{\alpha,\beta} (1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}. \tag{3}$$

It was shown by Hörmander [18] and G. I. Eskin [10] that if $a(x, \xi) \in S_{1,0}^0(\mathbb{R}^n)$ is smooth and compactly supported in x and if $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ is positively homogeneous of degree 1 in ξ and the mixed Hessian matrix of $\varphi(x, \xi)$ has a nonzero determinant on the support of $a(x, \xi)$ (the nondegeneracy condition), then the operator T_a^φ is L^2 -bounded. These type of operators are examples of the so-called *Fourier integral operators*, which were officially introduced in [18]. The global extension of the L^2 -boundedness result of Eskin and Hörmander to all possible amplitudes in the class $S_{\rho,\delta}^m(\mathbb{R}^n)$ was done by D. Dos Santos Ferreira and W. Staubach [9]. In that paper, it was shown that, if the phase function φ is positively homogeneous of degree 1 in ξ , the determinant of the mixed Hessian of φ is globally bounded from below by a nonzero constant (the strong nondegeneracy condition), and φ satisfies the bound

$$\sup_{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}} |\xi|^{-1+|\alpha|} \left| \partial_\xi^\alpha \partial_x^\beta \varphi(x, \xi) \right| \leq C_{\alpha,\beta}, \quad |\alpha + \beta| \geq 2,$$

then the Fourier integral operator T_a^φ is globally L^2 -bounded, provided that $\rho, \delta \in [0, 1]$, $\delta \neq 1$ and $m = \min(0, n(\rho - \delta)/2)$, or $\rho \in [0, 1]$, $\delta = 1$ and $m < n(\rho - 1)/2$. This result is sharp, and therefore completes the study of Hörmander-class Fourier integral operators with nondegenerate phase functions.

Regarding operators T_a^φ where the phase function φ is inhomogeneous, it was shown by D. Fujiwara [13] that if $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies the condition

$$\left| \partial_\xi^\alpha \partial_x^\beta \varphi(x, \xi) \right| \leq c_{\alpha,\beta}, \quad |\alpha + \beta| \geq 2, \tag{4}$$

mentioned satisfies the strong nondegeneracy condition, and if the amplitude belongs to the class $S_{0,0}^0(\mathbb{R}^n)$, then T_a^φ is globally L^2 -bounded. In [6], E. Cordero, F. Nicola and L. Rodino gave an elegant proof of this L^2 -boundedness result, which completely avoids many of the technicalities (e.g., the use of the Cotlar–Stein lemma) involved in the previous proofs and instead relies on techniques from the theory of modulation spaces. Therefore, one could assert that the L^2 -regularity of operators T_a^φ with smooth amplitudes and smooth nondegenerate phase functions has been brought to completion. However, the extent of the impact of [6] was not confined to the aforementioned L^2 -result, and indeed the investigations of Cordero–Nicola–Rodino also paved the way and inspired much activity in the field and, not least, some of the results of this paper.

Turning to the problem of L^p -regularity for $p \neq 2$, J. Peral [25] and A. Miyachi [22] studied the problem of L^p -boundedness of Fourier multipliers of the form $m(\xi) = e^{i\varphi(\xi)} \sigma(\xi)$, $\sigma(\xi) \in S_{1,0}^m(\mathbb{R}^n)$ and φ positively homogeneous of degree one. It was realised that for a $\sigma \in S_{1,0}^m(\mathbb{R}^n)$ the L^p -boundedness

can (in general) not hold if $m > -(n-1)|1/p - 1/2|$. M. Beals [2] extended their results to operators of the form (2) when the phase function is analytic in $\xi \in \mathbb{R}^n \setminus \{0\}$, nondegenerate and positively homogeneous of degree one in ξ .

For oscillatory Fourier multipliers of the form $m(\xi) = e^{i|\xi|^k} \psi(\xi) |\xi|^m$, ψ smooth and vanishing near the origin, $0 < k < 1$ and $m < 0$, C. Fefferman and E.M. Stein proved the L^p boundedness for $1 < p < \infty$ in their seminal paper [12] on H^p -spaces. The L^p -boundedness of multipliers $m(\xi)$ in the case of $k > 1$ was established by H. Ishii [19]. All these results were further extended in an influential paper by Miyachi [23], which has had a significant impact on the development of the regularity theory of oscillatory integral operators in Banach and quasi-Banach spaces. Miyachi showed that for $k = 1$ the Fourier multiplier defined by $m(\xi)$ is H^p -bounded for $0 < p < \infty$ if and only if $m \leq -(n-1)|1/p - 1/2|$, and for $k > 0$ (but $k \neq 1$) T_a^φ is H^p -bounded if and only if $m \leq -kn|1/p - 1/2|$. Moreover, Miyachi proves $L^\infty(\mathbb{R}^n) \rightarrow \text{BMO}(\mathbb{R}^n)$ estimates as well as boundedness results in Lipschitz (or Hölder) spaces for the aforementioned Fourier multipliers.

In the range $1 < p < \infty$, Peral's and Miyachi's results for the ordinary wave operator were generalised by A. Seeger, C. Sogge and E. Stein [29] to Fourier integral operators with amplitudes $a(x, \xi) \in S_{1,0}^m(\mathbb{R}^n)$, $m \leq -(n-1)|1/p - 1/2|$, with compact spatial support and homogeneous of degree one (nondegenerate) phase functions $\varphi(x, \xi)$. Using a novel method which was also partly inspired by Fefferman's paper [11], Seeger–Sogge–Stein thus proved the optimal local L^p -boundedness of Fourier integral operators. Extensions to global estimates in more general function spaces were carried out by A. Israelsson, S. Rodríguez-López and W. Staubach in [20]. Thus, the investigations mentioned above complete the picture regarding the regularity of Fourier integral operators of the form (2) with nondegenerate homogeneous of degree one phase functions and amplitudes in $S_{1,0}^m(\mathbb{R}^n)$.

For the so-called *Schrödinger integral operators* which are operators of the form (2) with nondegenerate phase functions $\varphi(x, \xi)$ satisfying equation (4), the question of boundedness has so far only been treated in the context of modulation spaces. Indeed Cordero, Nicola and Rodino [6] showed that Schrödinger integral operators with amplitudes in the class $S_{0,0}^0(\mathbb{R}^n)$ are bounded from the modulation space M_p to itself for $1 \leq p < \infty$. Regarding calculi for Schrödinger integral operators, Asada and Fujiwara [1] studied the action of pseudodifferential operators on their class of oscillatory integrals and showed that their class is closed under composition with pseudodifferential operators of order zero. In 2013, E. Cordero, K-H. Gröchenig, F. Nicola and L. Rodino showed in [5] that the class of operators that we here refer to as Schrödinger integral operators (of order zero) is actually closed under composition.

1.2. Some relevant results in the theory of linear partial differential equations

In the paper [29], the authors prove optimal local regularity results for the solutions of strictly hyperbolic variable coefficients in L^p -based Sobolev spaces for fixed time and their results were extended in [20] to global estimates in Besov–Lipschitz and Triebel–Lizorkin spaces.

The earlier investigations of B. Helffer and D. Robert [16] and those of Helffer [15] in connection to the study of propagators for Schrödinger equations (for example, the harmonic oscillator) have demonstrated the importance of Schrödinger integral operators; see, for example, the paper of Cordero–Nicola–Rodino [7]. Furthermore, in the remarkable paper [8], P. D'Ancona and F. Nicola established sharp L^p -frequency-truncated estimates (for certain p 's) for the Schrödinger group e^{itH} , where H is a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ whose heat operator e^{-tH} satisfies a suitable off-diagonal algebraic decay estimate.

For dispersive equations of the form (1) with initial data $f(x)$, the solutions are given by oscillatory integrals of the form

$$Tf(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi + it|\xi|^k} \widehat{f}(\xi) \, d\xi \quad (5)$$

for which one can apply the results of [12] and [23] to obtain certain regularity results in Hardy and Hölder spaces.

1.3. Synopsis of the results of this paper

Given the discussion above, some natural open problems are:

- The extension of the results of Seeger–Sogge–Stein to global results that also accommodate the case of Klein–Gordon-type equations and also to the setting of both Banach and quasi-Banach spaces.
- Development of the L^p -regularity theory for oscillatory integral operators with phase functions in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ that are associated to variable-coefficient Schrödinger equations.
- The extension of the results of Miyachi to the variable-coefficient setting in the case of $k \neq 1$. In other words, investigation of the regularity properties of oscillatory integral operators, with phase functions in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ that fall beyond the scope of the theory of Fourier integral operators.
- Estimates for variable coefficient Schrödinger as well as other dispersive equations in Triebel–Lizorkin and Besov–Lipschitz spaces.

In this paper, we have made progress in solving these problems and summarise our results as follows:

- i) Established an optimal L^p -regularity theory (and indeed even $L^p - L^q$, $1 < p \leq q < \infty$), for Schrödinger integral operators. We also go beyond these classes of operators and investigate the regularity of general oscillatory integral operators (see Definition 2.15) in classical function spaces. Furthermore, our results extend the range of validity of the estimates obtained by D’Ancona and Nicola in [8].
- ii) Investigated the boundedness problem in both Banach and quasi-Banach spaces. Our regularity results are valid in Besov–Lipschitz and Triebel–Lizorkin spaces with both Banach and quasi-Banach scales.
- iii) The abolition of the usual homogeneity assumption in the phase functions and improvement on the order of decay of the amplitudes. We show that it is enough for the amplitudes to merely belong to the class $S_{0,0}^m(\mathbb{R}^n)$, as opposed to the usual class $S_{1,0}^m(\mathbb{R}^n)$ (which is used in all previously obtained L^p -regularity results).
- vi) Sharp boundedness results were obtained, namely we show that our results are optimal for the specific order of decay m that we choose. Our results are not only sharp regarding the order of the amplitudes but also optimal regarding their type, which is measured by the lowercase indices of the classes of amplitudes. Indeed, for those oscillatory integral operators that are not Fourier integral operators, choosing an amplitude in the better class $S_{1,0}^m(\mathbb{R}^n)$ would not yield any improvement in the order of decay m (which is required for the regularity in various function spaces).
- v) Thorough analysis of the singular low-frequency part of globally defined oscillatory integral operators. The phase functions of oscillatory integrals such as those given by equation (5) have a singularity at origin when k is not an even positive integer. In previous investigations in the literature (i.e., Fefferman–Stein [12], Ishii [19], Miyachi [23], Peral [25] and Seeger–Sogge–Stein [29]), this singularity is always cut out in order to confine the analysis to the high-frequency part of the operator. However, in PDE theory, it is important to investigate the regularity of the whole operator (i.e., both the low- and high-frequency portions). In fact, the low-frequency part has a decisive impact on the range of the validity of global estimates in the quasi-Banach setting, as we have clearly demonstrated in the paper.
- vi) Steps towards construction of a calculus for oscillatory integral operators. We prove a basic composition theorem (similar to that of Hörmander’s in the Fourier integral operator setting) for the composition of a pseudodifferential operator and a general oscillatory integral operator. The advantages of our ‘calculus’ as compared to Hörmander’s composition formula for Fourier multipliers and Fourier integral operators are (a) our composition result is global, (b) the composition works for operators with inhomogeneous phase functions and (c) our composition is parameter dependent and the dependency on the parameter is carefully tracked. We show the usefulness of this composition

theorem in our demonstration of the regularity results in Besov–Lipschitz and Triebel–Lizorkin spaces.

- vii) Sharp estimates for linear dispersive PDEs in Banach and quasi-Banach spaces. The oscillatory integrals that are studied in this paper can be used to construct solutions to various evolutionary PDEs and our oscillatory integral estimates provide new sharp estimates for the solutions to initial value problems in Besov–Lipschitz and Triebel–Lizorkin spaces.

The proofs of the results are divided into essentially two categories. Those for oscillatory integral operators whose phases have a singularity at the origin (the simplest examples are provided by equation (5)), and those oscillatory integrals whose phases are smooth everywhere (e.g., the case of Schrödinger integral operators). In the former case, we split the operator into, low-, middle-, and high-frequency pieces and deal with each piece separately. It turns out that the low-frequency portion requires an additional condition which is also reflected in the optimal range of the validity of the boundedness results. The high-frequency portion can be dealt with methods based on Littlewood–Paley theory. However, the differences here compared to that of the case of Fourier integral operators treated in [29] are that we deal with a larger class of amplitudes and also have to deal with inhomogeneous phase functions. This forces a different approach towards the proof of the boundedness of general oscillatory integral operators. Indeed, the usual approach of first showing the $H^1 - L^1$ boundedness, which was carried out successfully for Fourier integral operators in [29], fails in our case. Moreover, the lack of a calculus (once again as opposed to the case of local Fourier integral operators in [29]) hampers the way of using duality arguments. To remedy this, we prove the boundedness of the operator and its adjoint in the quasi-Banach realm of the $h^p - L^p$ spaces ($0 < p < 1$ and h^p is the local Hardy space of D. Goldberg’s [14]). Thereafter, we lift these boundedness results to the $h^p - h^p$ boundedness and then use appropriate interpolation of Triebel–Lizorkin spaces to extend matters to all ranges of p ’s. Thus, utilising the quasi-Banach Hardy spaces in this context is crucial for our goals.

Concerning the proof of the boundedness of Schrödinger integral operators under the mere assumptions of strong nondegeneracy of the phase function and condition (4), it is not enough to simply use the Littlewood–Paley theory. Here, we once again use quasi-Banach spaces and use yet another frequency localisation superimposed on the first Littlewood–Paley decomposition adapted specifically to Schrödinger integral operators. In this connection, we have a domain of influence for Littlewood–Paley pieces of the operator and one has to use an atomic decomposition of the Hardy spaces to be able to estimate the L^p -norms of various pieces of the operator (and its adjoint) in the interior and the exterior of the aforementioned domain of influence. Putting the partial results together and summing up, we can show the global $h^p - L^p$ boundedness of the Schrödinger integral operators and use interpolation to obtain results for a wider range of p ’s.

In obtaining regularity results in Besov–Lipschitz and Triebel–Lizorkin spaces, it behooves one to consider the action of Fourier multipliers on various oscillatory integral operators. To achieve that, one needs some sort of a calculus (or asymptotic expansion for the composition). The proof of this special form of calculus, which is also extended to the composition of a pseudodifferential operator and an oscillatory integral operator, is a rather technical task that has been carried out by a suitable splitting of the operators into various pieces and delicate oscillatory integral estimates.

The paper is organised as follows: In Section 2, we recall some of the basic definitions and facts from the theory of function spaces. We also define our classes of phase functions, amplitudes and the corresponding operators which will be treated in this paper. In Section 3, we state the main results of the paper and outline the proofs of the theorems. This includes both local and global regularity results in Besov–Lipschitz and Triebel–Lizorkin spaces, $L^p - L^q$ estimates and also our parameter-dependent composition theorem. Furthermore, in the same section we also provide some examples regarding applications of our results within harmonic analysis and partial differential equations. For example, we show the regularity of operators with phase functions of the form $x \cdot \xi + t(x)|\xi|^k$ with $0 < k \leq 1$ and $x \cdot \xi + t(x)\langle \xi \rangle$, with $t(x)$ being smooth and bounded together with all of its derivatives. The former is

significant in the study of water-wave equation ($k = 1/2$), and the latter example is of significance in the study of Klein–Gordon equations. Since our regularity results are also valid for phase functions of the form $x \cdot \xi + |\xi|^k$ with $0 < k < \infty$, this enables us to prove sharp basic estimates (in both Banach and quasi-Banach scales) for the solutions of a large class of dispersive PDEs. Thereafter, we turn to variable-coefficient Schrödinger equations and show sharp estimates in classical function spaces for the solutions of Schrödinger equations with quadratic potentials (including the case of the harmonic oscillator).

Section 4 is devoted to the basic kernel estimates for oscillatory integral operators. In Section 5, we discuss the L^2 -regularity of the operators, and in Section 6, we deal with the boundedness of the low-frequency portion of the operators in Besov–Lipschitz and Triebel–Lizorkin spaces. Since we will sometimes divide the operators in question into low-frequency, middle-frequency and high-frequency portions, in Section 7 we treat the boundedness of the middle-frequency portion of the operators. In Section 8, we prove a local $h^p - L^p$ result for the oscillatory integral operators and Section 9 is devoted to the study of the $h^p - L^p$ boundedness of the high-frequency portion of the operators. The same problem for the Schrödinger integral operators is treated in Section 10. In Section 11, we prove a parameter-dependent composition formula and an expansion for the action of a pseudodifferential operator on an oscillatory integral operator. Section 12 and Section 13 are devoted to regularity results in Besov–Lipschitz and Triebel–Lizorkin spaces, respectively. The sharpness of the results is discussed in Section 14.

2. Definitions and preliminaries

In this section, we will collect all the definitions that will be used throughout this paper. We also state some useful results from both harmonic and microlocal analysis which will be used in the proofs.

2.1. Notations

We will denote constants which can be determined by known parameters in a given situation, but whose values are not crucial to the problem at hand, by C , or c or c_α and so on. Such parameters in this paper would be, for example, m , p , n , and the constants connected to the seminorms of various amplitudes or phase functions. The value of the constants may differ from line to line, but in each instance could be estimated if necessary. We also write $a \lesssim b$ as shorthand for $a \leq Cb$ and $a \sim b$ when $a \lesssim b$ and $b \lesssim a$.

Also, we shall denote the normalised Lebesgue measure $d\xi/(2\pi)^n$ by $d\xi$, $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$, the space of smooth functions with compact support by $C_c^\infty(\mathbb{R}^n)$, the space of smooth functions with bounded derivatives of all orders by $C_b^\infty(\mathbb{R}^n)$, the Schwartz class of rapidly decreasing smooth functions by $\mathcal{S}(\mathbb{R}^n)$ and the set of nonnegative integers $\{0, 1, 2, \dots\}$ by \mathbb{Z}_+ . In what follows, we use the notation

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx,$$

for the Fourier transform of the function f and ξ and η will denote frequency variables.

2.2. Function spaces

We start this section by defining the standard *Littlewood–Paley decomposition* which is a basic ingredient in our proofs and is also used to define the function spaces that we are concerned with here.

Definition 2.1. Let $\psi_0 \in C_c^\infty(\mathbb{R}^n)$ be equal to 1 on $B(0, 1)$ and have its support in $B(0, 2)$. Then let

$$\psi_j(\xi) := \psi_0(2^{-j}\xi) - \psi_0(2^{-(j-1)}\xi),$$

where $j \geq 1$ is an integer and $\psi(\xi) := \psi_1(\xi)$. Then $\psi_j(\xi) = \psi(2^{-(j-1)}\xi)$ and one has the following Littlewood–Paley partition of unity

$$\sum_{j=0}^{\infty} \psi_j(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^n.$$

Observe that ψ_j is supported inside the annulus $\{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. It is sometimes also useful to define a sequence of smooth and compactly supported functions Ψ_j with $\Psi_j = 1$ on the support of ψ_j and $\Psi_j = 0$ outside a slightly larger compact set. Explicitly, one could set

$$\Psi_j := \psi_{j+1} + \psi_j + \psi_{j-1},$$

with $\psi_{-1} := \psi_0$.

Next, we proceed with the definition of *local Hardy space*, $h^p(\mathbb{R}^n)$ due to D. Goldberg; see [14]. This space plays an important role in the paper since many of the subsequent results will be obtained by means of interpolation with the local Hardy spaces.

Definition 2.2. For $0 < p \leq 1$, the local Hardy space $h^p(\mathbb{R}^n)$ is the set of distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{h^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \sup_{0 < t < 1} |\psi_0(tD)f(x)|^p dx \right)^{1/p} < \infty,$$

where ψ_0 is given in Definition 2.1, and for $t \in \mathbb{R}$

$$\psi_0(tD)f(x) := \int_{\mathbb{R}^n} e^{ix \cdot \xi} \psi_0(t\xi) \widehat{f}(\xi) d\xi.$$

Another useful definition of the Hardy spaces is based on the so-called *atoms* and is given as follows.

Definition 2.3. For $0 < p \leq 1$, a function \mathbf{a} is called an h^p -atom (or a p -atom for short) if for some $x_0 \in \mathbb{R}^n$ and $r > 0$ the following three conditions are satisfied:

- i) $\text{supp } \mathbf{a} \subset B(x_0, r)$,
- ii) $|\mathbf{a}(x)| \leq |B(x_0, r)|^{-1/p}$,
- iii) if $r \leq 1$, then $\int_{\mathbb{R}^n} x^\alpha \mathbf{a}(x) dx = 0$, for any multi-index α with $|\alpha| \leq [n(1/p - 1)]$, and no further condition if $r > 1$. Here, $[x]$ denotes the integer part of x .

Then one has that a distribution $f \in h^p(\mathbb{R}^n)$ has an atomic decomposition

$$f = \sum_j \lambda_j \mathbf{a}_j,$$

where the λ_j are constants such that

$$\inf \left(\sum_j |\lambda_j|^p \right)^{1/p} \sim \|f\|_{h^p(\mathbb{R}^n)},$$

and the \mathbf{a}_j are p -atoms.

For $1 < p < \infty$, we identify $h^p(\mathbb{R}^n)$ with $L^p(\mathbb{R}^n)$. The dual of the local Hardy space $h^1(\mathbb{R}^n)$ is the *local BMO*(\mathbb{R}^n) and is denoted by $\text{bmo}(\mathbb{R}^n)$, which consists of locally integrable functions that verify

$$\|f\|_{\text{bmo}(\mathbb{R}^n)} := \|f\|_{\text{BMO}(\mathbb{R}^n)} + \|\psi_0(D)f\|_{L^\infty(\mathbb{R}^n)} < \infty,$$

where $BMO(\mathbb{R}^n)$ is the usual John–Nirenberg space of functions of bounded mean oscillation and ψ_0 is the cut-off function introduced in Definition 2.1.

Using the Littlewood–Paley decomposition above, we define the Besov–Lipschitz spaces.

Definition 2.4. Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. The Besov–Lipschitz spaces are defined by

$$B_{p,q}^s(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B_{p,q}^s(\mathbb{R}^n)} := \left(\sum_{j=0}^{\infty} 2^{jq_s} \|\psi_j(D)f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} < \infty \right\}.$$

It is worth mentioning that for $p = q = \infty$ and $0 < s \leq 1$ we obtain the familiar Lipschitz (or Hölder) space $\Lambda^s(\mathbb{R}^n)$, that is,

$$B_{\infty,\infty}^s(\mathbb{R}^n) = \Lambda^s(\mathbb{R}^n).$$

We will also produce boundedness results in the realm of Triebel–Lizorkin spaces which can be defined using Littlewood–Paley theory, as follows.

Definition 2.5. Let $0 < p < \infty, 0 < q \leq \infty$ and $s \in \mathbb{R}$. The Triebel–Lizorkin spaces are given by

$$F_{p,q}^s(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{F_{p,q}^s(\mathbb{R}^n)} := \left\| \left(\sum_{j=0}^{\infty} 2^{jq_s} |\psi_j(D)f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty \right\}.$$

It is well known (see, for example, [31, p. 51]) that

$$F_{p,q}^s(\mathbb{R}^n) = \begin{cases} L^p(\mathbb{R}^n), & s = 0, & 1 < p < \infty, q = 2, \\ h^p(\mathbb{R}^n), & s = 0, & 0 < p \leq 1, q = 2, \\ bmo(\mathbb{R}^n), & s = 0, & p = \infty, q = 2, \\ H^{s,p}(\mathbb{R}^n), & -\infty < s < \infty, & 1 < p < \infty, q = 2, \end{cases} \tag{6}$$

where $H^{s,p}(\mathbb{R}^n)$ are various Sobolev–Slobodetskij spaces.

Remark 2.6. Different choices of the basis $\{\psi_j\}_{j=0}^{\infty}$ give equivalent (quasi)-norms of $B_{p,q}^s(\mathbb{R}^n)$ and $F_{p,q}^s(\mathbb{R}^n)$ in Definition 2.4 and 2.5; see, for example, [31, p. 41]. We will use either $\{\psi_j\}_{j=0}^{\infty}$ or $\{\Psi_j\}_{j=0}^{\infty}$ to define the norm of $B_{p,q}^s(\mathbb{R}^n)$ and $F_{p,q}^s(\mathbb{R}^n)$.

Another fact which will be useful to us is that for $-\infty < s < \infty$ and $0 < p \leq \infty$

$$B_{p,p}^s(\mathbb{R}^n) = F_{p,p}^s(\mathbb{R}^n) \tag{7}$$

and that one has the two continuous embeddings

$$F_{p,q_0}^{s+\varepsilon}(\mathbb{R}^n) \hookrightarrow F_{p,q_1}^s(\mathbb{R}^n) \quad \text{and} \quad B_{p,q_0}^{s+\varepsilon}(\mathbb{R}^n) \hookrightarrow B_{p,q_1}^s(\mathbb{R}^n) \tag{8}$$

for $-\infty < s < \infty, 0 < p < \infty, 0 < q_0, q_1 \leq \infty$ and all $\varepsilon > 0$. Furthermore, for $s' \in \mathbb{R}$, the operator $(1 - \Delta)^{s'/2}$ maps $F_{p,q}^s(\mathbb{R}^n)$ isomorphically into $F_{p,q}^{s-s'}(\mathbb{R}^n)$ and $B_{p,q}^s(\mathbb{R}^n)$ isomorphically into $B_{p,q}^{s-s'}(\mathbb{R}^n)$; see [31, p. 58].

We will also make repeated use of the estimate; for and all s, p, q

$$\|fu\|_{A_{p,q}^s(\mathbb{R}^n)} \lesssim \left(\sum_{|\alpha| \leq M} \sup_{x \in \mathbb{R}^n} |\partial^\alpha f(x)| \right) \|u\|_{A_{p,q}^s(\mathbb{R}^n)}, \tag{9}$$

which is valid for $A = B$ (Besov–Lipschitz spaces) or $A = F$ (Triebel–Lizorkin spaces), and $M \in \mathbb{Z}_+$ large enough; see [28, p. 229, eq. (9), (10)]. For all the other facts about function spaces that are used in this paper, we refer the reader to [31].

We will state the following lemma which is a consequence of a lemma originally due to J. Peetre [24], which turns out to be useful in proving quasi-Banach Besov–Lipschitz/Triebel–Lizorkin boundedness of the low-frequency portions of oscillatory integral operators studied in forthcoming sections.

Lemma 2.7. *Let $f \in C^1(\mathbb{R}^n)$ with Fourier support inside the unit ball. Then for every $\rho > n$, and $r \in (n/\rho, 1]$ one has*

$$(\langle \cdot \rangle^{-\rho} * |f|)(x) \lesssim \left(M(|f|^r)(x) \right)^{1/r}, \quad x \in \mathbb{R}^n,$$

where M denotes the Hardy–Littlewood maximal function on \mathbb{R}^n .

Proof. As was shown by Peetre, see, for example, [28, Proposition 2, p. 22], one has for $r \geq n/\rho$ that

$$\sup_{y \in \mathbb{R}^n} \frac{|f(x-y)|}{\langle y \rangle^\rho} \lesssim \left(M(|f|^r)(x) \right)^{1/r}. \tag{10}$$

Now, taking $r \in (n/\rho, 1]$, and using equation (10) we obtain

$$\begin{aligned} |\langle \cdot \rangle^{-\rho} * f(x)| &\lesssim \int_{\mathbb{R}^n} \frac{|f(x-y)|}{\langle y \rangle^\rho} dy \leq \left(\sup_{y \in \mathbb{R}^n} \frac{|f(x-y)|}{\langle y \rangle^\rho} \right)^{1-r} \int_{\mathbb{R}^n} \frac{|f(x-y)|^r}{\langle y \rangle^{\rho r}} dy \\ &\lesssim \left(M(|f|^r)(x) \right)^{1/r-1} \left(M(|f|^r)(x) \right) = \left(M(|f|^r)(x) \right)^{1/r}. \quad \square \end{aligned}$$

In establishing the local boundedness of oscillatory integral operators in the range $0 < p < 1$, the following Bernstein-type estimate will be useful. The proof can be found in [31, p. 22].

Lemma 2.8. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a compact set, and let $0 < p \leq r \leq \infty$. Then*

$$\|\partial^\alpha f\|_{L^r(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

for all multi-indices α and all $f \in L^p_{\mathcal{K}}(\mathbb{R}^n)$, where

$$L^p_{\mathcal{K}}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{L^p(\mathbb{R}^n)} < \infty, \text{supp } \widehat{f} \subset \mathcal{K} \right\}.$$

2.3. Oscillatory integral operators

The class of amplitudes which are the basic building blocks of the oscillatory integral operators were first introduced by L. Hörmander in [17].

Definition 2.9. Let $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. An amplitude (symbol) $a(x, \xi)$ in the class $S^m_{\rho, \delta}(\mathbb{R}^n)$ is a function $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ that verifies the estimate

$$\left| \partial^\alpha_\xi \partial^\beta_x a(x, \xi) \right| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|},$$

for all multi-indices α and β and $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. We shall henceforth refer to m as the order of the amplitude and ρ, δ as its type.

Given the symbol classes defined above, one associates to the symbol its Kohn–Nirenberg quantisation as follows.

Definition 2.10. Let a be a symbol. Define a pseudodifferential operator (Ψ DO for short) as the operator

$$a(x, D)f(x) := \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(x, \xi) \widehat{f}(\xi) \, d\xi,$$

a priori defined on the Schwartz class $\mathcal{S}(\mathbb{R}^n)$.

In order to define the oscillatory integral operators that are studied in this paper, we also define classes of phase functions, which together with the amplitudes of Definition 2.9 are useful and natural conditions to assume in the study of oscillatory integral operators.

Definition 2.11. For $0 < k < \infty$, we say that a real-valued *phase function* $\varphi(x, \xi)$ belongs to the class \mathcal{P}^k , if $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ and satisfies the following estimates (depending on the range of k):

- for $k \geq 1$,

$$\left| \partial_\xi^\alpha (\varphi(x, \xi) - x \cdot \xi) \right| \leq c_\alpha |\xi|^{k-1}, \quad |\alpha| \geq 1,$$

- for $0 < k < 1$,

$$\left| \partial_\xi^\alpha \partial_x^\beta (\varphi(x, \xi) - x \cdot \xi) \right| \leq c_{\alpha, \beta} |\xi|^{k-|\alpha|}, \quad |\alpha + \beta| \geq 1,$$

for all $x \in \mathbb{R}^n$ and $|\xi| \geq 1$.

Remark 2.12. Allowing a singularity (in the frequency) at the origin in the phase functions is a natural assumption for both $k \geq 1$ and $k < 1$ and is motivated by the *PDE* applications. Indeed, the phase function associated to the wave equation is $x \cdot \xi + |\xi|$ ($k = 1$), the phase associated to the water-wave equation is $x \cdot \xi + |\xi|^{1/2}$ ($k < 1$) and the phase associated to the capillary waves is $x \cdot \xi + |\xi|^{3/2}$ ($k > 1$), all of which are nonsmooth at $\xi = 0$.

We will also need to consider phase functions that satisfy a certain *nondegeneracy condition*. To this end, we have

Definition 2.13. One says that the phase function $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ satisfies the strong nondegeneracy condition (or φ is *SND* for short) if

$$\left| \det \left(\partial_{x_j \xi_k}^2 \varphi(x, \xi) \right) \right| \geq \delta,$$

for some $\delta > 0$, all $x \in \mathbb{R}^n$ and all $|\xi| \geq 1$.

In case $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, then we require that the condition above is satisfied for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

In order to guarantee that our operators are globally L^2 -bounded we should also put yet another condition of the phase which we shall henceforth simply refer to as the L^2 -condition (motivated by D. Fujiwara’s result in [13]).

Definition 2.14. One says that the phase function $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ satisfies the L^2 -condition if

$$\left| \partial_\xi^\alpha \partial_x^\beta \varphi(x, \xi) \right| \leq c_{\alpha, \beta}, \quad |\alpha| \geq 1, \quad |\beta| \geq 1, \tag{11}$$

for all $x \in \mathbb{R}^n$ and all $|\xi| \geq 1$.

In case $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, then we require that the condition above is satisfied for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Having the definitions of the amplitudes and the phase functions at hand, one has

Definition 2.15. An oscillatory integral operator T_a^φ with amplitude $a \in S_{\rho,\delta}^m(\mathbb{R}^n)$ and a real-valued phase function φ is defined (once again a priori on $\mathcal{S}(\mathbb{R}^n)$) by

$$T_a^\varphi f(x) := \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi) \widehat{f}(\xi) \, d\xi. \tag{12}$$

If $\varphi \in \mathfrak{F}^k$ and is SND, then these operators will be referred to as oscillatory integral operators of order k .

The formal adjoint of T_a^φ is denoted by $(T_a^\varphi)^*$ and is given by

$$(T_a^\varphi)^* f(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi - i\varphi(y,\xi)} \overline{a(y,\xi)} f(y) \, dy \, d\xi. \tag{13}$$

To deal with the low-frequency portion of an oscillatory integral, which is frequency supported in a neighborhood of the origin, one would need a separate analysis because the phase function might be, and it usually is singular at the origin. This typically doesn't affect the Banach space results so much, but as we shall see, it certainly restricts the ranges of parameters in the quasi-Banach spaces. Therefore, to be able to prove regularity results for the low-frequency portions of the operators, one should put a mild condition on the phase functions. From the point of view of the applications into PDEs, this condition will always be satisfied and would not cause any loss of generality.

Definition 2.16. Assume that $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ is real-valued and $0 < \mu \leq 1$. We say that φ satisfies the low-frequency phase condition of order μ , (φ satisfies $LF(\mu)$ -condition for short), if one has

$$\left| \partial_\xi^\alpha \partial_x^\beta (\varphi(x, \xi) - x \cdot \xi) \right| \leq c_\alpha |\xi|^{\mu - |\alpha|}, \tag{14}$$

for all $x \in \mathbb{R}^n$, $0 < |\xi| \leq 2$ and all multi-indices α, β .

2.4. Schrödinger integral operators

Another important class of oscillatory integrals is the following.

Definition 2.17. An operator of the form (12) with a real-valued phase function $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ that verifies

$$\left| \partial_\xi^\alpha \partial_x^\beta \varphi(x, \xi) \right| \leq c_{\alpha,\beta}, \quad |\alpha + \beta| \geq 2, \tag{15}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$, will be referred to as a *Schrödinger integral operator*.

Remark 2.18. Observe that in one dimension $\sin x \sin \xi + \xi^2 + (2\xi + 1)x$ is an example of an SND phase function satisfying equation (15) which is not in \mathfrak{F}^2 .

Our motivation for such a name stems from the fact that the solution to the Cauchy problem with initial data f , for the free Schrödinger equation is given by the operator $e^{it\Delta} f$. Observe that for a fixed time (say $t = 1$), the phase function of the oscillatory integral defining the Schrödinger semigroup is given by $x \cdot \xi + |\xi|^2$ which satisfies equation (15) and is also SND, and its amplitude is identically equal to one which is trivially in the class $S_{1,0}^0(\mathbb{R}^n)$. A less naive example, which once again motivates our choice of designation above, stems from the Cauchy problem for the Schrödinger equation associated to the quantum mechanical harmonic oscillator $-\Delta + |x|^2$. In this case, the solution is given by $e^{it(-\Delta + |x|^2)} f$, which is also a Schrödinger integral operator according to Definition 2.17 with a phase function which is once again SND and verifies equation (15); see [15].

For the purpose of proving boundedness results for oscillatory integral operators, it turns out that, in most of the cases, the following order of the amplitude is the critical one, namely

$$m_k(p) := -kn \left| \frac{1}{p} - \frac{1}{2} \right|,$$

where $0 < p \leq \infty$ and $k > 0$ stems from the so-called \mathfrak{F}^k -condition, which is given in Definition 2.11. The corresponding critical order for the Schrödinger integral operators will be $m_2(p)$. This means that, we will be able to establish various boundedness results for the oscillatory integral operators (and Schrödinger integral operators) when the order of the amplitude is less than or equal to $m_k(p)$ (or $m_2(p)$), respectively.

A common method throughout the paper will be to split the amplitude $a(x, \xi)$ into several pieces with respect to ξ . This is used when there is a singularity at the origin $\xi = 0$ that needs to be treated separately. In some cases, we divide the amplitude into a low- and a high-frequency part

$$a(x, \xi) = \psi_0(\xi) a(x, \xi) + (1 - \psi_0(\xi)) a(x, \xi) =: a_L(x, \xi) + a_H(x, \xi),$$

where ψ_0 is given in Definition 2.1. In other cases, we divide the amplitude into three different pieces, a low-, middle- and high-frequency part

$$\begin{aligned} a(x, \xi) &= \psi_0(\xi) a(x, \xi) + (\psi_0(\xi/R) - \psi_0(\xi)) a(x, \xi) + (1 - \psi_0(\xi/R)) a(x, \xi) \\ &=: a_L(x, \xi) + a_M(x, \xi) + a_H(x, \xi), \end{aligned}$$

where R is some large constant that typically depends only on the dimension and the upper and lower bound of the mixed Hessian of φ .

Remark 2.19. We should emphasise here that the conditions that are put on the phases of the oscillatory- and the Schrödinger integral operators are quite natural and indeed without the SND-condition and boundedness of the mixed derivatives (11), the operators under consideration (i.e., with inhomogeneous phase functions) are not (in general) even L^2 -bounded. Assuming, say homogeneity of degree one in the frequency variable of the phase function, which is the case of Fourier integral operators, enables one to improve on the order of decay of the amplitude. This is, however, not possible for the Schrödinger- and general oscillatory integral operators. The other conditions on the phase functions are there to guarantee L^p -boundedness, and the ability to develop a calculus in order to be able to establish boundedness in Besov–Lipschitz and Triebel–Lizorkin spaces.

3. Main regularity results and applications

In this section, we gather the main regularity results of this paper and briefly outline the proofs or rather refer to the relevant sections where the various proofs could be found. At the end of this section, we shall discuss the application of our results to regularity problems in harmonic analysis and theory of partial differential equations.

3.1. Local regularity results

This subsection deals with local regularity of both Schrödinger integral operators and oscillatory integral operators on Besov–Lipschitz and Triebel–Lizorkin spaces. This, as usual, amounts to study the operators whose amplitude $a(x, \xi)$ is compactly supported in the spatial variables.

3.1.1. Local boundedness of oscillatory integrals

First, we start by the following basic theorem which is the counterpart of the available local L^p -boundedness result in the more familiar context of Fourier integral operators.

In what follows, the operator T_a^φ will denote an oscillatory integral of the form (12).

Theorem 3.1. *Let $k \geq 1$, $p \in (1, \infty)$ and $a(x, \xi) \in S_{0,0}^{m_k(p)}(\mathbb{R}^n)$ with compact support in the x -variable. Assume that one of the following assumptions hold true:*

- i) $\varphi(x, \xi) \in \mathfrak{F}^k \cap C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is SND and satisfies the L^2 -condition (11).
- ii) $\varphi(x, \xi) \in \mathfrak{F}^k$ is SND, satisfies equation (11) and additionally satisfies the estimate

$$|\partial_\xi^\alpha \partial_x^\beta \varphi(x, \xi)| \leq c_{\alpha,\beta} |\xi|^{-|\alpha|}, \quad |\alpha + \beta| \geq 1, \quad 0 < |\xi| \leq 2\sqrt{n}.$$

Then in either case, the operator T_a^φ maps $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ continuously. In the case $0 < k < 1$, all the results above are true provided that $a(x, \xi) \in S_{1,0}^{m_k(p)}(\mathbb{R}^n)$.

Outline of the proof.

For the high-frequency portion of the operator, we use Propositions 9.1 and 9.2 to show that for $a(x, \xi) \in S_{0,0}^{m_k(p_0)}(\mathbb{R}^n)$ (when $k \geq 1$), and for $a(x, \xi) \in S_{1,0}^{m_k(p_0)}(\mathbb{R}^n)$ (when $0 < k < 1$), the operators T_a^φ and $(T_a^\varphi)^*$ are bounded from $h^{p_0}(\mathbb{R}^n)$ to $L^{p_0}(\mathbb{R}^n)$ for all $0 < p_0 < 1$. Now, using analytic interpolation to the analytic family of operators in the Hardy space setting due to R. Macías (see, e.g., [4, Theorem E, p. 597]), one considers $T_{a_z}^\varphi$ and $(T_{a_z}^\varphi)^*$, with $0 \leq \text{Re } z \leq 1$ and

$$a_z(x, \xi) := |\xi|^{m_k(p_0)\varepsilon - (1+\varepsilon)m_k(p_0)z} a(x, \xi),$$

with $\varepsilon = (1/q - 1/2)/(1/p_0 - 1/2) - 1$ and $q < 1$ chosen such that $[n(1/q - 1/2)] = [n/2]$. Now, the method of proof of Propositions 9.1 and 9.2 reveals that $T_{a_{i\text{Im } z}}^\varphi$ and $(T_{a_{i\text{Im } z}}^\varphi)^*$ are bounded from $h^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with bounds that depend on a positive power of $(1 + |\text{Im } z|)$ while Theorem 5.1 yields that $T_{a_{1+i\text{Im } z}}^\varphi$ and $(T_{a_{1+i\text{Im } z}}^\varphi)^*$ are bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with constant bounds independent of z . This enables one to interpolate these results in accordance with [4, Theorem E, p. 597] to show that T_a^φ and $(T_a^\varphi)^*$ are bounded from $h^{p_0}(\mathbb{R}^n)$ to $L^{p_0}(\mathbb{R}^n)$ for all $0 < p_0 \leq 2$. Hence, the claimed L^p -boundedness follows by duality.

For the low-frequency portion of the operator, when the phase function is smooth we use Proposition 8.1, Lemma 6.2 and interpolation with Fujiwara’s L^2 -boundedness result in [13] (see the proof of Theorem 5.1 for details). For the low-frequency portion in the nonsmooth case, we just use Lemma 6.1. □

The next theorem deals with the local regularity of oscillatory integral operators on Besov–Lipschitz and Triebel–Lizorkin spaces.

Theorem 3.2. *Let $m, s \in \mathbb{R}$ and $a(x, \xi) \in S_{0,0}^m(\mathbb{R}^n)$ with compact support in the x -variable. Assume that $k \geq 1$, $\varphi \in \mathfrak{F}^k$ is SND, satisfies the L^2 -condition (11) and the $LF(\mu)$ -condition (14) for some $0 < \mu \leq 1$. Then the following statements hold true:*

- i) If $p \in (0, \infty]$, $q \in (0, \infty]$, then $T_a^\varphi : B_{p,q}^{s+m-m_k(p)}(\mathbb{R}^n) \rightarrow B_{p,q}^s(\mathbb{R}^n)$.
- ii) If $p \in (0, \infty)$, $q \in (0, \infty]$ and $\varepsilon > 0$, then $T_a^\varphi : F_{p,q}^{s+m-m_k(p)+\varepsilon}(\mathbb{R}^n) \rightarrow F_{p,q}^s(\mathbb{R}^n)$.
- iii) If $p \in (0, \infty)$, $\min(2, p) \leq q \leq \max(2, p)$, then $T_a^\varphi : F_{p,q}^{s+m-m_k(p)}(\mathbb{R}^n) \rightarrow F_{p,q}^s(\mathbb{R}^n)$.
- iv) $T_a^\varphi : F_{\infty,2}^{s+m-m_k(\infty)}(\mathbb{R}^n) \rightarrow F_{\infty,2}^s(\mathbb{R}^n)$.

In the case $0 < k < 1$, all the results above are true provided that $a(x, \xi) \in S_{1,0}^m(\mathbb{R}^n)$.

Outline of the proof.

- i) See Section 12.
- ii)-iv) See Section 13. □

3.1.2. Local boundedness of Schrödinger integrals

The following theorem deals with the question of the local regularity of the Schrödinger integral operators in Besov–Lipschitz and Triebel–Lizorkin spaces.

Theorem 3.3. *Let $m, s \in \mathbb{R}$ and $a(x, \xi) \in S_{0,0}^m(\mathbb{R}^n)$ with compact support in the x -variable. Assume that φ satisfies equation (15) and is SND. Then the following statements hold true:*

- i) *If $p \in (0, \infty]$, $q \in (0, \infty]$, then $T_a^\varphi : B_{p,q}^{s+m-m_2(p)}(\mathbb{R}^n) \rightarrow B_{p,q}^s(\mathbb{R}^n)$.*
- ii) *If $p \in (0, \infty)$, $q \in (0, \infty]$ and $\varepsilon > 0$, then $T_a^\varphi : F_{p,q}^{s+m-m_2(p)+\varepsilon}(\mathbb{R}^n) \rightarrow F_{p,q}^s(\mathbb{R}^n)$.*
- iii) *If $p \in (0, \infty)$, $\min(2, p) \leq q \leq \max(2, p)$, then $T_a^\varphi : F_{p,q}^{s+m-m_2(p)}(\mathbb{R}^n) \rightarrow F_{p,q}^s(\mathbb{R}^n)$.*
- iv) *$T_a^\varphi : F_{\infty,2}^{s+m-m_2(\infty)}(\mathbb{R}^n) \rightarrow F_{\infty,2}^s(\mathbb{R}^n)$.*

Outline of the proof.

- i) See Section 12.
- ii)-iv) See Section 13. □

3.2. Global regularity results

In this subsection, we deal with global regularity of both Schrödinger integral and oscillatory integral operators on Besov–Lipschitz and Triebel–Lizorkin spaces. We shall see that the global results concerning oscillatory integral operators (but not Schrödinger integrals) also require a further restriction of the range of p in case of operators with phase functions that are nonsmooth (at the origin) in the frequency variables.

3.2.1. Global boundedness of oscillatory integrals

We start with a global L^p -boundedness theorem.

Theorem 3.4. *Let $k \geq 1$, $p \in (1, \infty)$ and $a(x, \xi) \in S_{0,0}^{m_k(p)}(\mathbb{R}^n)$. Assume that $\varphi(x, \xi) \in \mathcal{F}^k$ is SND and satisfies the L^2 -condition (11), and for some $\mu > 0$ and some $R > n$ verifies the estimate*

$$|\partial_\xi^\alpha(\varphi(x, \xi) - x \cdot \xi)| \leq c_\alpha |\xi|^{\mu-|\alpha|}, \quad |\alpha| \geq 0, \quad 0 < |\xi| \leq 2R.$$

Then the operator T_a^φ as defined in equation (12) maps $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ continuously. In the case $0 < k < 1$, all the results above are true provided that $a(x, \xi) \in S_{1,0}^{m_k(p)}(\mathbb{R}^n)$.

Outline of the proof.

For the low-frequency part of the operator, using $a_L(x, \xi) = \psi_0(\xi/R) a(x, \xi)$, one applies Lemma 4.3 with condition (33).

For the high-frequency part, using $a_H(x, \xi) = (1 - \psi_0(\xi/R)) a(x, \xi)$, we shall use Propositions 9.1 and 9.2. To this end, we break up the operator T_a^φ into pieces $T_{a_\ell}^\varphi$ that satisfy $\partial_x^\beta \tilde{\varphi}(x, e_\ell) \in L^\infty(\mathbb{R}^n)$. To do this, we make the following construction: define the set of unit vectors $\{e_\ell\}_{\ell=1}^{2^n}$ by letting $\{e_{2^\gamma-1}\}_{\gamma=1}^n$ be the standard basis in \mathbb{R}^n and $e_{2^\gamma} := -e_{2^\gamma-1}$ for $1 \leq \gamma \leq n$. Next, let χ be a nonnegative function in $C_b^\infty(\mathbb{R})$ with

$$\text{supp } \chi = \{t \in \mathbb{R}; t \geq 1\}, \quad \chi(t) \geq 1 \text{ if } t \geq R/\sqrt{n},$$

with R as in the statement of the theorem, and let χ_ℓ be the functions in $C_b^\infty(\mathbb{R}^n)$ defined by

$$\chi_{2^\gamma-1}(\xi) = \chi(\xi_\gamma), \quad \chi_{2^\gamma}(\xi) = \chi(-\xi_\gamma), \quad 1 \leq \gamma \leq n,$$

where $\xi = (\xi_1, \dots, \xi_n)$.

Furthermore, define $\lambda_\ell(\xi) := \chi_\ell(\xi)/\sum_{\ell=1}^{2n} \chi_\ell(\xi)$ for $1 \leq \ell \leq 2n$ so that $\lambda_\ell \in C^\infty(\mathbb{R}^n)$, and $\sum_{\ell=1}^{2n} \lambda_\ell(\xi) = 1$ for every $\xi \in \mathbb{R}^n \setminus B(0, R)$. Observe that on the ξ -support of $a_H(x, \xi)$, the sum $\sum_{\ell=1}^{2n} \chi_\ell(\xi)$ is bounded from below by 1. This is because of the fact that if $|\xi| \geq R$, then at least one coordinate ξ_γ must satisfy $|\xi_\gamma| \geq R/\sqrt{n}$ and hence one of the χ_ℓ 's is bounded from below by 1. This yields that, for all multi-indices α , one has $|\partial^\alpha \lambda_\ell(\xi)| \lesssim 1$. Now, split

$$T_{a_H}^\varphi f(x) = \sum_{\ell=1}^{2n} \int_{\mathbb{R}^n} e^{i\varphi(x, \xi)} a_H(x, \xi) \lambda_\ell(\xi) \widehat{f}(\xi) \, d\xi =: \sum_{\ell=1}^{2n} T_{a_\ell}^\varphi f(x).$$

The proof reduces to showing the L^p -boundedness of each $T_{a_\ell}^\varphi$. By letting $\tilde{\varphi}(x, \xi) := \varphi(x, \xi) - \varphi(x, e_\ell)$, we can write $T_{a_\ell}^\varphi f(x) = e^{i\varphi(x, e_\ell)} T_{a_\ell}^{\tilde{\varphi}} f(x)$ with $\tilde{\varphi}(x, e_\ell) = 0$ and since L^p -norms are invariant under multiplications by factors of the form $e^{i\varphi(x, e_\ell)}$, the results are unchanged. Now, the rest of the argument goes exactly as in the proof of Theorem 3.1; however, this does not require compact support in the x -variable. In particular, Proposition 9.2 goes through since the new phase function $\tilde{\varphi}$ trivially satisfies $\partial_x^\beta \tilde{\varphi}(x, e_\ell) \in L^\infty(\mathbb{R}^n)$ for every integer $\ell \in [1, 2n]$.

Next, we prove the global Besov–Lipschitz and Triebel–Lizorkin regularity of oscillatory integral operators.

Theorem 3.5. *Let $m, s \in \mathbb{R}$ and $a(x, \xi) \in S_{0,0}^m(\mathbb{R}^n)$. Assume that $k \geq 1$, $\varphi \in \mathfrak{F}^k$ is SND, satisfies the L^2 -condition (11) and the $LF(\mu)$ -condition (14) for some $0 < \mu \leq 1$. Then the following statements hold true:*

- i) *If $p \in (n/(n + \mu), \infty]$, $q \in (0, \infty]$, then $T_a^\varphi : B_{p,q}^{s+m-m_k(p)}(\mathbb{R}^n) \rightarrow B_{p,q}^s(\mathbb{R}^n)$.*
- ii) *If $p \in (n/(n + \mu), \infty)$, $q \in (0, \infty]$ and $\varepsilon > 0$, then $T_a^\varphi : F_{p,q}^{s+m-m_k(p)+\varepsilon}(\mathbb{R}^n) \rightarrow F_{p,q}^s(\mathbb{R}^n)$.*
- iii) *If $p \in (n/(n + \mu), \infty)$, $\min(2, p) \leq q \leq \max(2, p)$, then $T_a^\varphi : F_{p,q}^{s+m-m_k(p)}(\mathbb{R}^n) \rightarrow F_{p,q}^s(\mathbb{R}^n)$.*
- iv) *$T_a^\varphi : F_{\infty,2}^{s+m-m_k(\infty)}(\mathbb{R}^n) \rightarrow F_{\infty,2}^s(\mathbb{R}^n)$.*

In the case $0 < k < 1$, all the results above are true provided that $a(x, \xi) \in S_{1,0}^m(\mathbb{R}^n)$.

If one deals with smooth phase functions, that is, if we assume that $\varphi \in \mathfrak{F}^k \cap C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, is SND and verifies the L^2 -condition (11) (both conditions for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$), and $|\nabla_x \varphi(x, 0)| \in L^\infty(\mathbb{R}^n)$, then the range of validity of the results above can be extended to $p > 0$.

Outline of the proof.

- i) See Section 12.
- ii)-iv) See Section 13. □

3.2.2. Global boundedness of Schrödinger integrals

For the case of Schrödinger integral operators, as in the case of smooth phase functions treated above, we only need to have control on $|\nabla_x \varphi(x, 0)|$, instead of the $LF(\mu)$ -condition (14) thanks to the smoothness of the phase and assumption (15). However, for the purpose of the L^p -boundedness, no assumption on the phase other than strong nondegeneracy and equation (15) are needed.

Theorem 3.6. *Let $m, s \in \mathbb{R}$ and $a(x, \xi) \in S_{0,0}^m(\mathbb{R}^n)$. Assume that φ satisfies equation (15), is SND and $|\nabla_x \varphi(x, 0)| \in L^\infty(\mathbb{R}^n)$. Then the following statements hold true:*

- i) *If $p \in (0, \infty]$, $q \in (0, \infty]$, then $T_a^\varphi : B_{p,q}^{s+m-m_2(p)}(\mathbb{R}^n) \rightarrow B_{p,q}^s(\mathbb{R}^n)$.*
- ii) *If $p \in (0, \infty)$, $q \in (0, \infty]$ and $\varepsilon > 0$, then $T_a^\varphi : F_{p,q}^{s+m-m_2(p)+\varepsilon}(\mathbb{R}^n) \rightarrow F_{p,q}^s(\mathbb{R}^n)$.*
- iii) *If $p \in (0, \infty)$, $\min(2, p) \leq q \leq \max(2, p)$, then $T_a^\varphi : F_{p,q}^{s+m-m_2(p)}(\mathbb{R}^n) \rightarrow F_{p,q}^s(\mathbb{R}^n)$.*
- iv) *$T_a^\varphi : F_{\infty,2}^{s+m-m_2(\infty)}(\mathbb{R}^n) \rightarrow F_{\infty,2}^s(\mathbb{R}^n)$.*

Outline of the proof.

- i) See Section 12.
- ii)-iv) See Section 13. □

Remark 3.7. Using the function space table (6), one immediately sees that the above regularity results yield in particular the local and global boundedness of the Schrödinger and oscillatory integral operators on $L^p(\mathbb{R}^n)$, $h^p(\mathbb{R}^n)$, $\text{bmo}(\mathbb{R}^n)$ and $\Lambda^s(\mathbb{R}^n)$.

Remark 3.8. In dealing with the L^p -boundedness in the smooth case of *Theorem 3.5* and *Theorem 3.6* the assumption on the boundedness of the gradient of $\varphi(x, 0)$ is superfluous. Indeed, if this is not the case, then we can simply replace $\varphi(x, \xi)$ by $\varphi(x, \xi) - \varphi(x, 0) + \varphi(x, 0)$. Now, the new phase function $\tilde{\varphi}(x, \xi) := \varphi(x, \xi) - \varphi(x, 0)$ is also SND, verifies equation (15) and last but not least $\tilde{\varphi}(x, 0) = 0$. Since L^p -norms are invariant under multiplications by factors of the form $e^{i\varphi(x,0)}$, the results are unchanged.

An interesting question here is whether one can prove global regularity results for Schrödinger integral operators when $|\nabla_x \varphi(x, 0)| \notin L^\infty(\mathbb{R}^n)$. This case already appears for the phase function associated to the propagator of the harmonic oscillator where $\varphi(x, 0)$ exhibits quadratic behaviour. The following theorem provides an answer to this question.

Theorem 3.9. *Assume that φ satisfies equation (15) and is SND. Then the following statements hold true:*

- i) *If $p \in (0, \infty)$ and $a(x, \xi) \in S_{0,0}^{m_2(p)}(\mathbb{R}^n)$, then $T_a^\varphi : h^p(\mathbb{R}^n) \rightarrow h^p(\mathbb{R}^n)$.*
- ii) *If $m \in \mathbb{R}$, $a(x, \xi) \in S_{0,0}^m(\mathbb{R}^n)$, $p \in [2, \infty)$ and $s \in [m_2(p), 0]$, then $T_a^\varphi : H^{s+m-m_2(p),p}(\mathbb{R}^n) \rightarrow H^{s,p}(\mathbb{R}^n)$. Furthermore, this estimate is sharp with respect to s .*

Outline of the proof.

- i) See Section 13.
- ii) Assume that T_a^φ is any Schrödinger integral operator with $a \in S_{0,0}^{m_2(p)}(\mathbb{R}^n)$ $m_2(p) \leq s \leq 0$ for $2 \leq p < \infty$. By [8, Theorem 5.3], one has $T_a^\varphi : L^p(\mathbb{R}^n) \rightarrow H^{m_2(p),p}(\mathbb{R}^n)$ if $m = 0$, which directly generalises to $T_a^\varphi : H^{m,p}(\mathbb{R}^n) \rightarrow H^{m_2(p),p}(\mathbb{R}^n)$ for any $m \in \mathbb{R}$. It follows from Theorem 10.1 that $T_a^\varphi : H^{m-m_2(p),p}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ and hence complex interpolation $H^{s,p}(\mathbb{R}^n) = (H^{0,p}(\mathbb{R}^n), H^{m_2(p),p}(\mathbb{R}^n))_\theta$ (taking $\theta = s/m_2(p)$) yields the desired estimate.

To prove the sharpness in s , define the operator $Tf(x) := e^{i|x|^2} f(x)$ and let $a(x, \xi) := \langle \xi \rangle^m$ and $\varphi(x, \xi) := x \cdot \xi + |x|^2$. Using the fact that $T_a^\varphi = T(1 - \Delta)^{m/2}$, we note that the estimate

$$\|T_a^\varphi f\|_{H^{s,p}(\mathbb{R}^n)} \lesssim \|f\|_{H^{s+m-m_2(p),p}(\mathbb{R}^n)},$$

is equivalent to

$$\|Tf\|_{H^{s,p}(\mathbb{R}^n)} \lesssim \|f\|_{H^{s-m_2(p),p}(\mathbb{R}^n)}.$$

Hence, from now on, we can take $m = 0$.

We start by assuming that $s > 0$. If $\mu := -s - n/p$ and $s' > 0$, then $(1 - \Delta)^{s'/2} \langle x \rangle^\mu \sim \langle x \rangle^\mu \in L^p(\mathbb{R}^n)$, but $(1 - \Delta)^{s/2} T \langle x \rangle^\mu \sim \langle x \rangle^{\mu+s} = \langle x \rangle^{-n/p} \notin L^p(\mathbb{R}^n)$. This shows that T does not map $H^{s',p}(\mathbb{R}^n)$ into $H^{s,p}(\mathbb{R}^n)$ continuously for any $s, s' > 0$ and in particular, if we choose $s' = s - m_2(p)$, then this is true.

We now assume that $s < m_2(p)$. Since $T^* f(x) = e^{-i|x|^2} f(x)$ we see by a duality argument that for any $s, s' < 0$, T does not map $H^{s',p}(\mathbb{R}^n)$ into $H^{s,p}(\mathbb{R}^n)$ continuously for any $s, s' < 0$. If one takes $s' = s - m_2(p)$, then $s < m_2(p)$ implies $s, s' < 0$, and this concludes the proof. □

One can also show the sharpness of the results in Theorem 3.9 in a much larger scale, as the following corollary shows.

Corollary 3.10. *If $s < m_2(p)$ or $s > 0$, then there is a Schrödinger integral operator T_a^φ of order $m_2(p)$ that is not $F_{p,q}^s$ - or $B_{p,q}^s$ -bounded for $1 < p < \infty$ and $0 < q \leq \infty$.*

Proof. The proofs for $F_{p,q}^s(\mathbb{R}^n)$ and $B_{p,q}^s(\mathbb{R}^n)$ can be done in one single step, so let A denote either F or B . We proceed using a proof by contradiction. Assume that $s < m_2(p)$ or $s > 0$ and that all T_a^φ are $A_{p,q}^s$ -bounded. Take $0 < \varepsilon < |s/4|$. Then according to the boundedness assumption one has

$$\begin{aligned} \|T_a^\varphi f\|_{H^{s-4\varepsilon,p}(\mathbb{R}^n)} &\lesssim \|T_a^\varphi f\|_{F_{p,p}^{s-3\varepsilon}(\mathbb{R}^n)} = \|T_a^\varphi f\|_{A_{p,p}^{s-3\varepsilon}(\mathbb{R}^n)} \lesssim \|T_a^\varphi f\|_{A_{p,q}^{s-2\varepsilon}(\mathbb{R}^n)} \\ &\lesssim \|f\|_{A_{p,q}^{s-2\varepsilon}(\mathbb{R}^n)} \lesssim \|f\|_{A_{p,p}^{s-\varepsilon}(\mathbb{R}^n)} = \|f\|_{F_{p,p}^{s-\varepsilon}(\mathbb{R}^n)} \lesssim \|f\|_{H^{s,p}(\mathbb{R}^n)}, \end{aligned}$$

using the embeddings in equations (7) and (8). Now, this is a contradiction since $s - 4\varepsilon < m_2(p)$ or $s - 4\varepsilon > 0$ and the Schrödinger integral operator $e^{i|x|^2(1-\Delta)^{m_2(p)/2}}$ is not bounded from $H^{s,p}(\mathbb{R}^n)$ to $H^{s-4\varepsilon,p}(\mathbb{R}^n)$ as was shown in the proof of Theorem 3.9 ii). \square

3.3. A parameter-dependent composition formula

The next result describes the action of a parameter-dependent pseudodifferential operator on a general oscillatory integral operator. Its significance is twofold. On one hand, it provides a step towards a calculus for the oscillatory integral operators. On the other, it enables one to prove regularity results for the operators on classical function spaces in both Banach and quasi-Banach scales. The result also generalises the asymptotic expansion that was obtained in [27].

Theorem 3.11. *Let $m, s \in \mathbb{R}$ and $\rho \in [0, 1]$. Suppose that $a(x, \xi) \in S_{\rho,0}^m(\mathbb{R}^n)$, $b(x, \xi) \in S_{1,0}^s(\mathbb{R}^n)$ and φ is a phase function that is smooth on $\text{supp } a$ and verifies the conditions*

- i) $|\xi| \lesssim |\nabla_x \varphi(x, \xi)| \lesssim |\xi|$ and
- ii) for all $|\alpha| \geq 0$ and all $|\beta| \geq 1$, $|\partial_\xi^\alpha \partial_x^\beta \varphi(x, \xi)| \lesssim \langle \xi \rangle^{1-|\alpha|}$,

for all $(x, \xi) \in \text{supp } a$. For $0 < t \leq 1$, consider the parameter-dependent pseudodifferential operator

$$b(x, tD)f(x) := \int_{\mathbb{R}^n} e^{ix \cdot \xi} b(x, t\xi) \widehat{f}(\xi) \, d\xi,$$

and the oscillatory integral operator

$$T_a^\varphi f(x) := \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x, \xi) \widehat{f}(\xi) \, d\xi.$$

Let σ_t be the amplitude of the composition operator $T_{\sigma_t}^\varphi := b(x, tD)T_a^\varphi$ given by

$$\sigma_t(x, \xi) := \iint_{\mathbb{R}^n \times \mathbb{R}^n} a(y, \xi) b(x, t\eta) e^{i(x-y) \cdot \eta + i\varphi(y, \xi) - i\varphi(x, \xi)} \, d\eta \, dy.$$

Then for any $M \geq 1$ and all $0 < \varepsilon < 1/2$, one can write σ_t as

$$\sigma_t(x, \xi) = b(x, t\nabla_x \varphi(x, \xi)) a(x, \xi) + \sum_{0 < |\alpha| < M} \frac{t^\varepsilon |\alpha|}{\alpha!} \sigma_\alpha(t, x, \xi) + t^\varepsilon M r(t, x, \xi), \tag{16}$$

where, for all multi-indices β, γ one has

$$\begin{aligned} |\partial_\xi^\beta \partial_x^\gamma \sigma_\alpha(t, x, \xi)| &\lesssim t^{\min(s,0)} \langle \xi \rangle^{s+m-(1/2-\varepsilon)|\alpha|-\rho|\beta|}, \quad \text{for } 0 < |\alpha| < M, \\ |\partial_\xi^\beta \partial_x^\gamma r(t, x, \xi)| &\lesssim t^{\min(s,0)} \langle \xi \rangle^{s+m-(1/2-\varepsilon)M-\rho|\beta|}. \end{aligned}$$

Outline of the proof.

See Section 11. □

Remark 3.12. We shall frequently use the previous theorem when t is replaced by 2^{-j} and $b(x, tD) = \psi(2^{-j}D)$ with ψ as in Definition 2.1. This yields the following formula for the composition $\psi(2^{-j}D)T_a^\varphi$. For any integer $M \geq 1$ and $0 < \varepsilon < 1/2$, one can write

$$\psi(2^{-j}D)T_a^\varphi = \sum_{|\alpha| < M} \frac{2^{-j\varepsilon|\alpha|}}{\alpha!} T_{\sigma_{\alpha,j}}^\varphi + 2^{-j\varepsilon M} T_{r_j}^\varphi, \tag{17}$$

with $\sigma_{0,j}(x, \xi) := \psi(2^{-j}\nabla_x \varphi(x, \xi))a(x, \xi)$ and

$$|\partial_\xi^\beta \partial_x^\gamma \sigma_{\alpha,j}(x, \xi)| \lesssim \langle \xi \rangle^{m-(1/2-\varepsilon)|\alpha|-\rho|\beta|}, \quad |\alpha| \geq 0 \tag{18}$$

$$\text{supp}_\xi \sigma_{\alpha,j}(x, \xi) = \{\xi \in \mathbb{R}^n : C_1 2^j \leq |\xi| \leq C_2 2^j\},$$

$$|\partial_\xi^\beta \partial_x^\gamma r_j(x, \xi)| \lesssim \langle \xi \rangle^{m-(1/2-\varepsilon)M-\rho|\beta|}. \tag{19}$$

Moreover, if $a(x, \xi)$ is supported outside the origin in the ξ -variable, then $r_j(x, \xi)$ also vanishes in a neighbourhood of $\xi = 0$. See the proof of Theorem 3.11 for the details.

3.4. Global $L^p - L^q$ estimates

In this section, we state and prove basic global $L^p - L^q$ estimates for the oscillatory integral and the Schrödinger integral operators. The $L^p - L^q$ estimates for the oscillatory integral operators are as follows.

Theorem 3.13. *Let $m, s \in \mathbb{R}$ and $a(x, \xi) \in S_{0,0}^m(\mathbb{R}^n)$. Assume that $k \geq 1$, $\varphi \in \mathcal{F}^k$ is SND, satisfies the L^2 -condition (11) and for some $\mu > 0$ and some $R > n$ verifies the estimate*

$$|\partial_\xi^\alpha (\varphi(x, \xi) - x \cdot \xi)| \leq c_\alpha |\xi|^{\mu-|\alpha|}, \quad |\alpha| \geq 0, \quad 0 < |\xi| \leq 2R.$$

Then for $1 < p \leq q < \infty$, $T_a^\varphi : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$, provided that $m \leq m_k(q) - n(1/p - 1/q)$. In the case $0 < k < 1$, all the results above are true provided that $a(x, \xi) \in S_{1,0}^m(\mathbb{R}^n)$.

Proof. We write

$$T_a^\varphi = T_a^\varphi (1 - \Delta)^{(m_k(q)-m)/2} (1 - \Delta)^{(m-m_k(q))/2}.$$

Then since $T_a^\varphi (1 - \Delta)^{(m_k(q)-m)/2}$ is an oscillatory integral operator with an amplitude in the class $S_{0,0}^{m_k(q)}(\mathbb{R}^n)$ for $k \geq 1$ and $S_{1,0}^{m_k(q)}(\mathbb{R}^n)$ when $0 < k < 1$, Theorem 3.4 yields that

$$\|T_a^\varphi (1 - \Delta)^{(m_k(q)-m)/2} u\|_{L^q(\mathbb{R}^n)} \lesssim \|u\|_{L^q(\mathbb{R}^n)}.$$

Therefore, applying the Sobolev embedding theorem and taking $u = (1 - \Delta)^{(m-m_k(q))/2} f$, we obtain

$$\|T_a^\varphi f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

provided that $1 < p \leq q < \infty$ and $1/p - 1/q \leq (m_k(q) - m)/n$. □

For the Schrödinger integral operators, we have

Theorem 3.14. *Assume that $a(x, \xi) \in S_{0,0}^m(\mathbb{R}^n)$ and φ satisfies equation (15) and is SND. Then for $1 < p \leq q < \infty$, $T_a^\varphi : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$, provided that $m \leq m_2(q) - n(1/p - 1/q)$.*

Proof. The proof is similar to that of Theorem 3.13. The only difference is that instead of using Theorem 3.4, we use Theorem 3.6 part *iii*), noting that due to the $L^p - L^q$ nature of our result, no requirement on the gradient of the phase function is needed. \square

3.5. Applications to harmonic analysis and PDEs

In this subsection, we outline some of the applications of the main results of this paper. We start by giving a couple of basic examples to highlight how the results obtain here can provide boundedness results for operators whose regularity has (hitherto) remained elusive.

For any $t(x) \in C_b^\infty(\mathbb{R}^n)$, the function

$$\varphi_1(x, \xi) := x \cdot \xi + t(x)\langle \xi \rangle$$

is in \mathcal{F}^1 . This example is not covered by the theory of Fourier integral operators due to lack of homogeneity and exhibits the simplest example of a phase function related to equations of Klein–Gordon type. Moreover if we also choose $t(x)$ such that $|\nabla t(x)|$ is small enough, then this phase function also satisfies the SND-condition. It is also easily checked that this phase verifies the L^2 -condition and $\partial_x^\beta \varphi_1(x, 0) = \partial^\beta t(x) \in L^\infty(\mathbb{R}^n)$.

Now, Theorem 3.5 (the part for smooth phase functions) shows that if $m \in \mathbb{R}$ and $a(x, \xi) \in S_{0,0}^m(\mathbb{R}^n)$, then for the oscillatory integral operator $T_a^{\varphi_1}$, $m_1(p) = -n|1/p - 1/2|$ and $s \in \mathbb{R}$, one has the following regularity results:

- i*) For $p \in (0, \infty]$, $q \in (0, \infty]$, $T_a^{\varphi_1} : B_{p,q}^{s+m-m_1(p)}(\mathbb{R}^n) \rightarrow B_{p,q}^s(\mathbb{R}^n)$ continuously.
- ii*) If $p \in (0, \infty)$, $q \in (0, \infty]$ and $\varepsilon > 0$, then $T_a^{\varphi_1} : F_{p,q}^{s+m-m_1(p)+\varepsilon}(\mathbb{R}^n) \rightarrow F_{p,q}^s(\mathbb{R}^n)$ continuously.
- iii*) If $p \in (0, \infty)$, $\min(2, p) \leq q \leq \max(2, p)$, then $T_a^{\varphi_1} : F_{p,q}^{s+m-m_1(p)}(\mathbb{R}^n) \rightarrow F_{p,q}^s(\mathbb{R}^n)$ continuously.
- iv*) $T_a^{\varphi_1} : F_{\infty,2}^{s+m+n/2}(\mathbb{R}^n) \rightarrow F_{\infty,2}^s(\mathbb{R}^n)$ continuously.

Another example is that of

$$\varphi_2(x, \xi) := x \cdot \xi + t(x)|\xi|^k$$

with $0 < k \leq 1$ which is in \mathcal{F}^k . Once again, if we choose $t(x)$ such that $|\nabla t(x)|$ is small enough, then φ_2 is also satisfies the SND-condition. Furthermore, we have that

$$|\partial_\xi^\alpha \partial_x^\beta (x \cdot \xi + t(x)|\xi|^k - x \cdot \xi)| \leq c_{\alpha,\beta} |\xi|^{k-|\alpha|}, \quad |\alpha + \beta| \geq 0, \quad |\xi| \leq 2,$$

which yields that the LF(μ) condition is satisfied with $\mu = k$. Finally,

$$|\partial_\xi^\alpha \partial_x^\beta (x \cdot \xi + t(x)|\xi|^k)| \leq c_{\alpha,\beta}, \quad |\alpha| \geq 1, \quad |\beta| \geq 1, \quad |\xi| \geq 1,$$

implies that the L^2 -condition is also satisfied. Thus, once again Theorem 3.5 shows that if $m \in \mathbb{R}$ and $a(x, \xi) \in S_{0,0}^m(\mathbb{R}^n)$, then for the oscillatory integral operator $T_a^{\varphi_2}$, $m_k(p) = -kn|1/p - 1/2|$ and $s \in \mathbb{R}$, one has similar regularity results in Besov–Lipschitz and Triebel–Lizorkin spaces, as above with the only difference that $m_1(p)$ is replaced by $m_k(p)$ and the range of validity of the results in p has to be taken larger than $n/(n+k)$.

The applications to partial differential equations concern local and global Besov–Lipschitz and Triebel–Lizorkin estimates for solutions to dispersive PDEs. First, let us consider the basic example of a dispersive equation in \mathbb{R}^{n+1}

$$\begin{cases} i\partial_t u(x, t) + \phi(D)u(x, t) = 0, & x \in \mathbb{R}^n, t \neq 0, \\ u(x, 0) = f(x), & x \in \mathbb{R}^n, \end{cases} \tag{20}$$

where $\widehat{\phi(D)u}(\xi, t) = \phi(\xi)\widehat{u}(\xi, t)$. It is well known that the solution to this Cauchy problem is given by

$$u(x, t) = \int_{\mathbb{R}^n} e^{ix \cdot \xi + it\phi(\xi)} \widehat{f}(\xi) \, d\xi. \tag{21}$$

Theorem 3.15. Assume that $0 < k < \infty$, $\phi(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and

$$|\partial^\alpha \phi(\xi)| \leq c_\alpha |\xi|^{k-|\alpha|} \text{ for } \xi \neq 0 \text{ and } |\alpha| \geq 1, \tag{22}$$

and $u(x, t)$ is the solution of the Cauchy problem (20) represented by the oscillatory integral above. Then for any $\tau > 0$ and each $t \in [-\tau, \tau]$ and all $p \in (n/(n + \min(1, k)), \infty]$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $m_k(p) = -kn|1/p - 1/2|$, one has

$$\sup_{t \in [-\tau, \tau]} \|u\|_{B_{p,q}^s(\mathbb{R}^n)} \leq C_\tau \|f\|_{B_{p,q}^{s-m_k(p)}(\mathbb{R}^n)}. \tag{23}$$

Similarly, we have for any $s \in \mathbb{R}$, $p \in (n/(n + \min(1, k)), \infty)$, $\min(2, p) \leq q \leq \max(2, p)$ that

$$\sup_{t \in [-\tau, \tau]} \|u\|_{F_{p,q}^s(\mathbb{R}^n)} \leq C_\tau \|f\|_{F_{p,q}^{s-m_k(p)}(\mathbb{R}^n)}. \tag{24}$$

All the results are sharp when $k > 1$.

Furthermore, one also has for $1 < p \leq q < \infty$ and $s \in \mathbb{R}$, the Sobolev space estimate

$$\sup_{t \in [-\tau, \tau]} \|u\|_{H^{s-n(1/p-1/q),q}(\mathbb{R}^n)} \leq C_\tau \|f\|_{H^{s-m_k(q),p}(\mathbb{R}^n)}. \tag{25}$$

Proof. Observe that, the phase function in the integral representation (21) is $x \cdot \xi + t\phi(\xi)$. Now, for any $\tau > 0$ and each $t \in [-\tau, \tau]$ the estimate (22) yields that this phase function is SND and in \mathcal{F}^k and also satisfies the LF(μ)-condition (14) with $\mu = \min(1, k)$. Moreover, the amplitude of the oscillatory integral (21) is identically equal to 1, which is trivially in $S_{1,0}^0(\mathbb{R}^n) \subset S_{0,0}^0(\mathbb{R}^n)$. Using equation (21) and Theorem 3.5, it follows that the solution equation (20) verifies equation (23). The proof of equation (24) is similar and hence omitted. For the proof of the sharpness, see Section 14. Finally, equation (25) follows from Theorem 3.13. \square

Remark 3.16. If the function ϕ in Theorem 3.15 is assumed to be positively homogeneous of degree 1, then the relevant order $m_1(p)$ in the theorem above could be improved to $-(n - 1)|1/p - 1/2|$; see [20, Section 10].

Concerning Schrödinger equations, let us consider the Cauchy problem for a variable-coefficient Schrödinger equation

$$\begin{cases} i\partial_t \Psi(x, t) + \mathcal{H}(x, D)\Psi(x, t) = 0, & x \in \mathbb{R}^n, t \neq 0, \\ \Psi(x, 0) = \Psi_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{26}$$

where $\mathcal{H}(x, D)$ is the Hamiltonian of the quantum mechanical system. For example, one can have $\mathcal{H}(x, D) = -\Delta + V(x)$, which corresponds to the Hamiltonian function $\mathcal{H}(x, \xi) = |\xi|^2 + V(x)$. Now, if in general \mathcal{H} is real valued and $|\partial_\xi^\alpha \partial_x^\beta \mathcal{H}(x, \xi)| \lesssim 1$ for $|\alpha + \beta| \geq 2$ (for example, the harmonic oscillator $-\Delta + |x|^2$ yields such a Hamiltonian), then the Cauchy problem above can be solved locally in time and modulo smoothing operators by

$$\Psi(x, t) = \int_{\mathbb{R}^n} e^{i\varphi(x, \xi, t)} a(x, \xi, t) \widehat{\Psi}_0(\xi) \, d\xi, \tag{27}$$

where for $t \in (-\tau, \tau)$, τ sufficiently small, one has that $|\partial_\xi^\alpha \partial_x^\beta \varphi(x, \xi, t)| \lesssim 1$ for $|\alpha + \beta| \geq 2$, φ is SND and $a(x, \xi, t) \in S_{0,0}^0(\mathbb{R}^n)$; see [7, Proposition 4.1]. This yields the following.

Theorem 3.17. Let $\Psi(x, t)$ be the solution of the Schrödinger Cauchy problem (26) with initial data Ψ_0 , where the Hamiltonian \mathcal{H} is real valued and satisfies the estimate $|\partial_\xi^\alpha \partial_x^\beta \mathcal{H}(x, \xi)| \lesssim 1$ for $|\alpha + \beta| \geq 2$. Then there exists $\tau > 0$ such that for all $p, q \in (0, \infty]$ and $s \in \mathbb{R}$, $m_2(p) = -2n|1/p - 1/2|$, we have the local Besov–Lipschitz space estimate

$$\sup_{t \in [-\tau, \tau]} \|\Psi\|_{B_{p,q}^{s,loc}(\mathbb{R}^n)} \leq C_\tau \|\Psi_0\|_{B_{p,q}^{s-m_2(p)}(\mathbb{R}^n)}.$$

Here, the superscript “loc” means that we first multiply the function (distribution) with a smooth cut-off function and then take the norm.

Similarly, for any $s \in \mathbb{R}$, $0 < p < \infty$, $\min(2, p) \leq q \leq \max(2, p)$, one has the local Triebel–Lizorkin estimate

$$\sup_{t \in [-\tau, \tau]} \|\Psi\|_{F_{p,q}^{s,loc}(\mathbb{R}^n)} \leq C_\tau \|\Psi_0\|_{F_{p,q}^{s-m_2(p)}(\mathbb{R}^n)},$$

which also holds when $p = \infty$ and $q = 2$. Moreover, if $m < m_2(p)$ then for all $s \in \mathbb{R}$ and $p, q \in (0, \infty]$ one has

$$\sup_{t \in [-\tau, \tau]} \|\Psi\|_{F_{p,q}^{s,loc}(\mathbb{R}^n)} \leq C_\tau \|\Psi_0\|_{F_{p,q}^{s-m}(\mathbb{R}^n)}.$$

Furthermore, we also have the following global (in space) sharp estimates

$$\begin{cases} \sup_{t \in [-\tau, \tau]} \|\Psi\|_{F_{p,2}^s(\mathbb{R}^n)} \leq C_\tau \|\Psi_0\|_{F_{p,2}^{s-m_2(p)}(\mathbb{R}^n)}, & 2 \leq p < \infty, \quad s \in [m_2(p), 0], \\ \sup_{t \in [-\tau, \tau]} \|\Psi\|_{F_{p,2}^0(\mathbb{R}^n)} \leq C_\tau \|\Psi_0\|_{F_{p,2}^{-m_2(p)}(\mathbb{R}^n)}, & 0 < p < \infty, \end{cases}$$

Proof. The local results all follow from the oscillatory integral representation (27) and Theorem 3.3. The global estimates are all consequences of Theorem 3.9 parts ii) and i), respectively. \square

4. Estimates for phases and kernels

In this section, we prove some basic kernel estimates for oscillatory integral operators.

The following lemma will enable us to use a composition formula and an asymptotic expansion for the action of a pseudodifferential operator on an oscillatory integral operator. It is also helpful in the proof of Proposition 9.2 below. Once this is done, we shall then prove Theorem 3.11 using only equations (11), (29) and (30).

Lemma 4.1. Assume that $a(x, \xi)$ is an amplitude, and let φ be a SND phase function satisfying

$$|\nabla_\xi \partial_x^\beta \varphi(x, \xi)| \leq c_\beta, \quad |\beta| \geq 1 \text{ and } |\xi| \geq 1. \tag{28}$$

Then for all $|\beta| \geq 1$, the following estimates

$$|\xi| \lesssim |\nabla_x \varphi(x, \xi)| \lesssim |\xi|, \tag{29}$$

$$|\partial_x^\beta \varphi(x, \xi)| \lesssim \langle \xi \rangle \tag{30}$$

hold true for the phase function φ , on the support of $a(x, \xi)$, provided that either

- i) the ξ -support of $a(x, \xi)$ lies outside the ball $B(0, R)$ for some large enough $R \gg 1$ and $\partial_x^\beta \varphi(x, \xi) \in L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})$, for $|\beta| \geq 1$
- or

ii) the amplitude $a(x, \xi)$ has compact x -support and has its ξ -support outside the ball $B(0, R)$ for some large enough $R \gg 1$

or

iii) $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $\varphi(x, 0) = 0$, and $|\nabla_\xi \partial_x^\beta \varphi(x, \xi)| \leq c_\beta$, $|\beta| \geq 1$, for $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proof. We would like to compare $\partial_x^\beta \varphi(x, \xi)$ with some $\partial_x^\beta \varphi(x, \xi_0)$ for $|\beta| \geq 1$. In *i)* and *ii)*, we choose $\xi_0 = \xi/|\xi|$. Note that the line segment $\xi_0 + t(\xi - \xi_0)$, with $t \in (0, 1)$ and $|\xi| \geq R$ does not intersect $B(0, 1)$ so we can use equation (28) without problem. In *iii)*, we choose $\xi_0 = 0$. Therefore, on the support of $a(x, \xi)$, using equation (28) and the mean-value theorem yield for $|\beta| \geq 1$ that

$$|\partial_x^\beta \varphi(x, \xi)| \leq |\partial_x^\beta \varphi(x, \xi) - \partial_x^\beta \varphi(x, \xi_0)| + |\partial_x^\beta \varphi(x, \xi_0)| \lesssim |\xi - \xi_0| + |\partial_x^\beta \varphi(x, \xi_0)| \lesssim |\xi| + |\partial_x^\beta \varphi(x, \xi_0)|.$$

Thus, for both cases *i)* and *ii)* one has that $|\partial_x^\beta \varphi(x, \xi_0)| \lesssim 1 \lesssim |\xi|$, uniformly in x on the support of $a(x, \xi)$, and the same is also true in case *iii)* due to the vanishing of the derivatives. This proves equation (30) and the second inequality of (29).

To prove the first inequality of equation (29), Schwartz’s global inverse function theorem can be used just as in the proof of in [9, Proposition 1.11] to obtain

$$|\xi| - |\xi_0| \leq |\xi - \xi_0| \lesssim |\nabla_x \varphi(x, \xi) - \nabla_x \varphi(x, \xi_0)| \leq |\nabla_x \varphi(x, \xi)| + |\nabla_x \varphi(x, \xi_0)|. \tag{31}$$

Therefore, to prove the desired lower bound for $|\nabla_x \varphi(x, \xi)|$ in case *i)* and *ii)*, let ξ_0 be defined above and insert it into equation (31). Then for a certain constant $A = A(n, \delta, c_1) > 0$ (where n is the dimension, δ is the lower bound in the SND-condition and c_1 is the upper bound on the norm of the mixed Hessian of $\varphi(x, \xi)$ when $|\xi| \geq 1$) equation (31) yields that

$$|\xi| \leq A \left(|\nabla_x \varphi(x, \xi)| + |\nabla_x \varphi(x, \xi_0)| \right) + 1.$$

However, since $|\xi| > R$, on the support of $a(x, \xi)$ and R can be chosen large enough by taking

$$R \geq 2A \left(\max_{x \in \text{supp } a(x, \xi_0)} |\nabla_x \varphi(x, \xi_0)| \right) + 2,$$

we obtain

$$|\xi| \lesssim |\nabla_x \varphi(x, \xi)|.$$

In case *iii)*, the same inequality is once gain valid since we take $\xi_0 = 0$ and $\nabla_x \varphi(x, 0) = 0$ in equation (31). □

Next, we turn to kernel estimates of the operators in various settings. A simple case is when the amplitude is spatially localised.

Lemma 4.2. *Let $m \in \mathbb{R}$, $\varphi(x, \xi)$ be a real-valued function and $a(x, \xi) \in S_{0,0}^m(\mathbb{R}^n)$ has compact support in the spatial variable x . Define*

$$K_j(x, y) := \int_{\mathbb{R}^n} a_j(x, \xi) e^{i\varphi(x, \xi)} e^{-iy \cdot \xi} \, d\xi,$$

where $a_j(x, \xi) := \psi_j(\xi) a(x, \xi)$ is a Littlewood–Paley piece of the amplitude a . Then for each $j \in \mathbb{Z}_+$ and all multi-indices β we have

$$\left\| \partial_y^\beta K_j(x, y) \right\|_{L_{x,y}^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \lesssim 2^{j(m+|\beta|+n)}.$$

Proof. Observe that

$$\begin{aligned}
 |\partial_y^\beta K_j(x, y)| &= \left| \partial_y^\beta \int_{\mathbb{R}^n} a_j(x, \xi) e^{i\varphi(x, \xi)} e^{-iy \cdot \xi} \, d\xi \right| = \left| \int_{\mathbb{R}^n} a_j(x, \xi) e^{i\varphi(x, \xi)} \xi^\beta e^{-iy \cdot \xi} \, d\xi \right| \\
 &\lesssim \int_{\mathbb{R}^n} |a_j(x, \xi)| |\xi|^{|\beta|} \, d\xi \lesssim \|a_j\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} 2^{j(|\beta|+n)} \lesssim 2^{j(|\beta|+m+n)},
 \end{aligned}$$

for any $x, y \in \mathbb{R}^n$. □

Next, we prove a kernel estimate for the low-frequency portion of oscillatory integral operators.

Lemma 4.3. *Let $\mu > 0$, $a_L(x, \xi)$ be a symbol that is compactly supported and smooth outside the origin in the ξ -variable and $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ be a phase function. Assume that one of the following conditions hold.*

$$\begin{cases} \|\partial_\xi^\alpha a_L(\cdot, \xi)\|_{L^\infty(\mathbb{R}^n)} \leq c_\alpha |\xi|^{\mu-|\alpha|}, & |\alpha| \geq 0, \\ \|\partial_\xi^\alpha (\varphi(x, \xi) - x \cdot \xi)\|_{L_x^\infty(\mathbb{R}^n)} \leq c_\alpha |\xi|^{-|\alpha|}, & |\alpha| \geq 1, \end{cases} \tag{32}$$

$$\begin{cases} \|\partial_\xi^\alpha a_L(\cdot, \xi)\|_{L^\infty(\mathbb{R}^n)} \leq c_\alpha, & |\alpha| \geq 0, \\ \|\partial_\xi^\alpha (\varphi(x, \xi) - x \cdot \xi)\|_{L_x^\infty(\mathbb{R}^n)} \leq c_\alpha |\xi|^{\mu-|\alpha|}, & |\alpha| \geq 0, \end{cases} \tag{33}$$

for $\xi \neq 0$ and on the support of $a_L(x, \xi)$. Then the modulus of the integral kernel

$$K(x, y) := \int_{\mathbb{R}^n} a_L(x, \xi) e^{i\varphi(x, \xi) - iy \cdot \xi} \, d\xi,$$

and that of $\overline{K(y, x)}$ are both bounded by $\langle x - y \rangle^{-n-\varepsilon\mu}$ for any $0 \leq \varepsilon < 1$.

Proof. Since $|K(x, y)| \lesssim 1$, it is enough to show that $|K(x, y)| \lesssim |x - y|^{-n-\varepsilon\mu}$.

In order to prove the lemma under assumptions (32), we set $\sigma(x, \xi) := a_L(x, \xi) e^{i\varphi(x, \xi) - ix \cdot \xi}$

$$K(x, y) := \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} \sigma(x, \xi) \, d\xi.$$

Observe that $|\partial_\xi^\alpha \sigma(x, \xi)| \lesssim |\xi|^{\mu-|\alpha|}$ for any $|\alpha| \geq 0$ and $\xi \in \text{supp}_\xi a_L(x, \xi)$. Now, one introduces a Littlewood–Paley partition of unity

$$\sum_{j=-\infty}^\infty \psi(2^{-j}\xi) = 1, \text{ for } \xi \neq 0, \text{ with } \text{supp } \psi(\xi) \subset \{1/2 \leq |\xi| \leq 2\},$$

and defines

$$K_j(x, y) := \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} \sigma(x, \xi) \psi(2^{-j}\xi) \, d\xi.$$

Integration by parts N times yields

$$\begin{aligned}
 |K_j(x, y)| &\lesssim |x - y|^{-N} \sum_{|\alpha|+|\beta|=N} \int_{\mathbb{R}^n} \left| \partial_\xi^\alpha \sigma(x, \xi) \right| \left| \partial_\xi^\beta \psi(2^{-j}\xi) \right| \, d\xi \\
 &\lesssim |x - y|^{-N} 2^{j(\mu+n-N)}. \end{aligned} \tag{34}$$

However, if H is any positive real number, then one can write H as the sum $N + \theta$, where N is a positive integer and $\theta \in [0, 1)$. Now, since equation (34) implies that

$$|K_j(x, y)| \lesssim 2^{j(n-N+\mu)} |x - y|^{-N}, \tag{35}$$

and

$$|K_j(x, y)| \lesssim 2^{j(n-(N+1)+\mu)} |x - y|^{-(N+1)}, \tag{36}$$

raising equation (35) to the power $1 - \theta$ and equation (36) to the power θ , and using the fact that $H = N + \theta$, yield that

$$|K_j(x, y)| = |K_j(x, y)|^{1-\theta} |K_j(x, y)|^\theta \lesssim 2^{j(n-H+\mu)} |x - y|^{-H},$$

for all $H \geq 0$.

Observe that there exists $M > 0$ such that $\text{supp}_\xi a_L(x, \xi) \subseteq B(0, 2^M)$. Therefore, we can write

$$K(x, y) = \sum_{j=-\infty}^M K_j(x, y),$$

and hence setting $H := n + \varepsilon\mu$, we obtain

$$|K(x, y)| \lesssim \sum_{j=-\infty}^M |x - y|^{-n-\varepsilon\mu} 2^{j\mu(1-\varepsilon)} \lesssim |x - y|^{-n-\varepsilon\mu}.$$

To prove the lemma under assumptions (33), split the kernel into $K(x, y) = K'(x, y) + K''(x, y)$, where

$$K'(x, y) := \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a_L(x, \xi) \, d\xi,$$

and

$$K''(x, y) := \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a_L(x, \xi) (e^{i\varphi(x, \xi) - ix\cdot\xi} - 1) \, d\xi.$$

Integration by parts N times yields that $|K'(x, y)| \lesssim |x - y|^{-N}$ for all N . To obtain the estimate for $K''(x, y)$, we set $\sigma(x, \xi) := a_L(x, \xi) (e^{i\varphi(x, \xi) - ix\cdot\xi} - 1)$ and note that for all $|\alpha| \geq 0$, $|\partial_\xi^\alpha (e^{i\varphi(x, \xi) - ix\cdot\xi} - 1)| \lesssim |\xi|^{\mu-|\alpha|}$ so that $|\partial_\xi^\alpha \sigma(x, \xi)| \lesssim |\xi|^{\mu-|\alpha|}$ for any $|\alpha| \geq 0$. Now, the rest of the proof proceeds as in the previous case above.

The proof for $K(y, x)$ is identical and hence omitted. □

Remark 4.4. Observe that for phase functions of the form $x \cdot \xi + |\xi|^k$ with $k > 1$ and symbols $a(x, \xi) = \chi(\xi) \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, a decay of the form $\langle x - y \rangle^{-n-1}$ was already proven in, for example, [3, Lemma 2.3].

The next lemma yields a sufficient condition for the $h^p - L^p$ boundedness of linear operators and will be quite useful in what follows.

Lemma 4.5. Assume that $0 < p < 1$ and T_a^φ is an L^2 -bounded oscillatory or Schrödinger integral operator. Let T_j be either $T_a^\varphi \psi_j(D)$ or $\psi_j(D)(T_a^\varphi)^*$ with ψ_j as in Definition 2.1 (i.e., the familiar j -th Littlewood–Paley piece of T_a^φ and its formal adjoint $(T_a^\varphi)^*$). We also assume that for a p -atom \mathbf{a} supported in a ball of radius r one has

$$\|T_j \mathbf{a}\|_{L^p(\mathbb{R}^n)} \lesssim r^{n-n/p} 2^{j(n-n/p)}. \tag{37}$$

Moreover, assume that whenever $r < 1$,

$$\|T_j \mathbf{a}\|_{L^p(\mathbb{R}^n)} \lesssim r^{N+1+n-n/p} 2^{j(N+1+n-n/p)}, \tag{38}$$

for some $N > n/p - n - 1$. Then T_a^φ (or $(T_a^\varphi)^*$ when the T_j 's are associated to the adjoint) is bounded from $h^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

Proof. Using the atomic characterisation of $h^p(\mathbb{R}^n)$, and following the strategy in [30, p. 402] for T_a^φ and the strategy in [29, p. 237] for $(T_a^\varphi)^*$, it is enough to show that for any p -atom \mathbf{a} , one has the uniform estimates

$$\|T_a^\varphi \mathbf{a}\|_{L^p(\mathbb{R}^n)} \lesssim 1,$$

or

$$\|(T_a^\varphi)^* \mathbf{a}\|_{L^p(\mathbb{R}^n)} \lesssim 1,$$

in each case. We only prove the result in the case of T_a^φ since the case of the adjoint is similar. We split the proof in two different cases, namely $r < 1$ and $r \geq 1$. For $r \geq 1$, equation (37) yields that

$$\|T_a^\varphi \mathbf{a}\|_{L^p(\mathbb{R}^n)}^p \lesssim \sum_{j=0}^\infty \|T_j \mathbf{a}\|_{L^p(\mathbb{R}^n)}^p \lesssim \sum_{j=0}^\infty r^{np-n} 2^{j(np-n)} \lesssim 1.$$

Assume now that $r < 1$. Choose $\ell \in \mathbb{Z}_+$ such that $2^{-\ell-1} \leq r < 2^{-\ell}$. Using the facts that $2^{-\ell} \sim r$, $N + 1 + n - n/p > 0$, $n - n/p < 0$, equations (37) and (38) we conclude that

$$\begin{aligned} \|T_a^\varphi \mathbf{a}\|_{L^p(\mathbb{R}^n)}^p &\lesssim \sum_{j=0}^\ell \left(r^{N+1+n-n/p} 2^{j(N+1+n-n/p)} \right)^p + \sum_{j=\ell+1}^\infty \left(r^{n-n/p} 2^{j(n-n/p)} \right)^p \\ &\lesssim \left(r^{N+1+n-n/p} 2^{\ell(N+1+n-n/p)} \right)^p + \left(r^{n-n/p} 2^{\ell(n-n/p)} \right)^p \\ &\sim \left(r^{N+1+n-n/p} r^{-(N+1+n-n/p)} \right)^p + \left(r^{n-n/p} r^{-(n-n/p)} \right)^p = 1. \end{aligned}$$

□

As an application of the previous lemma, we have the following $h^p - L^p$ boundedness result, based entirely on kernel estimates of the corresponding operators.

Lemma 4.6. *Let $0 < p \leq 1$, $k > 0$, $a(x, \xi) \in S_{0,0}^{m_k(p)}(\mathbb{R}^n)$, $\varphi(x, \xi) \in \mathcal{F}^k$, and let the operator T_j given in Lemma 4.5 have either the representation*

$$\int_{\mathbb{R}^n} K_{1,j}(x, x - y) f(y) \, dy, \tag{39}$$

or the representation

$$\int_{\mathbb{R}^n} K_{2,j}(y, x - y) f(y) \, dy. \tag{40}$$

○ *If*

$$\left\| (x - y)^\alpha \partial_y^\beta (K_{1,j}(x, x - y)) \right\|_{L_x^2(\mathbb{R}^n)} \lesssim 2^{j(|\alpha|(k-1) + |\beta| + m_k(p) + n/2)}, \tag{41}$$

uniformly in $y \in \mathbb{R}^n$, and $T_a^\varphi := \sum_{j=0}^\infty T_j$ (in the case of equation (39)) is L^2 -bounded, then T_a^φ is bounded from $h^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

○ *If*

$$\left\| (x - y)^\alpha \partial_y^\beta (K_{2,j}(y, x - y)) \right\|_{L_x^2(\mathbb{R}^n)} \lesssim 2^{j(|\alpha|(k-1) + |\beta| + m_k(p) + n/2)} \tag{42}$$

uniformly in $y \in \mathbb{R}^n$, and $(T_a^\varphi)^* := \sum_{j=0}^\infty T_j$ (in the case of equation (40)) is L^2 -bounded, then $(T_a^\varphi)^*$ is bounded from $h^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

Proof. Once again, we only treat the case of T_a^φ since that of the adjoint is done in a similar manner. Let \mathbf{a} be a p -atom supported in the ball $B := B(\bar{y}, r)$, and let $2B := B(\bar{y}, 2r)$. By Hölder’s inequality and the L^2 -boundedness of T_a^φ , we have

$$\begin{aligned} \|T_a^\varphi \mathbf{a}\|_{L^p(2B)} &\lesssim \|T_a^\varphi \mathbf{a}\|_{L^2(2B)} \|1\|_{L^{2p/(2-p)}(2B)} \lesssim \|\mathbf{a}\|_{L^2(\mathbb{R}^n)} r^{n(2-p)/2p} \\ &\lesssim r^{n(p-2)/2p} r^{n(2-p)/2p} = 1. \end{aligned}$$

We proceed to the boundedness of $\|T_a^\varphi \mathbf{a}\|_{L^p(\mathbb{R}^n \setminus 2B)}$, which is more subtle. By Lemma 4.5, it is enough to show estimates (37) and (38) for $\|T_j \mathbf{a}\|_{L^p(\mathbb{R}^n \setminus 2B)}$. For all multi-indices α , equation (41) yields

$$\left\| (2^{-j(k-1)}(x - y))^\alpha K_{1,j}(x, x - y) \right\|_{L_x^2(\mathbb{R}^n)} \lesssim 2^{j(n/2+m_k(p))}$$

so that for any integer M , if one sums over $|\alpha| \leq M$,

$$\left\| (1 + 2^{-j(k-1)}|x - y|)^M K_{1,j}(x, x - y) \right\|_{L_x^2(\mathbb{R}^n)} \lesssim 2^{j(n/2+m_k(p))}. \tag{43}$$

We now observe that for $t \in [0, 1]$, $x \in \mathbb{R}^n \setminus 2B$ and $y \in B$, one has

$$|x - \bar{y}| \lesssim |x - \bar{y} - t(y - \bar{y})|. \tag{44}$$

Next, we introduce

$$g(x) := \left(1 + 2^{-j(k-1)}|x - \bar{y}|\right)^{-M},$$

where $M > n/q$ and $1/q = 1/p - 1/2$. The Hölder and the Minkowski inequalities together with equations (43) and (44) (with $t = 1$) yield

$$\begin{aligned} \|T_j \mathbf{a}\|_{L^p(\mathbb{R}^n \setminus 2B)} &= \left\| \int_B K_{1,j}(x, x - y) \mathbf{a}(y) \, dy \right\|_{L_x^p(\mathbb{R}^n \setminus 2B)} \\ &\leq \left\| \frac{1}{g(x)} \int_B K_{1,j}(x, x - y) \mathbf{a}(y) \, dy \right\|_{L_x^2(\mathbb{R}^n \setminus 2B)} \|g\|_{L^q(\mathbb{R}^n)} \\ &\lesssim 2^{jn(k-1)/q} \int_B \left\| \frac{1}{g(x)} K_{1,j}(x, x - y) \mathbf{a}(y) \right\|_{L_x^2(\mathbb{R}^n \setminus 2B)} \, dy \\ &\lesssim 2^{jn(k-1)(1/p-1/2)} \int_B |\mathbf{a}(y)| \left\| (1 + 2^{-j(k-1)}|x - y|)^M K_{1,j}(x, x - y) \right\|_{L_x^2(\mathbb{R}^n \setminus 2B)} \, dy \\ &\lesssim r^{n-n/p} 2^{jn(k-1)(1/p-1/2)} 2^{j(n/2+m_k(p))}. \end{aligned}$$

Recalling that $m_k(p) = -kn(1/p - 1/2)$, we get equation (37).

We proceed to show estimate (38). Taking $N := [n(1/p - 1)]$ (note that $N > n/p - n - 1$), a Taylor expansion of the kernel at the point $y = \bar{y}$ yields that

$$\begin{aligned} K_{1,j}(x, x - y) &= \sum_{|\beta| \leq N} \frac{(y - \bar{y})^\beta}{\beta!} \partial_y^\beta (K_{1,j}(x, x - y))|_{y=\bar{y}} \\ &\quad + (N + 1) \sum_{|\beta|=N+1} \frac{(y - \bar{y})^\beta}{\beta!} \int_0^1 (1 - t)^N \partial_y^\beta (K_{1,j}(x, x - y))|_{y=\bar{y}+t(y-\bar{y})} \, dt, \end{aligned}$$

and due to vanishing moments of the atom in Definition 2.3, *iii*), we may express the operator as

$$T_j \mathbf{a}(x) = (N + 1) \sum_{|\beta|=N+1} \int_B \int_0^1 \frac{(y - \bar{y})^\beta}{\beta!} (1 - t)^N \partial_y^\beta (K_{1,j}(x, x - y))|_{y=\bar{y}+t(y-\bar{y})} \mathbf{a}(y) dt dy.$$

Now, noting that $|(y - \bar{y})^\beta| \lesssim r^{N+1}$ and applying the same procedure as above together with estimates (44) and (41), we obtain

$$\|T_j \mathbf{a}\|_{L^p(\mathbb{R}^n \setminus 2B)} \lesssim r^{N+1-n/p+n} 2^{j(N+1+m_k(p)+n/2+n(k-1)(1/p-1/2))},$$

which yields equation (38). □

5. L^2 -boundedness

In the forthcoming sections, we will also need the following important theorem about L^2 -boundedness of an oscillatory integral operator.

Theorem 5.1. *Let $a(x, \xi) \in S_{0,0}^0(\mathbb{R}^n)$, and assume that $\varphi(x, \xi)$ fulfills the L^2 -condition (11) and is SND. Then the oscillatory integral operator T_a^φ given by equation (12) is bounded from $L^2(\mathbb{R}^n)$ to itself under either of the following circumstances:*

- i) *The amplitude $a(x, \xi)$ is compactly supported in x .*
- ii) *One of the assumptions (32) or (33) holds true.*

Proof. We divide the proof into low- and high-frequency cases by writing $a(x, \xi) = \psi_0(\xi) a(x, \xi) + (1 - \psi_0(\xi)) a(x, \xi) =: a_L(x, \xi) + a_H(x, \xi)$, with ψ_0 is in Definition 2.1. For $T_{a_H}^\varphi$, the phase function is smooth and doesn't have any singularity. This enables one to use an L^2 -boundedness result for oscillatory integrals proven by D. Fujiwara in [13], since the assumptions of Theorem 5.1 fulfill conditions (A-I)–(A-IV) in [13], on the support of a_H .

For $T_{a_L}^\varphi$, part of the case *i*), using the compact support in ξ , Cauchy–Schwarz's inequality and Plancherel's theorem allow us to write

$$|T_{a_L}^\varphi f(x)| = \left| \int_{\mathbb{R}^n} a_L(x, \xi) e^{i\varphi(x, \xi)} \widehat{f}(\xi) d\xi \right| \lesssim \|f\|_{L^2(\mathbb{R}^n)}.$$

Now, the fact that $T_{a_L}^\varphi f(x)$ is compactly supported yields that

$$\|T_{a_L}^\varphi f(x)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}.$$

For the $T_{a_L}^\varphi$, part of the case *ii*), we use Lemma 4.3 to conclude that the kernel satisfies

$$|K(x, y)| \lesssim \langle x - y \rangle^{-n-\varepsilon\mu},$$

for any $\varepsilon \in [0, 1)$. Therefore, Schur's lemma applies in this case. □

Remark 5.2. In dimension one, for $k > 0$, if we take the phase function

$$\varphi(x, \xi) := x\xi - \frac{1}{2} \sin x \cos \xi + |\xi|^k,$$

then one can verify that for $k \geq 1$ the low-frequency assumption of equation (33) holds with $\mu = 1$ and for $0 < k < 1$ with $\mu = k$. Moreover,

$$\left| \partial_\xi^\alpha \partial_x^\beta \left(x\xi - \frac{1}{2} \sin x \cos \xi + |\xi|^k \right) \right| \lesssim 1, \quad |\alpha|, |\beta| \geq 1$$

and the SND-condition is also satisfied thanks to

$$\left| \partial_\xi \partial_x \left(x\xi - \frac{1}{2} \sin x \cos \xi + |\xi|^k \right) \right| \geq 1/2.$$

This together with an amplitude in $S_{0,0}^0(\mathbb{R}^n)$ gives rise to an L^2 -bounded operator. However, this is not entirely covered by the L^2 -boundedness results of Hörmander [18] (because of lack of homogeneity and also lack of compact support in the x -variable) or Fujiwara [13] (due to lack of smoothness). It is also important to note that the rather strong assumptions on the phase function are needed to deal with the lack of decay in the amplitude (i.e., an amplitude in $S_{0,0}^0(\mathbb{R}^n)$).

6. Boundedness of low-frequency portion

The kernel estimate obtained in Lemma 4.3 can be used to show that the corresponding oscillatory integral operators are bounded in various Banach, as well as quasi-Banach spaces. Now, as far as the L^p -regularity is concerned, the Mihlin multiplier theorem yields the following boundedness result for operators with amplitudes that are compactly supported in the spatial variables.

Lemma 6.1. *Let $a_L(x, \xi) \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be an amplitude and assume that $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ with*

$$|\partial_\xi^\alpha \partial_x^\beta \varphi(x, \xi)| \leq c_{\alpha,\beta} |\xi|^{-|\alpha|}, \quad |\alpha + \beta| \geq 1, \quad (x, \xi) \in \text{supp } a_L.$$

Then the operator $T_{a_L}^\varphi$ of the form (12) is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Proof. Set $\sigma(x, \xi) := a_L(x, \xi) e^{i(\varphi(x,\xi) - x \cdot \xi)}$, and observe that the condition on the phase function implies that $|\partial_\xi^\alpha \partial_x^\beta \sigma(x, \xi)| \lesssim |\xi|^{-|\alpha|}$. Now, we write

$$T_{a_L}^\varphi f(x) = \int_{\mathbb{R}^n} \sigma(x, \xi) e^{ix \cdot \xi} \widehat{f}(\xi) \, d\xi,$$

and using the fact that σ is compactly supported in x we have that for any integer $N > 0$

$$T_{a_L}^\varphi f(x) = \int_{\mathbb{R}^n} \langle \eta \rangle^{-2N} \left(\int_{\mathbb{R}^n} \langle \eta \rangle^{2N} \widehat{\sigma}(\eta, \xi) e^{ix \cdot \xi} \widehat{f}(\xi) \, d\xi \right) e^{ix \cdot \eta} \, d\eta. \tag{45}$$

The compact support of $\sigma(x, \xi)$ also implies that

$$\langle \eta \rangle^{2N} |\partial_\xi^\alpha \widehat{\sigma}(\eta, \xi)| = \left| \int_{\mathbb{R}^n} e^{-ix \cdot \eta} (1 - \Delta_x)^N \partial_\xi^\alpha \sigma(x, \xi) \, dx \right| \lesssim |\xi|^{-|\alpha|},$$

uniformly in η . Now, the boundedness of $\langle \eta \rangle^{2N} \widehat{\sigma}(\eta, \xi)$ and the estimate above show that the aforementioned function is a Mihlin multiplier and therefore bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Therefore, using Minkowski’s integral inequality to equation (45), which is valid for the Banach space scales of L^p -spaces and choosing N large enough yield the desired boundedness. \square

The following lemma establishes the local boundedness of the low-frequency portion of adjoint operator $(T_a^\varphi)^*$.

Lemma 6.2. *Let $0 < p < 1$. Moreover, assume $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $a_L(x, \xi) \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Then $(T_{a_L}^\varphi)^*$ given as in equation (13) is a bounded operator from $h^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

Proof. Set $\sigma(y, \xi) := \overline{a_L(y, \xi)} e^{-i(\varphi(y,\xi) + y \cdot \xi)}$, and consider the kernel of $(T_{a_L}^\varphi)^*$

$$K^*(y, x - y) = \int_{\mathbb{R}^n} \overline{a_L(y, \xi)} e^{-i(\varphi(y,\xi) - y \cdot \xi - (x-y) \cdot \xi)} \, d\xi = \int_{\mathbb{R}^n} \sigma(y, \xi) e^{i(x-y) \cdot \xi} \, d\xi.$$

Leibniz’s rule and integration by parts yield

$$\begin{aligned} (x - y)^\alpha \partial_y^\beta (K^*(y, x - y)) &= (x - y)^\alpha \int_{\mathbb{R}^n} \partial_y^\beta (\sigma(y, \xi) e^{i(x-y)\cdot\xi}) \, d\xi \\ &= \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ \beta_1 + \beta_2 = \beta \\ \beta_2 \geq \alpha_2}} C_{\alpha, \beta} \int_{\mathbb{R}^n} \partial_\xi^{\alpha_1} \partial_y^{\beta_1} \sigma(y, \xi) \xi^{\beta_2 - \alpha_2} e^{i(x-y)\cdot\xi} \, d\xi \\ &= (\rho(y, \cdot))^\wedge(y - x), \end{aligned}$$

where

$$\rho(y, \xi) := \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ \beta_1 + \beta_2 = \beta \\ \beta_2 \geq \alpha_2}} C_{\alpha, \beta} \partial_\xi^{\alpha_1} \partial_y^{\beta_1} \sigma(y, \xi) \xi^{\beta_2 - \alpha_2}.$$

Therefore, Plancherel’s formula yields that

$$\left\| (x - y)^\alpha \partial_y^\beta (K^*(y, x - y)) \right\|_{L_x^2(\mathbb{R}^n)} = \|\rho(y, \cdot)\|_{L_\xi^2(\mathbb{R}^n)} \lesssim \|\rho(y, \cdot)\|_{L_\xi^\infty(\mathbb{R}^n)} \lesssim 1.$$

Hence, Lemma 4.6 can be applied with $T_0 := (T_{a_L}^\varphi)^*$ and $T_j := 0, j \geq 1$, since by Theorem 5.1, T_0 is also bounded on $L^2(\mathbb{R}^n)$ and has an integral representation

$$\int_{\mathbb{R}^n} K^*(y, x - y) f(y) \, dy$$

with

$$K^*(y, x - y) = \int_{\mathbb{R}^n} \overline{a_L(y, \xi)} e^{-i(\varphi(y, \xi) - y \cdot \xi - (x - y) \cdot \xi)} \, d\xi. \quad \square$$

Next, we prove the main result concerning the regularity of the low-frequency portions of oscillatory integral operators.

Lemma 6.3. *Assume that $\psi_0(\xi) \in C^\infty(\mathbb{R}^n)$ is a smooth cut-off function supported in a neighborhood of the origin as in Definition 2.1, $a(x, \xi) \in S_{0,0}^m(\mathbb{R}^n)$ for some $m \in \mathbb{R}$, $a_L(x, \xi) := \psi_0(\xi) a(x, \xi)$, and let $\varphi(x, \xi)$ be a phase function. Finally, let the operator $T_{a_L}^\varphi$ be defined as in equation (12). Then the following statements hold:*

i) *If either equation (32) or (33) holds, then*

$$\|T_{a_L}^\varphi f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{\text{bmo}(\mathbb{R}^n)}.$$

ii) *Assume that $\varphi(x, \xi)$ satisfy the LF(μ)-condition (14) for $0 < \mu \leq 1$ and that $n/(n + \mu) < p \leq \infty$. Then for any $s_1, s_2 \in (-\infty, \infty)$, and $q_1, q_2 \in (0, \infty]$, one has*

$$\|T_{a_L}^\varphi f\|_{B_{p, q_2}^{s_2}(\mathbb{R}^n)} \lesssim \|f\|_{B_{p, q_1}^{s_1}(\mathbb{R}^n)}.$$

iii) *Assume that $\varphi(x, \xi)$ satisfy the LF(μ)-condition (14) for $0 < \mu \leq 1$ and that $a(x, \xi)$ has compact support in the x -variable. Then for any $s_1, s_2 \in (-\infty, \infty)$, and $p, q_1, q_2 \in (0, \infty]$*

$$\|T_{a_L}^\varphi f\|_{B_{p, q_2}^{s_2}(\mathbb{R}^n)} \lesssim \|f\|_{B_{p, q_1}^{s_1}(\mathbb{R}^n)}.$$

Moreover, all the Besov–Lipschitz estimates above can be replaced by the corresponding Triebel–Lizorkin estimates.

Proof of Lemma 4.3, i). We are going to show that $(T_{a_L}^\varphi)^* : L^1(\mathbb{R}^n) \rightarrow h^1(\mathbb{R}^n)$. By Lemma 4.3 and the definition of the h^1 -space (regarded as the Triebel–Lizorkin space $F_{1,2}^0(\mathbb{R}^n)$), we obtain

$$\begin{aligned} \|(T_{a_L}^\varphi)^* f\|_{h^1(\mathbb{R}^n)} &= \left\| \left(\sum_{j=0}^\infty |\psi_j(D)(T_{a_L}^\varphi)^* f|^2 \right)^{1/2} \right\|_{L^1(\mathbb{R}^n)} \sim \|(T_{a_L}^\varphi)^* f\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \|f\|_{L^1(\mathbb{R}^n)}, \end{aligned}$$

where we have also used the fact that $\psi_j(D)(T_{a_L}^\varphi)^* = (T_{a_L}^\varphi \psi_j(D))^* = 0$ when $j \geq 1$ and used the kernel estimate in Lemma 4.3 to deal with the last L^1 -estimate. □

Proof of Lemma 4.3, ii) – iii). Assume that $f_0 = \Psi_0(D)f$, where Ψ is a smooth cut-off function that is equal to one on the support of ψ_0 so that $T_{a_L}^\varphi f = T_{a_L}^\varphi f_0$. Define the self-adjoint operators

$$L_\xi := 1 - \Delta_\xi, \quad L_y := 1 - \Delta_y,$$

and note that

$$\langle \xi \rangle^{-2} L_y e^{i(x-y) \cdot \xi} = \langle x - y \rangle^{-2} L_\xi e^{i(x-y) \cdot \xi} = e^{i(x-y) \cdot \xi}.$$

Take integers N_1 and N_2 large enough. Integrating by parts, we have

$$\begin{aligned} \psi_j(D)T_{a_L}^\varphi f(x) &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y) \cdot \xi} \psi_j(\xi) T_{a_L}^\varphi f_0(y) \, dy \, d\xi \\ &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} \langle \xi \rangle^{-2N_1} L_y^{N_1} \left(\langle x - y \rangle^{-2N_2} L_\xi^{N_2} e^{i(x-y) \cdot \xi} \right) \psi_j(\xi) T_{a_L}^\varphi f_0(y) \, dy \, d\xi \\ &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y) \cdot \xi} L_\xi^{N_2} \left(\langle \xi \rangle^{-2N_1} \psi_j(\xi) \right) \langle x - y \rangle^{-2N_2} L_y^{N_1} T_{a_L}^\varphi f_0(y) \, dy \, d\xi. \end{aligned}$$

Since ψ_j is supported on an annulus of size 2^j , one has

$$\begin{aligned} \int_{\mathbb{R}^n} \left| L_\xi^{N_2} \langle \xi \rangle^{-2N_1} \psi_j(\xi) \right| \, d\xi &\lesssim \sum_{|\alpha| \leq 2N_2} \int_{|\xi| \sim 2^j} \left| \partial_\xi^\alpha \left(\langle \xi \rangle^{-2N_1} \psi_j(\xi) \right) \right| \, d\xi \\ &\lesssim 2^{jn} \sum_{|\alpha| \leq 2N_2} 2^{-j(2N_1 + |\alpha|)} \lesssim 2^{j(n-2N_1)}. \end{aligned}$$

Also, applying Leibniz’s rule and Faà di Bruno’s formulae, we have that

$$\begin{aligned} L_y^{N_1} T_{a_L}^\varphi f(y) &= \int_{\mathbb{R}^n} L_y^{N_1} (a_L(y, \eta) e^{i\varphi(y, \eta)}) \widehat{f_0}(\eta) \, d\eta \\ &= \int_{\mathbb{R}^n} \sigma(y, \eta) e^{i\varphi(y, \eta)} \widehat{f_0}(\eta) \, d\eta =: T_\sigma^\varphi f_0(y), \end{aligned}$$

with

$$\sigma(y, \eta) := \sum_{|\alpha| \leq 2N_1} \sum_{1 \leq |\beta| \leq 2N_1} \sum_{\ell \leq 2N_1} C_{\alpha, \beta, \ell} \partial_y^\alpha a_L(y, \eta) (\partial_y^\beta \varphi(y, \eta))^\ell. \tag{46}$$

Thus, we have

$$|\psi_j(D)T_{a_L}^\varphi f(x)| \lesssim 2^{j(n-2N_1)} \left(\langle \cdot \rangle^{-2N_2} * |T_\sigma^\varphi f_0| \right)(x). \tag{47}$$

Using the LF(μ) assumption, one has

$$\left| \partial_\eta^\alpha \partial_y^\beta \varphi(y, \eta) \right| \lesssim |\eta|^{\mu-|\alpha|},$$

for $|\alpha| \geq 0, |\beta| \geq 1$. The terms of equation (46), where $\ell = 0$ are bounded by 1 and the terms where $\ell \geq 1$ are bounded by $|\eta|^{\mu-|\alpha|}$.

Hence, Lemma 4.3, using both equations (33) ($\ell = 0$) and (32) ($\ell \geq 1$), yields that for all $0 < \varepsilon < 1$ the kernel of T_σ^φ satisfies the estimate

$$|K(x, y)| \lesssim \langle x - y \rangle^{-n-\varepsilon\mu}.$$

Now, it follows from equation (47), the kernel estimate above and Lemma 2.7 with $r > n/(n + \mu)$ that

$$\begin{aligned} |\psi_j(D)T_{a_L}^\varphi f(x)| &\lesssim 2^{j(n-2N_1)} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \langle x - z \rangle^{-2N_2} \langle z - y \rangle^{-n-\varepsilon\mu} dz \right) |f_0(y)| dy \\ &\lesssim 2^{j(n-2N_1)} \int_{\mathbb{R}^n} \langle x - y \rangle^{-n-\varepsilon\mu} |f_0(y)| dy \\ &\lesssim 2^{j(n-2N_1)} \left(M(|f_0|^r)(x) \right)^{1/r}. \end{aligned} \tag{48}$$

This yields that for $r < p \leq \infty$ one has

$$\|\psi_j(D)T_{a_L}^\varphi f\|_{L^p(\mathbb{R}^n)} \lesssim 2^{j(n-2N_1)} \|f_0\|_{L^p(\mathbb{R}^n)}. \tag{49}$$

In case *iii*), we would also like to extend (49) to the range $0 < p \leq \infty$ when $a(x, \xi)$ has compact support in x . If $\mathcal{K} := \text{supp}_y \sigma(y, \eta)$, then since f_0 is frequency localised, Lemma 2.7 and Peetre’s inequality yield that for $r > n/(n + \mu)$, we have the pointwise estimate

$$\begin{aligned} |\psi_j(D)T_{a_L}^\varphi f(x)| &\lesssim 2^{j(n-2N_1)} \left(\langle \cdot \rangle^{-2N_2} * \chi_{\mathcal{K}} \left(M(|f_0|^r) \right)^{1/r} \right)(x) \\ &\lesssim 2^{j(n-2N_1)} \langle x \rangle^{-2N_2} \int_{\mathcal{K}} \left(M(|f_0|^r)(y) \right)^{1/r} dy, \end{aligned} \tag{50}$$

where $\chi_{\mathcal{K}}$ is the characteristic function of \mathcal{K} . Now, taking the L^p -norm, choosing N_2 large enough, using the L^∞ -boundedness of the Hardy–Littlewood maximal operator, and finally using Lemma 2.8, we obtain for $0 < p \leq \infty$

$$\begin{aligned} \|\psi_j(D)T_{a_L}^\varphi f\|_{L^p(\mathbb{R}^n)} &\lesssim 2^{j(n-2N_1)} \| |f_0|^r \|_{L^\infty(\mathbb{R}^n)}^{1/r} \lesssim 2^{j(n-2N_1)} \|f_0\|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim 2^{j(n-2N_1)} \|f_0\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{51}$$

Thus, equations (49) and (51) yield for N_1 large enough

$$\begin{aligned} \|T_{a_L}^\varphi f\|_{B_{p,q_2}^{s_2}(\mathbb{R}^n)} &= \left(\sum_{j=0}^\infty 2^{js_2q_2} \|\psi_j(D)T_{a_L}^\varphi f\|_{L^p(\mathbb{R}^n)}^{q_2} \right)^{1/q_2} \\ &\lesssim \left(\sum_{j=0}^\infty 2^{jq_2(s_2+n-2N_1)} \|f_0\|_{L^p(\mathbb{R}^n)}^{q_2} \right)^{1/q_2} = \|f_0\|_{L^p(\mathbb{R}^n)} \left(\sum_{j=0}^\infty 2^{jq_2(s_2+n-2N_1)} \right)^{1/q_2} \\ &\lesssim \|f_0\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{B_{p,q_1}^{s_1}(\mathbb{R}^n)}. \end{aligned}$$

In the case of boundedness in Triebel–Lizorkin spaces for *ii*), we use equation (48) and the assumption that $p > r > n/(n + \mu)$ which yield for N_1 large enough that

$$\begin{aligned} \|T_{a_L}^\varphi f\|_{F_{p,q_2}^{s_2}(\mathbb{R}^n)} &= \left\| \left(\sum_{j=0}^\infty 2^{js_2q_2} |\psi_j(D)T_{a_L}^\varphi f|^{q_2} \right)^{1/q_2} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \left\| \left(\sum_{j=0}^\infty 2^{js_2q_2} \left| 2^{j(n-2N_1)} \left(M(|f_0|^r) \right)^{1/r} \right|^{q_2} \right)^{1/q_2} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \left(\sum_{j=0}^\infty 2^{jq_2(s_2+n-2N_1)} \right)^{1/q_2} \left\| \left(M(|f_0|^r) \right)^{1/r} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \|f_0\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{F_{p,q_1}^{s_1}(\mathbb{R}^n)}. \end{aligned}$$

In the case of boundedness in Triebel–Lizorkin spaces for *iii*), we use equation (50) and Lemma 2.8 to see that for all $p > 0$ one has

$$\begin{aligned} \|T_{a_L}^\varphi f\|_{F_{p,q_2}^{s_2}(\mathbb{R}^n)} &= \left\| \left(\sum_{j=0}^\infty 2^{js_2q_2} |\psi_j(D)T_{a_L}^\varphi f|^{q_2} \right)^{1/q_2} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \left(\sum_{j=0}^\infty 2^{jq_2(s_2+n-2N_2)} \right)^{1/q_2} \|\langle \cdot \rangle^{-2N_2}\|_{L^p(\mathbb{R}^n)} \int_{\mathcal{K}} \left(M(|f_0|^r)(x) \right)^{1/r} dx \\ &\lesssim \|f_0\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{F_{p,q_1}^{s_1}(\mathbb{R}^n)}, \end{aligned}$$

by choosing N_2 large enough. □

Remark 6.4. Note that the type of the phase (i.e., the μ in the $LF(\mu)$ -condition (14)) enters the picture only at the level of quasi-Banach boundedness of the oscillatory integral operators.

7. Boundedness of middle-frequency portion

In this section, we show that for the portion of the operator where the frequency support of the amplitude is bounded below, away from the origin and also bounded from above by a fixed $R \gg 1$, then the middle portion of the operator is bounded on Besov–Lipschitz and Triebel–Lizorkin spaces, as the following lemma shows.

Lemma 7.1. *Assume that $\psi_0(\xi) \in C_c^\infty(\mathbb{R}^n)$ is a smooth cut-off function supported in a neighborhood of the origin as in Definition 2.1, $a(x, \xi) \in S_{0,0}^m(\mathbb{R}^n)$ for some $m \in \mathbb{R}$, $a_M(x, \xi) := (\psi_0(\xi/R) - \psi_0(\xi)) a(x, \xi)$ for some $R > 1$, and let $\varphi(x, \xi)$ be a phase function satisfying the ψ^k -condition. Finally, let the operator $T_{a_M}^\varphi$ be defined as in equation (12). Then the following statements hold:*

i) $T_{a_M}^\varphi$ satisfies

$$\|T_{a_M}^\varphi f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{\text{bmo}(\mathbb{R}^n)}.$$

ii) Assume that $\partial_x^\beta \varphi(x, \xi) \in L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})$, for any $|\beta| \geq 1$ and that φ satisfy the L^2 -condition (11). Then for any $s_1, s_2 \in (-\infty, \infty)$ and $p, q_1, q_2 \in (0, \infty]$ one has

$$\|T_{a_M}^\varphi f\|_{B_{p,q_2}^{s_2}(\mathbb{R}^n)} \lesssim \|f\|_{B_{p,q_1}^{s_1}(\mathbb{R}^n)}.$$

iii) Assume that $a(x, \xi)$ has compact support in the x -variable and that $\varphi(x, \xi)$ satisfies the L^2 -condition (11). Then for any $s_1, s_2 \in (-\infty, \infty)$, and $p, q_1, q_2 \in (0, \infty]$

$$\|T_{a_M}^\varphi f\|_{B_{p,q_2}^{s_2}(\mathbb{R}^n)} \lesssim \|f\|_{B_{p,q_1}^{s_1}(\mathbb{R}^n)}.$$

Moreover, all the Besov–Lipschitz estimates above can be replaced by the corresponding Triebel–Lizorkin estimates.

Proof. The proof is similar to that of Lemma 6.3. The only difference is that we cannot use any of equation (14), (32) or (33) to obtain kernel estimates. Instead, we observe that for any $N \geq 0$

$$\begin{aligned} |K(x, y)| &= \left| \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} e^{i\varphi(x,\xi)-ix\cdot\xi} a_M(x, \xi) \, d\xi \right| \\ &= \left| \langle x-y \rangle^{-2N} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} (1-\Delta_\xi)^N e^{i\varphi(x,\xi)-ix\cdot\xi} a_M(x, \xi) \, d\xi \right| \\ &\lesssim \langle x-y \rangle^{-2N}, \end{aligned} \tag{52}$$

using the \mathfrak{F}^k -condition. This is enough to conclude the result in i).

For ii) and iii), we need to replace a_L with a_M in equation (46) and obtain estimate (52) for the kernel of T_σ^φ . The only problem here is to control the factors of the form $\partial_\eta^\alpha \partial_y^\beta \varphi(y, \eta)$, where $|\alpha| \geq 0$ and $|\beta| \geq 1$. But they are uniformly bounded because of the L^2 -condition when $|\alpha| \geq 1$ and

$$|\partial_y^\beta \varphi(y, \eta)| \leq |\partial_y^\beta \varphi(y, \eta) - \partial_y^\beta \varphi(y, \eta_0)| + |\partial_y^\beta \varphi(y, \eta_0)| \lesssim |\eta - \eta_0| + |\partial_y^\beta \varphi(y, \eta_0)| \lesssim 1,$$

for $|\alpha| = 0$ and $\eta \in \text{supp } a_M$, if we choose $\eta_0 := \eta/|\eta|$. Hence, $|\partial_\eta^\alpha \sigma(y, \eta)| \lesssim 1$ which yields the estimate (52), and we can proceed as in Lemma 6.3 from equation (48) onwards. \square

8. Local $h^p - L^p$ boundedness

In this section, we prove the local $h^p - L^p$ boundedness of oscillatory integral operators. As it turns out, for the case of $0 < p < 1$ and the local $h^p - L^p$ boundedness of T_a^φ , no condition on the phase function is required. Moreover, the order of the amplitude could also be larger than the critical order $m_k(p)$. More explicitly, we have

Proposition 8.1. *Let $0 < p < 1$ and $m = -n/p$, and suppose that $\varphi(x, \xi)$ is a measurable real-valued function, $a(x, \xi) \in S_{0,0}^m(\mathbb{R}^n)$ with compact support in the x -variable. Then T_a^φ as given in equation (12) is a bounded operator from $h^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

Proof. Fix a p -atom \mathbf{a} supported in the ball $B := B(\bar{y}, r)$, with $\bar{y} \in \mathbb{R}^n$ and $r > 0$. Also, make the Littlewood–Paley decomposition using Definition 2.1 so that $T_a^\varphi = \sum_{j=0}^\infty T_j$, where $T_j := T_a^\varphi \psi_j(D)$. By Lemma 4.5 and since $T_j \mathbf{a}$ has compact support, it is enough to show that

$$\|T_j \mathbf{a}\|_{L^\infty(\mathbb{R}^n)} \lesssim r^{n-n/p} 2^{j(n-n/p)}, \tag{53}$$

and whenever $r < 1$

$$\|T_j \mathbf{a}\|_{L^\infty(\mathbb{R}^n)} \lesssim r^{N+1+n-n/p} 2^{j(N+1+n-n/p)}, \tag{54}$$

for some $N > n/p - n - 1$.

First of all, Lemma 4.2 taken with $\beta = 0$, yields for all $x \in \mathbb{R}^n$

$$|T_j \mathbf{a}(x)| \leq \int_B |K_j(x, y)| |\mathbf{a}(y)| dy \lesssim r^{n-n/p} 2^{j(n+m)},$$

which gives equation (53).

On the other hand, if $r < 1$, we Taylor expand the kernel as follows:

$$\begin{aligned} K_j(x, y) &= \sum_{|\beta| \leq N} \frac{(y - \bar{y})^\beta}{\beta!} (\partial_y^\beta K_j)(x, \bar{y}) \\ &\quad + (N + 1) \sum_{|\beta|=N+1} \frac{(y - \bar{y})^\beta}{\beta!} \int_0^1 (1-t)^N (\partial_y^\beta K_j)(x, ty + (1-t)\bar{y}) dt, \end{aligned}$$

and taking advantage of the vanishing moments of the atom, we obtain

$$T_j \mathbf{a}(x) = (N + 1) \sum_{|\beta|=N+1} \int_B \int_0^1 \frac{(y - \bar{y})^\beta}{\beta!} (1-t)^N (\partial_y^\beta K_j)(x, ty + (1-t)\bar{y}) \mathbf{a}(y) dt dy.$$

Therefore, applying once again Lemma 4.2, with $|\beta| = N + 1$ and $N := [n(1/p - 1)]$, we obtain

$$|T_j \mathbf{a}(x)| \lesssim r^{N+1+n-n/p} 2^{j(N+1+n+m)},$$

which yields equation (54). □

Remark 8.2. We observe that interpolating the result of Proposition 8.1 with the L^2 -boundedness of operators with amplitudes in $S_{0,0}^0(\mathbb{R}^n)$ yields that T_a^φ is bounded from $h^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $0 < p \leq 2$ with a SND phase function verifying equation (11) and $m < m_1(p)$.

9. Boundedness of high-frequency portion

In this section, we treat the global regularity of the high-frequency portion of oscillatory integral operators. Here we prove $h^p - L^p$ boundedness results.

Proposition 9.1. Suppose that $\varphi \in \mathfrak{F}^k$ is SND, for some $k \geq 1$ and satisfy the L^2 -condition (11). Let $a(x, \xi) \in S_{0,0}^{m_k(p)}(\mathbb{R}^n)$ and $a_H(x, \xi) := (1 - \psi_0(\xi)) a(x, \xi)$, where ψ_0 is given in Definition 2.1. Then, $T_{a_H}^\varphi$ as in equation (12) is a bounded operator from $h^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $0 < p < 1$. In the case $0 < k < 1$, the result above is true provided that $a(x, \xi) \in S_{1,0}^{m_k(p)}(\mathbb{R}^n)$.

Proof. We consider a generic Littlewood–Paley piece of the kernel of $T_{a_H}^\varphi$:

$$K_j(x, x - y) := \int_{\mathbb{R}^n} a_j(x, \xi) e^{i(\varphi(x, \xi) - x \cdot \xi + (x-y) \cdot \xi)} d\xi, \tag{55}$$

where $a_j(x, \xi) := a_H(x, \xi) \psi_j(\xi)$. In light of Lemma 4.6, we only need to show that $T_{a_H}^\varphi$ is L^2 -bounded, which is indeed the case by Theorem 5.1, and that

$$\|(x - y)^\alpha \partial_y^\beta K_j(x, x - y)\|_{L_x^2(\mathbb{R}^n)} \lesssim 2^{j(|\alpha| + (k-1) + |\beta| + m_k(p) + n/2)}.$$

However, since differentiating equation (55) β times in y will only introduce factors of the size $2^{j|\beta|}$, it is enough to establish the above estimate for $\beta = 0$. To this end, take Ψ_j as in Definition 2.1, integrate by parts, and rewrite

$$\begin{aligned}
 (x - y)^\alpha K_j(x, x - y) &= \int_{\mathbb{R}^n} a_j(x, \xi) e^{i\varphi(x, \xi) - ix \cdot \xi} \partial_\xi^\alpha e^{i(x-y) \cdot \xi} \, d\xi \\
 &= \int_{\mathbb{R}^n} \partial_\xi^\alpha \left[a_j(x, \xi) e^{i\varphi(x, \xi) - ix \cdot \xi} \right] e^{i(x-y) \cdot \xi} \Psi_j(\xi) \, d\xi \\
 &= \sum_{\alpha_1 + \alpha_2 = \alpha} C_{\alpha_1, \alpha_2} \int_{\mathbb{R}^n} \partial_\xi^{\alpha_1} a_j(x, \xi) \partial_\xi^{\alpha_2} e^{i\varphi(x, \xi) - ix \cdot \xi} e^{i(x-y) \cdot \xi} \Psi_j(\xi) \, d\xi \\
 &= \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ \lambda_1 + \dots + \lambda_r = \alpha_2}} C_{\alpha_1, \alpha_2, \lambda_1, \dots, \lambda_r} \int_{\mathbb{R}^n} \partial_\xi^{\alpha_1} a_j(x, \xi) \\
 &\quad \times \partial_\xi^{\lambda_1} (\varphi(x, \xi) - x \cdot \xi) \dots \partial_\xi^{\lambda_r} (\varphi(x, \xi) - x \cdot \xi) e^{i\varphi(x, \xi)} e^{-iy \cdot \xi} \Psi_j(\xi) \, d\xi \\
 &= \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ \lambda_1 + \dots + \lambda_r = \alpha_2}} C_{\alpha_1, \alpha_2, \lambda_1, \dots, \lambda_r} 2^{j(m_k(p) + (k-1)|\alpha|)} \int_{\mathbb{R}^n} b_j^{\alpha_1, \alpha_2, \lambda_1, \dots, \lambda_r}(x, \xi) e^{i\varphi(x, \xi)} e^{-iy \cdot \xi} \Psi_j(\xi) \, d\xi \\
 &=: \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ \lambda_1 + \dots + \lambda_r = \alpha_2}} C_{\alpha_1, \alpha_2, \lambda_1, \dots, \lambda_r} 2^{j(m_k(p) + (k-1)|\alpha|)} S_j^{\alpha_1, \alpha_2, \lambda_1, \dots, \lambda_r}(\tau_{-y} \widehat{\Psi}_j)(x),
 \end{aligned}$$

where τ_{-y} is a translation by $-y$, $|\lambda_j| \geq 1$ and

$$\begin{aligned}
 b_j^{\alpha_1, \alpha_2, \lambda_1, \dots, \lambda_r}(x, \xi) &:= 2^{-j(m_k(p) + (k-1)|\alpha|)} \\
 &\quad \times \partial_\xi^{\alpha_1} a_j(x, \xi) \partial_\xi^{\lambda_1} (\varphi(x, \xi) - x \cdot \xi) \dots \partial_\xi^{\lambda_r} (\varphi(x, \xi) - x \cdot \xi).
 \end{aligned}$$

Now, we claim that $b_j^{\alpha_1, \alpha_2, \lambda_1, \dots, \lambda_r}(x, \xi) \in S_{0,0}^0(\mathbb{R}^n)$ uniformly in j . Indeed, since $a \in S_{0,0}^{m_k(p)}(\mathbb{R}^n)$ and $\varphi \in \mathcal{F}^k$ (with $k \geq 1$), we can write

$$\begin{aligned}
 \left| b_j^{\alpha_1, \alpha_2, \lambda_1, \dots, \lambda_r}(x, \xi) \right| &\lesssim 2^{-j(m_k(p) + (k-1)|\alpha|)} 2^{jm_k(p)} 2^{j(k-1)r} \\
 &\leq 2^{-j(k-1)|\alpha|} 2^{j(k-1)r} \lesssim 1.
 \end{aligned}$$

In a similar way, using the \mathcal{F}^k -condition, we can also check that, for any multi-indices γ and β ,

$$\left| \partial_\xi^\gamma \partial_x^\beta b_j^{\alpha_1, \alpha_2, \lambda_1, \dots, \lambda_r}(x, \xi) \right| \lesssim 1, \tag{56}$$

hence $b_j^{\alpha_1, \alpha_2, \lambda_1, \dots, \lambda_r} \in S_{0,0}^0(\mathbb{R}^n)$. In the case $0 < k < 1$, the hypothesis on a and φ yield that

$$\left| \partial_\xi^\alpha \partial_x^\beta a_j(x, \xi) \right| \lesssim 2^{j(m_k(p) - |\alpha|)},$$

and on the support of a_j

$$\left| \partial_\xi^\alpha \partial_x^\beta (\varphi(x, \xi) - x \cdot \xi) \right| \lesssim 2^{j(k-|\alpha|)},$$

which together imply equation (56).

Therefore, Theorem 5.1 yields that

$$\begin{aligned} \|(x - y)^\alpha K_j(x, x - y)\|_{L^2_x(\mathbb{R}^n)} &\lesssim \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ \lambda_1 + \dots + \lambda_r = \alpha_2}} 2^{j(m_k(p) + (k-1)|\alpha|)} \|S_j^{\alpha_1, \alpha_2, \lambda_1, \dots, \lambda_r}(\tau_y \widehat{\Psi}_j)\|_{L^2(\mathbb{R}^n)} \\ &\lesssim 2^{j(|\alpha|(k-1) + m_k(p))} \|\widehat{\Psi}_j\|_{L^2(\mathbb{R}^n)} \lesssim 2^{j(|\alpha|(k-1) + m_k(p) + n/2)}, \end{aligned}$$

and the proof is completed. □

We would like to have a similar result for the adjoint operator, but in this case, we need to add an extra condition. However, this extra condition is automatically fulfilled if one assumes LF(μ)-condition (14), and it turns out to be superfluous as far as the L^p -boundedness is concerned. Since the result is only applied in these two cases, this extra condition will not have any impact on any of the main results. In the following proposition, we let e_ℓ be the unit vectors as in the proof of Theorem 3.4 on page 16.

Proposition 9.2. *Let $a(x, \xi) \in S_{0,0}^{m_k(p)}(\mathbb{R}^n)$ and $a_H(x, \xi) := (1 - \psi_0(\xi/\sqrt{n})) a(x, \xi)$, where ψ_0 is given in Definition 2.1. Suppose that, for $k \geq 1$, $\varphi \in \mathcal{F}^k$ is SND and satisfies the L^2 -condition (11). Moreover, assume that for all $\xi \in \text{supp}_\xi a_H(x, \xi)$, there exists $1 \leq \ell \leq 2n$ such that the line segment between ξ and e_ℓ does not pass through the unit ball $B(0, 1)$ and such that $\partial_x^\beta \varphi(x, e_\ell) \in L^\infty(\mathbb{R}^n)$, for all $|\beta| \geq 1$. Then, $(T_{a_H}^\varphi)^*$ given as in equation (13) is a bounded operator from $h^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $0 < p < 1$. In the case $0 < k < 1$, the result above is true provided that $a(x, \xi) \in S_{1,0}^{m_k(p)}(\mathbb{R}^n)$.*

Proof. The proof follows the same lines as that of Lemma 6.2. Indeed, since $(T_{a_H}^\varphi)^*$ is L^2 -bounded, we only need to show that

$$\|\rho_j(y, \cdot)\|_{L^2_\xi(\mathbb{R}^n)} \lesssim 2^{j(|\alpha|(k-1) + |\beta| + m_k(p) + n/2)}, \tag{57}$$

where

$$\rho_j(y, \xi) := \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ \beta_1 + \beta_2 = \beta \\ \beta_2 \geq \alpha_2}} C_{\alpha, \beta} \partial_\xi^{\alpha_1} \partial_y^{\beta_1} \sigma_j(y, \xi) \xi^{\beta_2 - \alpha_2}, \tag{58}$$

$$\sigma_j(y, \xi) := \overline{a_j(y, \xi)} e^{-i\varphi(y, \xi) + iy \cdot \xi},$$

and $a_j(y, \xi) := a_H(y, \xi) \psi_j(\xi)$ is the usual Littlewood–Paley piece. To this end, Leibniz’s rule yields that

$$\begin{aligned} |\partial_\xi^{\alpha_1} \partial_y^{\beta_1} \sigma_j(y, \xi)| &= |\partial_\xi^{\alpha_1} \partial_y^{\beta_1} \overline{a_j(y, \xi)} e^{-i\varphi(y, \xi) + iy \cdot \xi}| \\ &\lesssim \sum_{\substack{\alpha'_1 + \alpha''_1 = \alpha_1 \\ \beta'_1 + \beta''_1 = \beta_1}} |\partial_\xi^{\alpha'_1} \partial_y^{\beta'_1} \overline{a_j(y, \xi)}| |\partial_\xi^{\alpha''_1} \partial_y^{\beta''_1} e^{-i\varphi(y, \xi) + iy \cdot \xi}|. \end{aligned}$$

Now, if we let $\Phi(y, \xi) := \varphi(y, \xi) - y \cdot \xi$, then Faà di Bruno’s formulae implies

$$|\partial_\xi^\alpha \partial_y^\beta e^{-i\Phi(y, \xi)}| \lesssim \sum_{(\gamma_1, \delta_1) + \dots + (\gamma_r, \delta_r) = (\alpha, \beta)} |\partial_\xi^{\gamma_1} \partial_y^{\delta_1} \Phi(y, \xi)| \dots |\partial_\xi^{\gamma_r} \partial_y^{\delta_r} \Phi(y, \xi)|, \tag{59}$$

where the sum above runs over all possible partitions of (α, β) such that $|\gamma_\nu| + |\delta_\nu| \geq 1$ for $\nu = 1, \dots, r$.

The \mathcal{F}^k -condition isn't enough to estimate the terms in equation (59), and we also need to derive estimates for the derivatives in x . For any $\xi \in \text{supp}_\xi a_H(y, \xi)$, take ℓ as in the statement of this theorem. Then the L^2 -condition and the mean-value theorem yield that

$$\begin{aligned} |\partial_y^\gamma \varphi(y, \xi)| &\leq |\partial_y^\gamma \varphi(y, \xi) - \partial_y^\gamma \varphi(y, e_\ell)| + |\partial_y^\gamma \varphi(y, e_\ell)| \\ &\lesssim |\xi - e_\ell| + |\partial_y^\gamma \varphi(y, e_\ell)| \lesssim |\xi|. \end{aligned} \tag{60}$$

Hence, on the support of a_j one has, for $k \geq 1$,

$$|\partial_\xi^\gamma \partial_y^\delta \Phi(y, \xi)| = \begin{cases} O(2^j), & \gamma = 0, \\ O(2^{j(k-1)}), & \gamma \neq 0, \end{cases}$$

and for $0 < k < 1$

$$|\partial_\xi^\gamma \partial_y^\delta \Phi(y, \xi)| = \begin{cases} O(2^j), & \gamma = 0, \\ O(2^{j(k-|\gamma|)}), & \gamma \neq 0, \end{cases}$$

where we have used the \mathcal{F}^k -condition, L^2 -condition and equation (60). Therefore, for $k \geq 1$, using equation (59) we get

$$|\partial_\xi^{\alpha_1} \partial_y^{\beta_1} \sigma_j(y, \xi)| \lesssim 2^{j(m_k(p)+(k-1)|\alpha_1|+|\beta_1|)}.$$

On the other hand, in the case $0 < k < 1$ using the assumption $a \in S_{1,0}^{m_k(p)}(\mathbb{R}^n)$ we obtain

$$\begin{aligned} |\partial_\xi^{\alpha_1} \partial_y^{\beta_1} \sigma_j(y, \xi)| &\lesssim \sum_{\substack{\alpha'_1 + \alpha''_1 = \alpha_1 \\ \beta'_1 + \beta''_1 = \beta_1}} |\partial_\xi^{\alpha'_1} \partial_y^{\beta'_1} \overline{a_j(y, \xi)}| |\partial_\xi^{\alpha''_1} \partial_y^{\beta''_1} e^{-i\varphi(y, \xi) + iy \cdot \xi}| \\ &\lesssim \sum_{\substack{\alpha'_1 + \alpha''_1 = \alpha_1 \\ \beta'_1 + \beta''_1 = \beta_1}} 2^{j(m_k(p) - |\alpha'_1| + |\alpha''_1|k - |\alpha''_1| + |\beta''_1|)} \\ &\lesssim 2^{j(m_k(p)+(k-1)|\alpha_1|+|\beta_1|)}. \end{aligned}$$

Thus, in both cases

$$\begin{aligned} |\partial_\xi^{\alpha_1} \partial_y^{\beta_1} \sigma_j(y, \xi)| &\lesssim 2^{j(m_k(p)+(k-1)|\alpha_1|+|\beta_1|)} \\ &= 2^{j(m_k(p)+(k-1)|\alpha|-(k-1)|\alpha_2|+|\beta_1|)} \\ &\lesssim 2^{j(m_k(p)+(k-1)|\alpha|+|\alpha_2|+|\beta_1|)}. \end{aligned} \tag{61}$$

Finally, combining equations (61) and (58) we obtain equation (57). Hence, Lemma 4.6 holds and the proof is concluded. \square

10. The $h^p - L^p$ boundedness of Schrödinger integral operators

This section deals with the regularity of the Schrödinger integral operators. An important tool in the proof of the following theorem is a Littlewood–Paley decomposition of the amplitude, where each Littlewood–Paley annulus is further decomposed into a union of balls with constant radii, in contrast to the second frequency localisation introduced by C. Fefferman in [11], where different pieces of the amplitude are supported in ‘angular-radial rectangles’.

Theorem 10.1. *Let T_a^φ be a Schrödinger integral operator according to Definition 2.17 with amplitude $a(x, \xi) \in S_{0,0}^{m_2(p)}(\mathbb{R}^n)$ and phase function φ that is SND. Then T_a^φ is a bounded operator from $h^p(\mathbb{R}^n)$*

to $L^p(\mathbb{R}^n)$ for $0 < p < \infty$. Moreover, if $|\nabla_x \varphi(x, 0)| \in L^\infty(\mathbb{R}^n)$, then T_a^φ is bounded from $L^\infty(\mathbb{R}^n)$ to $\text{bmo}(\mathbb{R}^n)$.

Proof. We start by the analysis of the case of $0 < p < 1$. We make the following decomposition of the integral kernel

$$K(x, y) = \int_{\mathbb{R}^n} a(x, \xi) e^{i\varphi(x, \xi) - iy \cdot \xi} \, d\xi$$

of the operator T_a^φ . We introduce a standard Littlewood–Paley partition of unity $\sum_{j=0}^\infty \psi_j(\xi) = 1$ with $\text{supp } \psi_0 \subset B(0, 2)$, and $\text{supp } \psi_j \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ for $j \geq 1$. Then for every $j \geq 0$ we cover $\text{supp } \psi_j$ with open balls C_j^ν with radius 1 and center ξ_j^ν , where ν runs from 1 to $O(2^{jn})$. Observe that $|C_j^\nu| \lesssim 1$ uniformly in j and ν . Now, take $u \in C_c^\infty(\mathbb{R}^n)$, with $0 \leq u \leq 1$ and supported in $B(0, 2)$ with $u = 1$ on $\overline{B(0, 1)}$. Define $\lambda_j^\nu(\xi) \in C_c^\infty(\mathbb{R}^n)$ to be equal to $u(\xi - \xi_j^\nu)$. Next, set $\chi_j^\nu(\xi) := \lambda_j^\nu(\xi) / \sum_\nu \lambda_j^\nu(\xi)$ and observe that for each $\xi \in \text{supp } \psi_j$ the sum $\sum_\nu \lambda_j^\nu(\xi) \geq 1$, and also $\sum_{j=0}^\infty \sum_\nu \chi_j^\nu(\xi) \psi_j(\xi) = 1$. Now, consider the kernel

$$K_j^\nu(x, y) := \int_{\mathbb{R}^n} \psi_j(\xi) \chi_j^\nu(\xi) e^{i\varphi(x, \xi) - iy \cdot \xi} a(x, \xi) \, d\xi.$$

Therefore, for any multi-index α and any $j \geq 0$ we have

$$\begin{aligned} \partial_y^\alpha K_j^\nu(x, y) &= \int_{\mathbb{R}^n} \psi_j(\xi) \chi_j^\nu(\xi) \partial_y^\alpha e^{i(\varphi(x, \xi) - y \cdot \xi)} a(x, \xi) \, d\xi \\ &= \int_{\mathbb{R}^n} e^{i(\varphi(x, \xi) - y \cdot \xi)} \sigma_j^{\alpha, \nu}(x, \xi) \, d\xi, \end{aligned}$$

where

$$\sigma_j^{\alpha, \nu}(x, \xi) := \psi_j(\xi) \chi_j^\nu(\xi) (-i\xi)^\alpha a(x, \xi).$$

Using the assumption that $a(x, \xi) \in S_{0,0}^{m_2(p)}(\mathbb{R}^n)$, we deduce that for any multi-index γ , any $j \geq 0$ and any ν one has

$$|\partial_\xi^\gamma \sigma_j^{\alpha, \nu}(x, \xi)| \lesssim 2^{j(m_2(p) + |\alpha|)}. \tag{62}$$

If we now set $\vartheta(x, \xi) := \varphi(x, \xi) - \xi \cdot \nabla_\xi \varphi(x, \xi_j^\nu)$, then we can write

$$\partial_y^\alpha K_j^\nu(x, y) = \int_{\mathbb{R}^n} e^{i(\nabla_\xi \varphi(x, \xi_j^\nu) - y) \cdot \xi} e^{i\vartheta(x, \xi)} \sigma_j^{\alpha, \nu}(x, \xi) \, d\xi.$$

Now, we claim that the derivatives of ϑ in ξ are uniformly bounded on the support of $\sigma_j^{\alpha, \nu}(x, \xi)$. To this end, the mean-value theorem and equation (15) yield

$$\begin{aligned} |\partial_{\xi_l} \vartheta(x, \xi)| &= \left| \partial_{\xi_l} \varphi(x, \xi) - \partial_{\xi_l} \varphi(x, \xi_j^\nu) \right| = \left| (\xi - \xi_j^\nu) \cdot \int_0^1 \partial_{\xi_l} \nabla_\xi \varphi(x, t\xi + (1-t)\xi_j^\nu) \, dt \right| \\ &\lesssim \left| \xi - \xi_j^\nu \right| \leq 1, \end{aligned}$$

and

$$\left| \partial_\xi^\mu \vartheta(x, \xi) \right| = \left| \partial_\xi^\mu \varphi(x, \xi) \right| \lesssim 1, \quad \text{for all } |\mu| \geq 2.$$

Defining the differential operator L by

$$L := 1 - i(\nabla_{\xi} \varphi(x, \xi_j^{\nu}) - y) \cdot \nabla_{\xi},$$

one can easily verify that

$$\langle \nabla_{\xi} \varphi(x, \xi_j^{\nu}) - y \rangle^{-2M} L^M e^{i(\nabla_{\xi} \varphi(x, \xi_j^{\nu}) - y) \cdot \xi} = e^{i(\nabla_{\xi} \varphi(x, \xi_j^{\nu}) - y) \cdot \xi},$$

for all integers $M \geq 0$. Therefore, integrating by parts yields

$$\partial_y^{\alpha} K_j^{\nu}(x, y) = \langle \nabla_{\xi} \varphi(x, \xi_j^{\nu}) - y \rangle^{-2M} \int_{\mathbb{R}^n} e^{i(\nabla_{\xi} \varphi(x, \xi_j^{\nu}) - y) \cdot \xi} (L^*)^M e^{i\vartheta(x, \xi)} \sigma_j^{\alpha, \nu}(x, \xi) \, d\xi.$$

This equality, the observation that $\text{supp } \sigma_j^{\alpha, \nu} \subset C_j^{\nu}$, with $|C_j^{\nu}| = O(1)$ uniformly in ν and j , the estimates for the derivatives of ϑ and equation (62) yield

$$|\partial_y^{\alpha} K_j^{\nu}(x, y)| \lesssim \frac{2^{j(m_2(p) + |\alpha|)}}{\langle \nabla_{\xi} \varphi(x, \xi_j^{\nu}) - y \rangle^M}, \tag{63}$$

for all multi-indices α and all $j \geq 0$.

Let T_j^{ν} be the operators corresponding to the kernels K_j^{ν} and \mathbf{a} be a p -atom supported in the ball $B := B(\bar{y}, r)$ with $\bar{y} \in \mathbb{R}^n$ and $r > 0$. Define the *domain of influence* of T_j^{ν} by

$$B_j^{\nu} := \{x : |\nabla_{\xi} \varphi(x, \xi_j^{\nu}) - \bar{y}| \leq 2r\}.$$

Since $0 < p < 1$ we have

$$\begin{aligned} \|T_a^{\varphi} \mathbf{a}\|_{L^p(\mathbb{R}^n)}^p &\lesssim \sum_j \sum_{\nu} \|T_j^{\nu} \mathbf{a}\|_{L^p(\mathbb{R}^n)}^p \\ &= \sum_j \sum_{\nu} \left(\|T_j^{\nu} \mathbf{a}\|_{L^p(B_j^{\nu})}^p + \|T_j^{\nu} \mathbf{a}\|_{L^p(\mathbb{R}^n \setminus B_j^{\nu})}^p \right). \end{aligned} \tag{64}$$

We start with the first term in equation (64). Since, by the SND-condition, the map $x \mapsto \nabla_{\xi} \varphi(x, \xi_j^{\nu})$ is a global diffeomorphism, one has that $|B_j^{\nu}| \lesssim r^n$ uniformly in j and ν . Therefore, the L^2 -boundedness of $2^{-jm_2(p)} T_j^{\nu}$ proven in [13] and Hölder's inequality yield

$$\begin{aligned} \|T_j^{\nu} \mathbf{a}\|_{L^p(B_j^{\nu})} &\lesssim r^{n(1/p-1/2)} 2^{jm_2(p)} \|2^{-jm_2(p)} T_j^{\nu} \mathbf{a}\|_{L^2(\mathbb{R}^n)} \\ &\lesssim r^{n(1/p-1/2)} 2^{jm_2(p)} \|\mathbf{a}\|_{L^2(\mathbb{R}^n)} \lesssim 2^{jm_2(p)}, \end{aligned}$$

where the L^2 -boundedness of the second inequality is uniform in j and ν . This is because the symbol of $2^{-jm_2(p)} T_j^{\nu}$ fulfills

$$|\partial_{\xi}^{\alpha} \partial_x^{\beta} (2^{-jm_2(p)} \psi_j(\xi) \chi_j^{\nu}(\xi) a(x, \xi))| \lesssim 1,$$

uniformly in j and ν and is hence an element of $S_{0,0}^0(\mathbb{R}^n)$.

We turn to the second term in equation (64) and estimate that in two different ways. First, we observe that for $x \in \mathbb{R}^n \setminus B_j^{\nu}$ and $y \in B$ one has

$$|\nabla_{\xi} \varphi(x, \xi_j^{\nu}) - y| \geq \frac{1}{2} |\nabla_{\xi} \varphi(x, \xi_j^{\nu}) - \bar{y}|.$$

Now, using the SND-condition of the phase and equation (63) with $\alpha = 0$, we obtain (taking M large enough)

$$\begin{aligned} \|T_j^y \mathbf{a}\|_{L^p(\mathbb{R}^n \setminus B_j^y)}^p &\lesssim \int_{\mathbb{R}^n \setminus B_j^y} \left(\int_B |K_j^y(x, y)| |\mathbf{a}(y)| \, dy \right)^p dx \\ &\lesssim \int_{\mathbb{R}^n} \left(\int_B \frac{2^{jm_2(p)} r^{-n/p}}{\langle \nabla_\xi \varphi(x, \xi_j^y) - \bar{y} \rangle^M} \, dy \right)^p dx \lesssim 2^{j(np-2n)} r^{np-n}. \end{aligned} \tag{65}$$

Second, if $r < 1$, Taylor expansion of K_j^y in the y -variable around \bar{y} , using the moment conditions of \mathbf{a} , and finally equation (63) yield that for $N := [n(1/p - 1)]$

$$\begin{aligned} \|T_j^y \mathbf{a}\|_{L^p(\mathbb{R}^n \setminus B_j^y)}^p &\lesssim \sum_{|\alpha|=N+1} \int_{\mathbb{R}^n \setminus B_j^y} \left(\int_B |\partial_y^\alpha K_j^y(x, y^*)| |y - \bar{y}|^{N+1} |\mathbf{a}(y)| \, dy \right)^p dx \\ &\lesssim 2^{j(np-2n+p(N+1))} r^{np-n+p(N+1)}, \end{aligned}$$

where y^* is a point on the line segment connecting y and \bar{y} . Note that we have also used that for $x \in \mathbb{R}^n \setminus B_j^y$, one has

$$|\nabla_\xi \varphi(x, \xi_j^y) - \bar{y}| \lesssim |\nabla_\xi \varphi(x, \xi_j^y) - y^*|.$$

Since $r < 1$, take the unique integer $\ell \in \mathbb{Z}_+$ such that $2^{-\ell-1} \leq r < 2^{-\ell}$. Then recalling that there are $O(2^{j\ell n})$ terms in the sum in ν , we have

$$\begin{aligned} \sum_{j=0}^\infty \sum_\nu \left(\|T_j^y \mathbf{a}\|_{L^p(B_j^y)}^p + \|T_j^y \mathbf{a}\|_{L^p(\mathbb{R}^n \setminus B_j^y)}^p \right) &\lesssim \sum_{j \geq \ell} \sum_\nu \left(2^{-2jn(1-p/2)} + 2^{j(np-2n)} r^{np-n} \right) \\ &\quad + \sum_{j < \ell} \sum_\nu \left(2^{-2jn(1-p/2)} + 2^{j(np-2n+pN+p)} r^{np-n+pN+p} \right) \\ &\lesssim \sum_{j \geq \ell} \left(2^{-jn(1-p)} + 2^{j(np-n)} r^{np-n} \right) \\ &\quad + \sum_{j < \ell} \left(2^{-jn(1-p)} + 2^{j(np-n+pN+p)} r^{np-n+pN+p} \right) \\ &\lesssim 1 + 2^{\ell(np-n)} r^{np-n} + 2^{\ell(np-n+pN+p)} r^{np-n+pN+p} \sim 1. \end{aligned}$$

Now, if $r \geq 1$, we do the same calculation as above, except that we take $\ell = 0$ and do not consider the case $j < \ell$. Hence, only equation (65) is needed to estimate $\|T_j^y \mathbf{a}\|_{L^p(\mathbb{R}^n \setminus B_j^y)}^p$, and we conclude that $\|T_a^\varphi \mathbf{a}\|_{L^p(\mathbb{R}^n)}^p$ is also uniformly bounded when $r \geq 1$.

Interpolating this with the L^2 -boundedness result in [13] yields the result for $0 < p \leq 2$.

For the L^p -boundedness of T_a^φ in the range $2 \leq p < \infty$, using Remark 3.8 we can without loss of generality assume that $\varphi(x, 0) = 0$ in T_a^φ . Now, using duality and interpolation, the L^p -boundedness of T_a^φ (with this kind of phase function) would be a consequence of the $h^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ boundedness of the adjoint operator $(T_a^\varphi)^*$, for $0 < p \leq 2$.

Therefore, we start by showing the $h^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ boundedness of the adjoint operator $(T_a^\varphi)^*$ (with $\varphi(x, 0) = 0$), for $0 < p < 1$ and make the following observations. The kernel of $(T_a^\varphi)^*$ is given by

$$K_j^{y*}(x, y) = \int_{\mathbb{R}^n} \psi_j(\xi) \chi_j^y(\xi) e^{-i(\varphi(y, \xi) - x \cdot \xi)} \overline{a(y, \xi)} \, d\xi,$$

therefore for any multi-index α we have

$$\begin{aligned} \partial_y^\alpha K_j^{v*}(x, y) &= \int_{\mathbb{R}^n} \psi_j(\xi) \chi_j^v(\xi) \partial_y^\alpha \left(e^{-i(\varphi(y, \xi) - x \cdot \xi)} \overline{a(y, \xi)} \right) d\xi \\ &= \int_{\mathbb{R}^n} e^{-i(\varphi(y, \xi) - x \cdot \xi)} \sigma_j^{\alpha, v*}(y, \xi) d\xi, \end{aligned}$$

where

$$\sigma_j^{\alpha, v*}(y, \xi) := \psi_j(\xi) \chi_j^v(\xi) \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ \lambda_1 + \dots + \lambda_r = \alpha_2}} C_{\alpha_1, \alpha_2, \lambda_1, \dots, \lambda_r} \partial_y^{\alpha_1} \overline{a(y, \xi)} \partial_y^{\lambda_1} \varphi(y, \xi) \cdots \partial_y^{\lambda_r} \varphi(y, \xi),$$

and $|\lambda_j| \geq 1$. Now, for $|\lambda_j + \beta| \geq 2$,

$$|\partial_y^{\lambda_j} \partial_\xi^\beta \varphi(y, \xi)| \lesssim 1,$$

and using that $\varphi(y, 0) = 0$ and the mean-value theorem, we obtain

$$|\nabla_y \varphi(y, \xi)| \lesssim |\xi|.$$

From these estimates, we deduce that for any multi-index γ one has $|\partial_\xi^\gamma \sigma_j^{\alpha, v*}(y, \xi)| \lesssim 2^{j(m_2(p) + |\alpha|)}$. Therefore, following the same line of reasoning as for the case of T_a^φ yields for all multi-indices α and all $j \geq 0$ that

$$|\partial_y^\alpha K_j^{v*}(x, y)| \lesssim \frac{2^{j(m_2(p) + |\alpha|)}}{\langle \nabla_\xi \varphi(y, \xi_j^v) - x \rangle^M}.$$

Now, the rest of the proof proceeds almost exactly as in the case of T_a^φ .

Having established the $h^p - L^p$ boundedness of $(T_a^\varphi)^*$ for $0 < p < 1$, we can use interpolation to extend this to the desired range $0 < p \leq 2$. Summing up, this (together with duality and interpolation) shows the $h^p - L^p$ boundedness of T_a^φ for $0 < p < \infty$.

Now, for the boundedness of T_a^φ from $L^\infty(\mathbb{R}^n)$ to $\text{bmo}(\mathbb{R}^n)$ one can write $T_a^\varphi = e^{i\varphi(x, 0)} T_a^{\tilde{\varphi}}$ with $\tilde{\varphi}(x, 0) = 0$. Then given the assumption on the phase function of Schrödinger integral operators and the extra assumption $|\nabla_x \varphi(x, 0)| \in L^\infty(\mathbb{R}^n)$ on the phase, one can use equation (9) to reduce matters to the boundedness of $T_a^{\tilde{\varphi}}$. But the boundedness of $T_a^{\tilde{\varphi}}$ from $L^\infty(\mathbb{R}^n)$ to $\text{bmo}(\mathbb{R}^n)$ is a consequence of the boundedness of $(T_a^{\tilde{\varphi}})^*$ from $h^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ which is achieved in the same way, as in the analysis of $(T_a^\varphi)^*$ above. The details are left to the interested reader. \square

11. Action of parameter-dependent pseudodifferential operators on oscillatory integrals

Here, we prove the result concerning the composition of parameter-dependent pseudodifferential operators and oscillatory integral operators and also derive an asymptotic expansion for the composition operator.

Proof of Theorem 3.11. The idea of the proof is similar to that of the asymptotic expansion proved in [26]; however, the details are somewhat different. Let $\chi(x - y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that $0 \leq \chi \leq 1$, $\chi(x - y) \equiv 1$ for $|x - y| < \kappa/2$ and $\chi(x - y) = 0$ for $|x - y| > \kappa$, for some small κ to be specified later. We now decompose $\sigma_t(x, \xi)$ into two parts $\mathbf{I}_1(t, x, \xi)$ and $\mathbf{I}_2(t, x, \xi)$, where

$$\mathbf{I}_1(t, x, \xi) := \iint_{\mathbb{R}^n \times \mathbb{R}^n} a(y, \xi) b(x, t\eta) (1 - \chi(x - y)) e^{i(x-y) \cdot \eta + i\varphi(y, \xi) - i\varphi(x, \xi)} d\eta dy,$$

and

$$\mathbf{I}_2(t, x, \xi) := \iint_{\mathbb{R}^n \times \mathbb{R}^n} a(y, \xi) b(x, t\eta) \chi(x - y) e^{i(x-y) \cdot \eta + i\varphi(y, \xi) - i\varphi(x, \xi)} \, d\eta \, dy.$$

Step 1 – The analysis of $\mathbf{I}_1(t, x, \xi)$

To this end, we introduce the differential operators

$$L_\eta := -i \frac{x - y}{|x - y|^2} \cdot \nabla_\eta \quad \text{and} \quad L_y := \frac{1}{\langle \nabla_y \varphi(y, \xi) \rangle^2 - i \Delta_y \varphi(y, \xi)} (1 - \Delta_y).$$

Because of equation (29), one has

$$|\langle \nabla_y \varphi(y, \xi) \rangle^2 - i \Delta_y \varphi(y, \xi)| \geq \langle \nabla_y \varphi(y, \xi) \rangle^2 \gtrsim \langle \xi \rangle^2.$$

Now, integration by parts yields

$$\begin{aligned} \mathbf{I}_1(t, x, \xi) &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} (L_y^*)^{N_2} \{ e^{-iy \cdot \eta} a(y, \xi) (L_\eta^*)^{N_1} [(1 - \chi(x - y)) b(x, t\eta)] \} \\ &\quad \times e^{ix \cdot \eta + i\varphi(y, \xi) - i\varphi(x, \xi)} \, d\eta \, dy. \end{aligned}$$

Now, since $0 < t \leq 1$, provided $0 < N_3 < N_1 - s$, we have

$$\begin{aligned} \left| \partial_{\eta_j}^{N_1} b(x, t\eta) \right| &\lesssim t^{N_1} \langle t\eta \rangle^{s - N_1} = t^{N_1} \langle t\eta \rangle^{-N_3} \langle t\eta \rangle^{s - (N_1 - N_3)} \\ &\lesssim t^{N_1} \left(t^2 + |t\eta|^2 \right)^{-N_3/2} \langle t\eta \rangle^{s - (N_1 - N_3)} \lesssim t^{N_1 - N_3} \langle \eta \rangle^{-N_3}. \end{aligned}$$

Therefore, choosing $N_1 > n$ and $2N_2 < N_3 - n$

$$\begin{aligned} |\mathbf{I}_1(t, x, \xi)| &\lesssim t^{N_1 - N_3} \langle \xi \rangle^{-2N_2 + m} \iint_{|x - y| > \kappa} \langle \eta \rangle^{2N_2} |x - y|^{-N_1} \langle \eta \rangle^{-N_3} \, d\eta \, dy \\ &\lesssim t^{N_1 - N_3} \langle \xi \rangle^{-2N_2 + m}. \end{aligned}$$

Estimating derivatives of $\mathbf{I}_1(t, x, \xi)$ with respect to x and ξ may introduce factors estimated by powers of $\langle \xi \rangle$, $\langle \eta \rangle$, and $|x - y|$, which can all be handled by choosing N_1 and N_2 appropriately. Therefore, for all N and any $\nu > 0$

$$\left| \partial_\xi^\alpha \partial_x^\beta \mathbf{I}_1(t, x, \xi) \right| \lesssim t^\nu \langle \xi \rangle^{-N},$$

and so $\mathbf{I}_1(t, x, \xi)$ forms part of the error term $t^{\varepsilon M} r(t, x, \xi)$ in equation (16).

Step 2 – The analysis of $\mathbf{I}_2(t, x, \xi)$

First, we make the change of variables $\eta = \nabla_x \varphi(x, \xi) + \zeta$ in the integral defining $\mathbf{I}_2(t, x, \xi)$ and then expand $b(x, t\eta)$ in a Taylor series to obtain

$$b(x, t\nabla_x \varphi(x, \xi) + t\zeta) = \sum_{0 \leq |\alpha| < M} t^{|\alpha|} \frac{\zeta^\alpha}{\alpha!} \left(\partial_\eta^\alpha b \right) (x, t\nabla_x \varphi(x, \xi)) + t^M \sum_{|\alpha| = M} C_\alpha \zeta^\alpha r_\alpha(t, x, \xi, \zeta),$$

where

$$r_\alpha(t, x, \xi, \zeta) := \int_0^1 (1 - \tau)^{M-1} \left(\partial_\eta^\alpha b \right) (x, t\nabla_x \varphi(x, \xi) + \tau t\zeta) \, d\tau. \tag{66}$$

If we set

$$\Phi(x, y, \xi) := \varphi(y, \xi) - \varphi(x, \xi) + (x - y) \cdot \nabla_x \varphi(x, \xi),$$

we obtain

$$\mathbf{I}_2(t, x, \xi) = \sum_{|\alpha| < M} \frac{t^{\varepsilon|\alpha|}}{\alpha!} \sigma_\alpha(t, x, \xi) + t^{\varepsilon M} \sum_{|\alpha|=M} C_\alpha R_\alpha(t, x, \xi),$$

where, using integration by parts, we have

$$\begin{aligned} \sigma_\alpha(t, x, \xi) &:= t^{(1-\varepsilon)|\alpha|} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y) \cdot \zeta + i\Phi(x, y, \xi)} \zeta^\alpha a(y, \xi) \chi(x - y) (\partial_\eta^\alpha b)(x, t \nabla_x \varphi(x, \xi)) \, dy \, d\zeta \\ &= t^{(1-\varepsilon)|\alpha|} \left(\partial_\eta^\alpha b \right) (x, t \nabla_x \varphi(x, \xi)) (i)^{-|\alpha|} \partial_y^\alpha \left[e^{i\Phi(x, y, \xi)} a(y, \xi) \chi(x - y) \right] \Big|_{y=x}, \end{aligned}$$

and

$$R_\alpha(t, x, \xi) := t^{(1-\varepsilon)|\alpha|} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y) \cdot \zeta} e^{i\Phi(x, y, \xi)} \zeta^\alpha a(y, \xi) \chi(x - y) r_\alpha(t, x, \xi, \zeta) \, dy \, d\zeta.$$

Step 2.1 – The analysis of $\sigma_\alpha(t, x, \xi)$

We now claim that

$$\left| \partial_y^\gamma e^{i\Phi(x, y, \xi)} \Big|_{y=x} \right| \lesssim \langle \xi \rangle^{|\gamma|/2}. \tag{67}$$

We first observe that when $\gamma = 0$, equation (67) is obvious. To obtain equation (67) for $\gamma \neq 0$, we recall Faà di Bruno’s formulae

$$\partial_y^\gamma e^{i\Phi(x, y, \xi)} = \sum_{\gamma_1 + \dots + \gamma_k = \gamma} C_\gamma (\partial_y^{\gamma_1} \Phi(x, y, \xi)) \cdots (\partial_y^{\gamma_k} \Phi(x, y, \xi)) e^{i\Phi(x, y, \xi)},$$

where the sum ranges of γ_j such that $|\gamma_j| \geq 1$ for $j = 1, 2, \dots, k$ and $\gamma_1 + \dots + \gamma_k = \gamma$ for some $k \in \mathbb{Z}_+$. Since $\Phi(x, x, \xi) = 0$ and $\partial_y \Phi(x, y, \xi) \Big|_{y=x} = 0$, setting $y = x$ in the expansion above leaves only terms in which $|\gamma_j| \geq 2$ for all $j = 1, 2, \dots, k$. But $\sum_{j=1}^k |\gamma_j| \leq |\gamma|$, so we actually have $2k \leq |\gamma|$, that is $k \leq |\gamma|/2$. Estimate (30) on the phase tells us that $|\partial_y^{\gamma_j} \Phi(x, y, \xi)| \lesssim \langle \xi \rangle$, so

$$\left| \partial_y^\gamma e^{i\Phi(x, y, \xi)} \Big|_{y=x} \right| \lesssim \langle \xi \rangle \cdots \langle \xi \rangle \lesssim \langle \xi \rangle^k \lesssim \langle \xi \rangle^{|\gamma|/2},$$

which is equation (67).

If we use the fact that $t \leq 1$ and the assumption *i*) of Theorem 3.11 on the phase function φ , then we have

$$\begin{aligned} |\sigma_\alpha(t, x, \xi)| &\lesssim t^{(1-\varepsilon)|\alpha|} \langle t \nabla_x \varphi(x, \xi) \rangle^{s-|\alpha|} \langle \xi \rangle^{|\alpha|/2} \langle \xi \rangle^m \\ &\lesssim t^{(1-\varepsilon)|\alpha|} \langle t \xi \rangle^{s-(1-\varepsilon)|\alpha|} \langle t \xi \rangle^{-\varepsilon|\alpha|} \langle \xi \rangle^{m+|\alpha|/2} \\ &\lesssim t^{\min(s, 0)} \langle \xi \rangle^{s+m-(1/2-\varepsilon)|\alpha|}, \end{aligned}$$

when $|\alpha| > 0$.

By the assumptions of the theorem, the derivatives of σ_α with respect to x or ξ do not change the estimates when applied to b , and the same is true when derivatives are applied to $\partial_y^\alpha e^{i\Phi(x,y,\xi)}|_{y=x}$. Therefore, for all multi-indices $\beta, \gamma \in \mathbb{Z}_+$,

$$\left| \partial_\xi^\beta \partial_x^\gamma \sigma_\alpha(t, x, \xi) \right| \lesssim t^{\min(s,0)} \langle \xi \rangle^{s+m-(1/2-\varepsilon)|\alpha|-\rho|\beta|},$$

as required.

Step 2.2 – The analysis of $R_\alpha(t, x, \xi)$

Take $g \in C_c^\infty(\mathbb{R}^n)$ such that $g(x) = 1$ for $|x| < \delta/2$ and $g(x) = 0$ for $|x| > \delta$, for some small $\delta > 0$ to be chosen later. We then decompose

$$\begin{aligned} R_\alpha(t, x, \xi) &= t^{(1-\varepsilon)|\alpha|} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\cdot \zeta} g\left(\frac{\zeta}{\langle \xi \rangle}\right) \partial_y^\alpha \left[e^{i\Phi(x,y,\xi)} \chi(x-y) a(y, \xi) r_\alpha(t, x, \xi, \zeta) \right] dy d\zeta \\ &\quad + t^{(1-\varepsilon)|\alpha|} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\cdot \zeta} \left(1 - g\left(\frac{\zeta}{\langle \xi \rangle}\right) \right) \\ &\quad \times \partial_y^\alpha \left[e^{i\Phi(x,y,\xi)} \chi(x-y) a(y, \xi) r_\alpha(t, x, \xi, \zeta) \right] dy d\zeta \\ &=: R_\alpha^I(t, x, \xi) + R_\alpha^{II}(t, x, \xi). \end{aligned}$$

Step 2.2.1 – The analysis of $R_\alpha^I(t, x, \xi)$

Note that the inequality

$$\langle \xi \rangle \leq 1 + |\xi| \leq \sqrt{2} \langle \xi \rangle,$$

and equation (29) yield

$$\langle t \nabla_x \varphi(x, \xi) + t \tau \zeta \rangle \leq (C_2 \sqrt{2} + \delta) \langle t \xi \rangle,$$

and

$$\begin{aligned} \sqrt{2} \langle t \nabla_x \varphi(x, \xi) + t \tau \zeta \rangle &\geq 1 + |t \nabla_x \varphi| - |t \zeta| \\ &\geq 1 + C_1 |t \xi| - t \delta \langle \xi \rangle \\ &\geq (1 - \delta) + (C_1 - \delta) |t \xi| \geq (\min\{1, C_1\} - \delta) \langle t \xi \rangle. \end{aligned}$$

Therefore, if we choose $\delta < \min\{1, C_1\}$, then for any $\tau \in (0, 1)$, $\langle t \nabla_x \varphi(x, \xi) + t \tau \zeta \rangle$ and $\langle t \xi \rangle$ are equivalent.

This yields that for $|\zeta| \leq r \langle \xi \rangle$, $\partial_\zeta^\beta r_\alpha(t, x, \xi, \zeta)$ are dominated by $t^{|\beta|} \langle t \xi \rangle^{s-|\alpha|-|\beta|}$. Furthermore, for $t \leq 1$, it follows from the representation (66) for r_α that

$$\begin{aligned} \left| \partial_\zeta^\beta \left(g\left(\frac{\zeta}{\langle \xi \rangle}\right) r_\alpha(t, x, \xi, \zeta) \right) \right| &\lesssim \sum_{\gamma \leq \beta} \left| \partial_\zeta^\gamma g\left(\frac{\zeta}{\langle \xi \rangle}\right) \partial_\zeta^{\beta-\gamma} r_\alpha(t, x, \xi, \zeta) \right| \\ &\leq C_{\alpha,\beta} \sum_{\gamma \leq \beta} t^{|\beta|-|\gamma|} \langle \xi \rangle^{-|\gamma|} \langle t \xi \rangle^{s-|\alpha|-|\beta|+|\gamma|} \\ &\lesssim \sum_{\gamma \leq \beta} t^{\min(s,0)+|\beta|-|\gamma|-(1-\varepsilon)|\alpha|} \langle \xi \rangle^{-|\gamma|} \langle \xi \rangle^{s-(1-\varepsilon)|\alpha|} t^{-(|\beta|-|\gamma|)} \langle \xi \rangle^{-(|\beta|-|\gamma|)} \\ &\lesssim t^{\min(s,0)-(1-\varepsilon)|\alpha|} \langle \xi \rangle^{s-(1-\varepsilon)|\alpha|-|\beta|}. \end{aligned} \tag{68}$$

At this point, we also need estimates for $\partial_y^\alpha e^{i\Phi(x,y,\xi)}$ off the diagonal, that is, when $x \neq y$. This derivative has at most $|\alpha|$ powers of terms $\nabla_y \varphi(y, \xi) - \nabla_x \varphi(x, \xi)$, possibly also multiplied by at most

$|\alpha|$ higher order derivatives $\partial_y^\beta \varphi(y, \xi)$, which can be estimated by $(|y - x| \langle \xi \rangle)^{|\alpha|}$ using equation (30). The term containing the difference $\nabla_y \varphi(y, \xi) - \nabla_x \varphi(x, \xi)$ is the product of at most $|\alpha|/2$ terms of the type $\partial_y^\beta \varphi(y, \xi)$, which can be estimated by $\langle \xi \rangle^{|\alpha|/2}$ in view of equation (30). These observations yield

$$|\partial_y^\alpha e^{i\Phi(x,y,\xi)}| \lesssim (1 + |x - y| \langle \xi \rangle)^{|\alpha|} \langle \xi \rangle^{|\alpha|/2},$$

and therefore we also have

$$\left| \partial_y^\alpha \left[e^{i\Phi(x,y,\xi)} \chi(x - y) \right] \right| \lesssim (1 + |x - y| \langle \xi \rangle)^{|\alpha|} \langle \xi \rangle^{|\alpha|/2}. \tag{69}$$

Let

$$L_\zeta := \frac{(1 - \langle \xi \rangle^2 \Delta_\zeta)}{1 + \langle \xi \rangle^2 |x - y|^2}, \quad \text{so} \quad L_\zeta^N e^{i(x-y) \cdot \zeta} = e^{i(x-y) \cdot \zeta}.$$

Integration by parts with L_ζ yields

$$\begin{aligned} R_\alpha^I(t, x, \xi) &= t^{(1-\varepsilon)|\alpha|} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{e^{i(x-y) \cdot \zeta} \partial_y^\alpha [\chi(x - y) a(y, \xi) e^{i\Phi(x,y,\xi)}]}{(1 + \langle \xi \rangle^2 |x - y|^2)^N} \\ &\quad \times (1 - \langle \xi \rangle^2 \Delta_\zeta)^N \left\{ g\left(\frac{\zeta}{\langle \xi \rangle}\right) r_\alpha(t, x, \xi, \zeta) \right\} dy d\zeta \\ &= t^{(1-\varepsilon)|\alpha|} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{e^{i(x-y) \cdot \zeta} \partial_y^\alpha [\chi(x - y) a(y, \xi) e^{i\Phi(x,y,\xi)}]}{(1 + \langle \xi \rangle^2 |x - y|^2)^N} \\ &\quad \times \sum_{|\beta| \leq 2N} c_\beta \langle \xi \rangle^{|\beta|} \left\{ \partial_\zeta^\beta \left(g\left(\frac{\zeta}{\langle \xi \rangle}\right) r_\alpha(t, x, \xi, \zeta) \right) \right\} dy d\zeta. \end{aligned}$$

Using estimates (68), (69) and that the size of the support of $g(\zeta/\langle \xi \rangle)$ in ζ is bounded by $(\delta \langle \xi \rangle)^n$ yield

$$\begin{aligned} |R_\alpha^I(t, x, \xi)| &\lesssim t^{\min(s,0)} \sum_{|\beta| \leq 2N} \langle \xi \rangle^{n+|\beta|} \langle \xi \rangle^{-(1-\varepsilon)|\alpha|-|\beta|} \langle \xi \rangle^{|\alpha|/2+s+m} \int_{|x-y| < \kappa} \frac{(1 + |x - y| \langle \xi \rangle)^{|\alpha|}}{(1 + \langle \xi \rangle^2 |x - y|^2)^N} dy \\ &\lesssim t^{\min(s,0)} \sum_{|\beta| \leq 2N} \langle \xi \rangle^{n+|\beta|} \langle \xi \rangle^{s-(1-\varepsilon)|\alpha|-|\beta|} \langle \xi \rangle^{|\alpha|/2+m} \langle \xi \rangle^{-n} \int_0^\infty \frac{\tau^{n-1} (1 + \tau)^{|\alpha|}}{(1 + \tau^2)^N} d\tau \\ &\lesssim t^{\min(s,0)} \langle \xi \rangle^{s+m-(1/2-\varepsilon)|\alpha|}, \end{aligned}$$

if we choose $N > (n + |\alpha|)/2$, and the hidden constants in the estimates are independent of t (because of (68)). The derivatives of $R_\alpha^I(t, x, \xi)$ with respect to x and ξ give an extra power of ζ under the integral. This amounts to taking more y -derivatives, yielding a higher power of $\langle \xi \rangle$. However, for a given number of derivatives of the remainder $R_\alpha^I(t, x, \xi)$, we are free to choose $M = |\alpha|$ as large as we like and therefore the higher power of $\langle \xi \rangle$ will not cause a problem. Thus, for all multi-indices β, γ and $|\alpha|$ large enough we have

$$|\partial_\xi^\beta \partial_x^\gamma R_\alpha^I(t, x, \xi)| \lesssim t^{\min(s,0)} \langle \xi \rangle^{s+m-(1/2-\varepsilon)|\alpha|-\rho|\beta|},$$

where the hidden constant in the estimate does not depend on t .

Step 2.2.2 – The analysis of $R_\alpha^II(t, x, \xi)$

Define

$$\Psi(x, y, \xi, \zeta) := (x - y) \cdot \zeta + \Phi(x, y, \xi) = (x - y) \cdot (\nabla_x \varphi(x, \xi) + \zeta) + \varphi(y, \xi) - \varphi(x, \xi).$$

It follows from equations (29) and (30) that if we choose $\kappa < \delta/8C_0$, then since $|x - y| < \kappa$ on the support of χ , one has (using that we are in the region $|\zeta| \geq \delta\langle\xi\rangle/2$)

$$|\nabla_y \Psi| = |-\zeta + \nabla_y \varphi - \nabla_x \varphi| \leq 2C_2(|\zeta| + \langle\xi\rangle), \quad \text{and}$$

$$|\nabla_y \Psi| \geq |\zeta| - |\nabla_y \varphi - \nabla_x \varphi| \geq \frac{1}{2}|\zeta| + \left(\frac{\delta}{4} - C_0|x - y|\right)\langle\xi\rangle \geq C(|\zeta| + \langle\xi\rangle).$$

Now, using equation (30), for any β we have the estimate

$$\left| \partial_y^\beta \left(e^{-i\Phi(x,y,\xi)} \partial_y^\gamma e^{i\Phi(x,y,\xi)} \right) \right| \lesssim \langle\xi\rangle^{|\gamma|}. \tag{70}$$

For $M = |\alpha| > s$, we also observe that

$$|r_\alpha(t, x, \xi, \zeta)| \lesssim 1. \tag{71}$$

For the differential operator defined to be

$$L_y := i|\nabla_y \Psi|^{-2} \sum_{j=1}^n (\partial_{y_j} \Psi) \partial_{y_j},$$

induction shows that L_y^N has the form

$$(L_y^*)^N = \frac{1}{|\nabla_y \Psi|^{4N}} \sum_{|\beta| \leq N} P_{\beta,N} \partial_y^\beta,$$

where

$$P_{\beta,N} := \sum_{|\mu|=2N} c_{\beta\mu} \delta_j (\nabla_y \Psi)^\mu \partial_y^{\delta_1} \Psi \dots \partial_y^{\delta_N} \Psi,$$

$|\delta_j| \geq 1$ and $\sum_{j=M}^N |\delta_j| + |\beta| = 2N$. It follows from equation (30) that $|P_{\beta,N}| \leq C(|\zeta| + \langle\xi\rangle)^{3N}$. Now, Leibniz's rule yields

$$\begin{aligned} R_\alpha^H(t, x, \xi) &= t^{(1-\varepsilon)|\alpha|} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\cdot\zeta} \left(1 - g\left(\frac{\zeta}{\langle\xi\rangle}\right) \right) r_\alpha(x, \xi, \zeta) \\ &\quad \times \partial_y^\alpha \left[e^{i\Phi(x,y,\xi)} a(y, \xi) \chi(x - y) \right] dy \, d\zeta \\ &= t^{(1-\varepsilon)|\alpha|} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\Psi(x,y,\xi,\zeta)} \left(1 - g\left(\frac{\zeta}{\langle\xi\rangle}\right) \right) r_\alpha(t, x, \xi, \zeta) \\ &\quad \times \sum_{\gamma_1+\gamma_2+\gamma_3=\alpha} \left(e^{-i\Phi(x,y,\xi)} \partial_y^{\gamma_1} e^{i\Phi(x,y,\xi)} \right) \partial_y^{\gamma_2} \chi(x - y) \partial_y^{\gamma_3} a(y, \xi) dy \, d\zeta \\ &= t^{(1-\varepsilon)|\alpha|} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\Psi(x,y,\xi,\zeta)} |\nabla_y \Psi|^{-4N} \sum_{|\beta| \leq N} P_{\beta,N}(x, y, \xi, \zeta) \\ &\quad \times \left(1 - g\left(\frac{\zeta}{\langle\xi\rangle}\right) \right) r_\alpha(t, x, \xi, \zeta) \sum_{\gamma_1+\gamma_2+\gamma_3=\alpha} \partial_y^\beta \left[\left(e^{-i\Phi(x,y,\xi)} \right) \right. \\ &\quad \left. \times \partial_y^{\gamma_1} e^{i\Phi(x,y,\xi)} \partial_y^{\gamma_2} \chi(x - y) \partial_y^{\gamma_3} a(y, \xi) \right] dy \, d\zeta. \end{aligned}$$

It follows now from equation (70) and (71) that

$$\begin{aligned}
 |R_\alpha^{\text{II}}(t, x, \xi)| &\lesssim t^{(1-\varepsilon)|\alpha|} \int_{|\zeta| \geq \delta \langle \xi \rangle / 2} \int_{|x-y| < \kappa} (|\zeta| + \langle \xi \rangle)^{-N} \langle \xi \rangle^{|\alpha|+m} dy \, d\zeta \\
 &\lesssim t^{(1-\varepsilon)|\alpha|} \langle \xi \rangle^{|\alpha|+m} \int_{|\zeta| \geq \delta \langle \xi \rangle / 2} |\zeta|^{-N} d\zeta \leq C \langle \xi \rangle^{|\alpha|+n+m-N},
 \end{aligned}$$

which yields the desired estimate when $N > |\alpha| + n$. For the derivatives of $R_\alpha^{\text{II}}(t, x, \xi)$, we can get, in a similar way to the case for R_α^{I} , an extra power of ζ , which can be taken care of by choosing N large and using the fact that $|x - y| < \kappa$. Therefore, for all multi-indices $\beta, \gamma \in \mathbb{Z}_+$,

$$|\partial_\xi^\beta \partial_x^\gamma R_\alpha^{\text{II}}(t, x, \xi)| \lesssim \langle \xi \rangle^{|\alpha|+n+m-N},$$

where the constant hidden in the estimate does not depend on t . The proof of Theorem 3.11 is now complete. □

12. Regularity on Besov–Lipschitz spaces

In this section, we prove sharp boundedness results of oscillatory integral operators on Besov–Lipschitz spaces. The idea here is to boost all $h^p - L^p$ results in above sections to $B_{p,q}^{s+m-m_k(p)} - B_{p,q}^s$ using the calculus of Theorem 3.11. To this end, we prove the following proposition.

Proposition 12.1. *Let $k \geq 1, 0 < p \leq \infty$ and $a(x, \xi) \in S_{0,0}^{m_k(p)}(\mathbb{R}^n)$ with compact support in the x -variable. Assume that $\varphi \in \mathcal{F}^k$ is SND satisfies the L^2 -condition (11) and the $LF(\mu)$ -condition (14) for some $0 < \mu \leq 1$. If $0 < p < \infty$, then $T_a^\varphi : h^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ and for $p = \infty$ one has $T_a^\varphi : L^\infty(\mathbb{R}^n) \rightarrow \text{bmo}(\mathbb{R}^n)$. If one removes the condition of compact support of $a(x, \xi)$ in x , then the aforementioned boundedness result is valid, but p has to be taken strictly larger than $n/(n + \mu)$. In the case $0 < k < 1$, the results above are true provided that $a(x, \xi) \in S_{1,0}^{m_k(p)}(\mathbb{R}^n)$.*

Proof. For the high-frequency portion of the operator (here is the compact support in the spatial variable not relevant), we use Propositions 9.1 and 9.2 to show that the operators T_a^φ and $(T_a^\varphi)^*$ are bounded from $h^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $0 < p < 1$. Observe that the condition $\partial_x^\beta \varphi(x, e_\ell) \in L^\infty(\mathbb{R}^n)$ is satisfied for all ℓ due to the $LF(\mu)$ -condition. Now, using analytic interpolation, duality and the L^2 -boundedness provided in Theorem 5.1, yields the desired result for the high-frequency portion of the operator T_a^φ .

For the low- and middle-frequency portions of the operator, we just use Lemma 6.3 and Lemma 7.1 in the Triebel–Lizorkin case with $s = 0$ and $q = 2$. □

Lemma 12.2. *Let $k \geq 1, 0 < p \leq \infty, m \in \mathbb{R}$ and $a(x, \xi) \in S_{0,0}^m(\mathbb{R}^n)$. Assume that φ is SND and satisfies the L^2 -condition (11). If ψ_j is defined as in Definition 2.1, then the operator T_j given by*

$$T_j f(x) := \int_{\mathbb{R}^n} e^{i\varphi(x, \xi)} a(x, \xi) \psi_j(\xi) \widehat{f}(\xi) \, d\xi$$

satisfies

$$\|T_j f\|_{L^p(\mathbb{R}^n)} \lesssim 2^{j(m-m_k(p))} \|\Psi_j(D)f\|_{L^p(\mathbb{R}^n)},$$

for $j \in \mathbb{Z}_+$, provided that one of the following holds true:

- i) $\varphi \in \mathcal{F}^k, a(x, \xi)$ is compactly supported in the x -variable and has frequency support in $\mathbb{R}^n \setminus B(0, R)$, for the R given in Lemma 4.1.
- ii) $\varphi \in \mathcal{F}^k, \partial_x^\beta \varphi(x, \xi) \in L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})$, for any $|\beta| \geq 1$ and $a(x, \xi)$ has frequency support in $\mathbb{R}^n \setminus B(0, R)$, for the R given in Lemma 4.1.

- iii) $\varphi \in \mathcal{F}^k \cap C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.
- iv) $k = 2$ and φ satisfies equation (15).

If one removes the requirement on the frequency support of $a(x, \xi)$ and adds the $LF(\mu)$ -condition (14) in i) – ii), then one obtains the result for $0 < p < \infty$ in i) and $n/(n + \mu) < p < \infty$ in ii).

In the case $0 < k < 1$, the results for i) – iii) above are true provided that $a(x, \xi) \in S_{1,0}^{m_k}(\mathbb{R}^n)$.

Proof. Using Remark 3.8, we can without loss of generality assume that $\varphi(x, 0) = 0$ in iii) – iv). Observe that using the mean value theorem, and either L^2 -condition (11) or equation (15), yields that $\partial_x^\beta \varphi(x, \xi) \in L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})$ for $|\beta| \geq 1$.

To simplify the calculation, we set $\sigma(x, \xi) := \langle \xi \rangle^{m_k(p)-m} a(x, \xi)$ so that $\sigma \in S_{0,0}^{m_k(p)}(\mathbb{R}^n)$.

We start with the case $p \in (0, \infty)$. Proposition 12.1 in i) – iii) and Theorem 10.1 in iv) yields that $T_\sigma^\varphi : h^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$. Next, we use the definition of the local Hardy space $h^p(\mathbb{R}^n)$ (see Definition 2.5) and Definition 2.1 to obtain

$$\begin{aligned} \|T_j f\|_{L^p(\mathbb{R}^n)} &\lesssim 2^{j(m-m_k(p))} \|T_\sigma^\varphi \psi_j(D) f\|_{L^p(\mathbb{R}^n)} \lesssim 2^{j(m-m_k(p))} \|\psi_j(D) f\|_{h^p(\mathbb{R}^n)} \\ &\sim 2^{j(m-m_k(p))} \left\| \left(\sum_{\ell=0}^\infty |\Psi_\ell(D) \psi_j(D) f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim 2^{j(m-m_k(p))} \|\Psi_j(D) f\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

If one removes the requirement on the frequency support of $a(x, \xi)$ and adds either equation (32) or (33) in i) – ii), then Lemma 6.3 iii) yields the $h^p - L^p$ result for the low-frequency part of T_σ^φ .

We turn to the case when $p = \infty$, which can only be proved under the assumption $|\xi| \geq R$ in i) – ii). Observe that Proposition 12.1 in i) – iii) and Theorem 10.1 in case iv) give us the $h^1 - L^1$ boundedness of the adjoint operator $(T_\sigma^\varphi)^*$. We set $f_\ell := \psi_\ell(D) f$. Now, the assumptions on the phase and Lemma 4.1 enable us to apply formula (17) to $(\Psi_\ell(D) T_\sigma^\varphi)^*$, which in turn yields that

$$\begin{aligned} \|T_j^* f\|_{L^1(\mathbb{R}^n)} &= 2^{j(m-m_k(p))} \|\psi_j(D) (T_\sigma^\varphi)^* f\|_{L^1(\mathbb{R}^n)} \\ &= 2^{j(m-m_k(p))} \left\| \sum_{\ell=0}^\infty \psi_j(D) (T_\sigma^\varphi)^* \Psi_\ell(D) \psi_\ell(D) f \right\|_{L^1(\mathbb{R}^n)} \\ &\lesssim 2^{j(m-m_k(p))} \sum_{|\alpha| < M} \left\| \sum_{\ell=0}^N 2^{-\ell \varepsilon |\alpha|} \psi_j(D) (T_{\sigma_{\alpha,\ell}}^\varphi)^* f_\ell \right\|_{L^1(\mathbb{R}^n)} \\ &\quad + 2^{j(m-m_k(p))} \left\| \sum_{\ell=0}^\infty 2^{-\ell \varepsilon M} \psi_j(D) (T_{r_\ell}^\varphi)^* f_\ell \right\|_{L^1(\mathbb{R}^n)} \\ &=: \text{I} + \text{II}, \end{aligned} \tag{72}$$

where $N < \infty$ by the properties of the Littlewood–Paley sums. We consider the main terms I above. Observe that using the seminorm estimate (18) for $\sigma_{\alpha,\ell}$, we can claim that

$$\begin{aligned} \left\| \sum_{\ell=0}^N 2^{-\ell \varepsilon |\alpha|} \psi_j(D) (T_{\sigma_{\alpha,\ell}}^\varphi)^* f_\ell \right\|_{L^1(\mathbb{R}^n)} &\lesssim \sum_{\ell=0}^N \left\| 2^{-\ell \varepsilon |\alpha|} \psi_j(D) (T_{\sigma_{\alpha,\ell}}^\varphi)^* f_\ell \right\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \sum_{\ell=0}^N \|f_\ell\|_{h^1(\mathbb{R}^n)} = \sum_{\ell=0}^N \left\| \left(\sum_{j=0}^\infty |\psi_j(D) \Psi_\ell(D) f|^2 \right)^{1/2} \right\|_{L^1(\mathbb{R}^n)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

To prove this claim, we first observe that $\psi_j(D)$ maps L^1 into itself (with a norm independent of j). Then we use Proposition 12.1 in i) – iii) and Theorem 10.1 in case iv) to obtain the desired result.

For the remainder term II we use the representation

$$(T_{r_\ell}^\varphi)^* f(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy.$$

Then in case *i) – –iii)*, integration by parts yields

$$\begin{aligned} |K(x, y)| &= \left| \int_{\mathbb{R}^n} e^{-i(x-y)\cdot\xi} e^{i\varphi(y,\xi)-iy\cdot\xi} \overline{r_\ell(y, \xi)} d\xi \right| \\ &\lesssim \langle x - y \rangle^{-2N} \int_{\mathbb{R}^n} \left| (1 + i(x - y) \cdot \nabla_\xi)^N \left(e^{i\varphi(y,\xi)-iy\cdot\xi} \overline{r_\ell(y, \xi)} \right) \right| d\xi \\ &\lesssim \langle x - y \rangle^{-2N} \int_{|\xi| \geq 1} \sum_{|\alpha_1| + \dots + |\alpha_N| \leq N} \left| \partial_\xi^{\alpha_N} \overline{r_\ell(y, \xi)} \right| \\ &\quad \times \prod_{\nu=1}^{N-1} C_\alpha \left| (x - y)^{\alpha_\nu} \partial_\xi^{\alpha_\nu} (\varphi(y, \xi) - y \cdot \xi) \right| d\xi. \end{aligned}$$

Now, since by estimate (19) we have that $r_\ell \in S_{0,0}^{m-(1/2-\varepsilon)M}(\mathbb{R}^n)$, choosing M large enough, the \mathfrak{F}^k -condition yields that

$$|K(x, y)| \lesssim \langle x - y \rangle^{-N},$$

for any $N > 0$. In case *iv)*, we estimate

$$\begin{aligned} |K(x, y)| &= \left| \int_{\mathbb{R}^n} e^{i(x-\nabla_\xi\varphi(y,0))\cdot\xi} e^{i\nabla_\xi\varphi(y,0)\cdot\xi-i\varphi(y,\xi)} \overline{r_\ell(y, \xi)} d\xi \right| \\ &\lesssim \langle x - \nabla_\xi\varphi(y, 0) \rangle^{-2N} \int_{\mathbb{R}^n} \left| (1 + i(x - \nabla_\xi\varphi(y, 0)) \cdot \nabla_\xi)^N \right. \\ &\quad \left. \times e^{i\varphi(y,\xi)-i\nabla_\xi\varphi(y,0)\cdot\xi} \overline{r_\ell(y, \xi)} \right| d\xi \\ &\lesssim \langle x - \nabla_\xi\varphi(y, 0) \rangle^{-2N} \int_{\mathbb{R}^n} \sum_{|\alpha_1| + \dots + |\alpha_N| \leq N} \left| \partial_\xi^{\alpha_N} \overline{r_\ell(y, \xi)} \right| \\ &\quad \times \prod_{\nu=1}^{N-1} \left| (x - \nabla_\xi\varphi(y, 0))^{\alpha_\nu} \partial_\xi^{\alpha_\nu} (\varphi(y, \xi) - \nabla_\xi\varphi(y, 0) \cdot \xi) \right| d\xi. \end{aligned}$$

Here, we observe that

$$|\partial_\xi^{\alpha_\nu} (\varphi(y, \xi) - \nabla_\xi\varphi(y, 0) \cdot \xi)| = \begin{cases} O(1), & |\alpha_\nu| \geq 2, \\ O(|\xi|), & |\alpha_\nu| = 1, \end{cases}$$

where we have used the fact that when $|\alpha_\nu| \geq 2$, then equation (15) yields the first estimate and when $|\alpha_\nu| = 1$, then the mean-value theorem yields the second. Therefore, once again choosing M large enough, we have for any $N > 0$ that

$$|K(x, y)| \lesssim \langle x - \nabla_\xi\varphi(y, 0) \rangle^{-N}$$

and hence

$$\|(T_{r_\ell}^\varphi)^* f\|_{L^1(\mathbb{R}^n)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}.$$

Now, we estimate the remainder term of equation (72). It is bounded by

$$\sum_{\ell=0}^{\infty} 2^{-\ell \varepsilon M} \|\psi_j(D)(T_{r_\ell}^\varphi)^* f_\ell\|_{L^1(\mathbb{R}^n)} \lesssim \sum_{\ell=0}^{\infty} 2^{-\ell \varepsilon M} \|f\|_{L^1(\mathbb{R}^n)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}.$$

Therefore,

$$\|T_j^* f\|_{L^1(\mathbb{R}^n)} \lesssim \|f\|_{L^1(\mathbb{R}^n)},$$

and a duality argument yields

$$\|T_j f\|_{L^\infty(\mathbb{R}^n)} = \|T_j \Psi_j(D) f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|\Psi_j(D) f\|_{L^\infty(\mathbb{R}^n)}.$$

Now, if one removes the requirement on the frequency support of $a(x, \xi)$ and adds either equation (32) or (33) in $i) - ii)$, then Lemma 6.3 $i)$ yields $L^\infty - bmo$ boundedness and the rest of the argument proceeds as before. □

Now, we are finally ready to prove the regularity of oscillatory integral operators on Besov–Lipschitz spaces.

Proof of Theorems 3.2 and 3.5, part $i)$. For the low- and middle-frequency portions of the operator, we just use Lemma 6.3 and Lemma 7.1 parts $ii)$ and $iii)$. Observe that $\partial_x^\beta \varphi(x, \xi) \in L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})$ is a consequence of the LF(μ)-condition (14). Thus, from now on we concentrate on the high-frequency portion of the operator. We divide the proof into three steps. In Step 1, we invoke a composition formula which yields a sum of two terms (a main term and a remainder term) that need to be analysed separately and conclude that the main term is L^p -bounded (in the sense of Lemma 12.2). In Step 2, we show $B_{p,q}^s - L^p$ boundedness for the remainder term, and in Step 3, we complete the proof by deducing the $B_{p,q}^{s+m-m_k(p)} - B_{p,q}^s$ boundedness. In Step 4, we deal with the case when the phase function is smooth everywhere in Theorem 3.5.

Step 1 – A composition formula and boundedness of the main term

In the definition of the Besov–Lipschitz norm, the expression $\psi_j(D)T_a^\varphi f$ plays a central role. To obtain favourable estimates for $\psi_j(D)T_a^\varphi f$ we use formula (17) with M chosen large enough, which states

$$\psi(2^{-j}D)T_a^\varphi = \sum_{|\alpha| \leq M-1} \frac{2^{-j\varepsilon|\alpha|}}{\alpha!} T_{\sigma_{\alpha,j}}^\varphi + 2^{-j\varepsilon M} T_{r_j}^\varphi. \tag{73}$$

From Lemma 12.2, we have, after a change of variables, that

$$\|T_{\sigma_{\alpha,j}}^\varphi f\|_{L^p(\mathbb{R}^n)} \lesssim 2^{j(m-m_k(p))} \|\Psi_j(D) f\|_{L^p(\mathbb{R}^n)}. \tag{74}$$

Step 2 – The remainder term

-Paley pieces as follows:

$$T_{r_j}^\varphi f(x) = \sum_{\ell=0}^{\infty} \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} r_j(x, \xi) \psi_\ell(\xi) \widehat{f}(\xi) \, d\xi =: \sum_{\ell=0}^{\infty} T_{r_{j,\ell}}^\varphi f(x),$$

where the ψ_ℓ 's are given in Definition 2.1. We use the fact that for $0 < p \leq \infty$,

$$\|f + g\|_{L^p(\mathbb{R}^n)} \leq 2^{C_p} (\|f\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)}), \tag{75}$$

where $C_p := \max(0, 1/p - 1)$. Now, Fatou’s lemma and iteration of equation (75) yield that

$$\begin{aligned} \|T_{r_j}^\varphi f\|_{L^p(\mathbb{R}^n)} &= \left\| \sum_{\ell=0}^\infty T_{r_j,\ell}^\varphi f \right\|_{L^p(\mathbb{R}^n)} \leq \liminf_{N \rightarrow \infty} \left\| \sum_{\ell=0}^N T_{r_j,\ell}^\varphi f \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \liminf_{N \rightarrow \infty} \sum_{\ell=0}^N 2^{\ell C_p} \|T_{r_j,\ell}^\varphi f\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{\ell=0}^\infty 2^{\ell C_p} \|T_{r_j,\ell}^\varphi f\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

where the hidden constant in the last estimate depends only on p . Therefore, applying Lemma 12.2 with $m - (1/2 - \varepsilon)M$ instead of m (recall that r_j vanishes for all ξ , for which a vanishes), we obtain

$$\begin{aligned} \|T_{r_j}^\varphi f\|_{L^p(\mathbb{R}^n)} &\lesssim \sum_{\ell=0}^\infty 2^{\ell C_p} \|T_{r_j,\ell}^\varphi f\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \sum_{\ell=0}^\infty 2^{\ell(C_p+m-m_k(p)-(1/2-\varepsilon)M)} \|\Psi_\ell(D)f\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{76}$$

Note that the estimate (76) is uniform in j . Now, we claim that

$$T_{r_j}^\varphi : B_{p,q}^s(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n). \tag{77}$$

To see this, we shall analyse the cases $0 < q < 1$ and $1 \leq q \leq \infty$ separately. Starting with the former, we have

$$\begin{aligned} \|T_{r_j}^\varphi f\|_{L^p(\mathbb{R}^n)} &\lesssim \sum_{\ell=0}^\infty 2^{\ell(C_p+m-m_k(p)-(1/2-\varepsilon)M)} \|\Psi_\ell(D)f\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \sum_{\ell=0}^\infty 2^{\ell(s+m-m_k(p))} \|\Psi_\ell(D)f\|_{L^p(\mathbb{R}^n)} \\ &\leq \left(\sum_{\ell=0}^\infty 2^{\ell q(s+m-m_k(p))} \|\Psi_\ell(D)f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} = \|f\|_{B_{p,q}^{s+m-m_k(p)}(\mathbb{R}^n)}, \end{aligned}$$

where we used (76) for the first inequality and that M is large enough for the second. For $1 \leq q \leq \infty$, Hölder’s inequality in the sum over ℓ and picking M large enough yield

$$\begin{aligned} \|T_{r_j}^\varphi f\|_{L^p(\mathbb{R}^n)} &\lesssim \sum_{\ell=0}^\infty 2^{\ell(C_p+m-m_k(p)-(1/2-\varepsilon)M)} \|\Psi_\ell(D)f\|_{L^p(\mathbb{R}^n)} \\ &= \sum_{\ell=0}^\infty 2^{\ell(-s+C_p-(1/2-\varepsilon)M)} \left(2^{\ell(s+m-m_k(p))} \|\Psi_\ell(D)f\|_{L^p(\mathbb{R}^n)} \right) \\ &\lesssim \left(\sum_{\ell=0}^\infty 2^{\ell q'(-s+C_p-(1/2-\varepsilon)M)} \right)^{1/q'} \left(\sum_{\ell=0}^\infty 2^{\ell q(s+m-m_k(p))} \|\Psi_\ell(D)f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \\ &\lesssim \|f\|_{B_{p,q}^{s+m-m_k(p)}(\mathbb{R}^n)}, \end{aligned}$$

which implies equation (77). Note that the calculation above also holds for $q = \infty$ with the usual interpretation of Hölder’s inequality.

Step 3 – The $B_{p,q}^{s+m-m_k(p)} - B_{p,q}^s$ boundedness

The results in (74) and (77) yield that

$$\begin{aligned} \|T_a^\varphi f\|_{B_{p,q}^s(\mathbb{R}^n)} &= \left(\sum_{j=0}^\infty (2^{js} \|\psi(2^{-j}D)T_a^\varphi f\|_{L^p(\mathbb{R}^n)})^q \right)^{1/q} \\ &\lesssim \left(\sum_{j=0}^\infty \left(\sum_{|\alpha| \leq M-1} 2^{js} \|T_{\sigma_{\alpha,j}} f\|_{L^p(\mathbb{R}^n)} + 2^{-j(\varepsilon M-s)} \|T_{r_j} f\|_{L^p(\mathbb{R}^n)} \right)^q \right)^{1/q} \\ &\lesssim \left(\sum_{j=0}^\infty \left(2^{j(s+m-m_k(p))} \|\Psi_j(D)f\|_{L^p(\mathbb{R}^n)} + 2^{-j(\varepsilon M-s)} \|f\|_{B_{p,q}^{s+m-m_k(p)}(\mathbb{R}^n)} \right)^q \right)^{1/q} \\ &\lesssim \left(\sum_{j=0}^\infty 2^{jq(s+m-m_k(p))} \|\Psi_j(D)f\|_{L^p(\mathbb{R}^n)}^q + \sum_{j=0}^\infty 2^{-jq(\varepsilon M-s)} \|f\|_{B_{p,q}^{s+m-m_k(p)}(\mathbb{R}^n)}^q \right)^{1/q} \\ &\lesssim \|f\|_{B_{p,q}^{s+m-m_k(p)}(\mathbb{R}^n)}. \end{aligned}$$

Step 4 – The smooth case

For the smooth version, we don't need separate proofs for low, middle and high frequencies. Note that $T_a^\varphi = e^{i\varphi(x,0)}T_a^{\tilde{\varphi}}$, where $\tilde{\varphi}(x,0) = 0$. Therefore, using the condition $|\nabla_x \varphi(x,0)| \in L^\infty(\mathbb{R}^n)$ and the L^2 -condition (for all x and ξ), we have by equation (9) that $\|T_a^\varphi f\|_{B_{p,q}^s} \lesssim \|T_a^{\tilde{\varphi}} f\|_{B_{p,q}^s}$. Now, apply Lemma 12.2 iii) and Lemma 4.1 iii) to $T_a^{\tilde{\varphi}}$ and continue as above and the proof is complete. \square

We can also establish the boundedness of Schrödinger integral operators on Besov–Lipschitz spaces.

Proof of Theorems 3.3 and 3.6, part i). Theorem 3.3 is a special case of Theorem 3.6 so it is enough to consider the latter. This is identical to Step 4 in the previous proof, except that Lemma 12.2 iv) is used instead. \square

13. Regularity on Triebel–Lizorkin spaces

In this section, we prove various Triebel–Lizorkin regularity results as corollaries of the previous Besov–Lipschitz results. We observe that, if we do not let the order m of the amplitude to go all the way to the endpoint, then we have Triebel–Lizorkin boundedness for all p 's and q 's.

Proof of Theorems 3.2, 3.3, 3.5 and 3.6, part ii). Using the embedding (8), equality (7) and part i) of the theorems, we have that

$$\begin{aligned} \|T_a^\varphi f\|_{F_{p,q}^s(\mathbb{R}^n)} &\lesssim \|T_a^\varphi f\|_{F_{p,p}^{s+\varepsilon/2}(\mathbb{R}^n)} \lesssim \|f\|_{F_{p,p}^{s+m-m_k(p)+\varepsilon/2}(\mathbb{R}^n)} \\ &\lesssim \|f\|_{F_{p,q}^{s+m-m_k(p)+\varepsilon}(\mathbb{R}^n)}. \end{aligned}$$

\square

Proof of Theorems 3.2, 3.3, 3.5 and 3.6, parts iii) and iv). We divide the proof into different steps.

Step 1 – The diagonal $p = q$

The theorem is true for the diagonal $p = q$ because of the Besov–Lipschitz results in Theorems 3.2–3.5, 3.6, part i) and the fact that $F_{p,p}^s(\mathbb{R}^n) = B_{p,p}^s(\mathbb{R}^n)$.

Step 2 – The $h^p - h^p$ boundedness

For Theorems 3.2 and 3.5 iii) – iv), we split the proof into low-, middle- and high-frequency parts. The low- and middle-frequency parts were treated in Lemma 6.3 and Lemma 7.1. Observe that $\partial_x^\beta \varphi(x, \xi) \in L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})$ is a consequence of the LF(μ)-condition (14).

For the high-frequency cases, $a_H(x, \xi) := (1 - \psi_0(\xi/R)) a(x, \xi)$, recall that Proposition 12.1 yields the $h^p - L^p$ boundedness, which we will now lift to the $h^p - h^p$ level.

To this end, it is enough to show that if $b(D)$ is a Fourier multiplier with $b \in S^0_{1,0}(\mathbb{R}^n)$, and t a parameter in $(0, 1]$, then the composition $b(tD)T^\varphi_{a_H}$ is $h^p - L^p$ bounded with a norm that doesn't depend on t . But this is indeed the case, since using the composition formula (16) with $M = 1$ we see that

$$b(tD)T^\varphi_{a_H} = T^\varphi_{a_H b(t\cdot)} + t^\varepsilon T^\varphi_r,$$

where $|\partial^\alpha_\xi \partial^\beta_x r(t, x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m_k(p) - (1/2 - \varepsilon)}$. Now, since $a_H b(t\cdot) \in S^{m_k(p)}_{0,0}(\mathbb{R}^n)$ uniformly in $t \in (0, 1]$, Proposition 12.1 yields the $h^p - L^p$ boundedness of $b(tD)T^\varphi_{a_H}$ with a norm that is independent of t , and the proof for the oscillatory integral operators is concluded.

For Schrödinger integral operators (Theorems 3.3 and 3.6 *iii*) – iv) and the smooth version of Theorem 3.5, there is no need to divide the amplitude different frequency portions, and we once again note that $T^\varphi_a = e^{i\varphi(x,0)} T^\varphi_{\tilde{a}}$ where $\tilde{a}(x, 0) = 0$. Therefore, using the condition $|\nabla_x \varphi(x, 0)| \in L^\infty(\mathbb{R}^n)$ and condition (15), we see by equation (9) and the definition of the local Hardy space as a Triebel-Lizorkin space that $\|T^\varphi_a f\|_{h^p} \lesssim \|T^\varphi_{\tilde{a}} f\|_{h^p}$. Now, using Lemma 4.1, Theorem 3.11 and Theorem 10.1, we can proceed as above to show the $h^p - L^p$ boundedness of $b(tD)T^\varphi_a$ (for $0 < p < \infty$) with a norm that is independent of t , and the proof for the Schrödinger integral operators is also concluded.

Step 3 – Boosting $F^0_{p,2}$ -boundedness to arbitrary regularity

Once again for oscillatory integral operators, we decompose into low-, middle- and high-frequency portions. For the low- and middle-frequency parts, we apply Lemma 6.3 and Lemma 7.1. For the high-frequency parts, we proceed as follows. Write $a_H(x, \xi) = \sigma(x, \xi) \langle \xi \rangle^{m - m_k(p)}$, with $\sigma(x, \xi) \in S^{m_k(p)}_{0,0}(\mathbb{R}^n)$, and use Theorem 3.11 to conclude that $(1 - \Delta)^{s/2} T^\varphi_\sigma (1 - \Delta)^{-s/2}$ is the same kind of oscillatory integral operator as T^φ_σ . Therefore, by Step 2 above

$$\begin{aligned} \|T^\varphi_{a_H} f\|_{F^s_{p,2}(\mathbb{R}^n)} &= \left\| (1 - \Delta)^{s/2} T^\varphi_\sigma (1 - \Delta)^{-s/2} (1 - \Delta)^{(m - m_k(p) + s)/2} f \right\|_{F^0_{p,2}(\mathbb{R}^n)} \\ &\lesssim \left\| (1 - \Delta)^{(m - m_k(p) + s)/2} f \right\|_{F^0_{p,2}(\mathbb{R}^n)} = \|f\|_{F^{s + m - m_k(p)}_{p,2}(\mathbb{R}^n)}. \end{aligned}$$

Now, for the Schrödinger integral operator case and the smooth phase function case, there is no need to decompose the operator into high- and low-frequency cases; instead, we just use equation (9) to once again reduce to the case of $T^\varphi_{a_H}$ for which it is true, thanks to Lemma 4.1, that $(1 - \Delta)^{s/2} T^\varphi_\sigma (1 - \Delta)^{-s/2}$ is the same kind of oscillatory integral operator as T^φ_σ . Therefore, we can once again run the same argument as above and achieve the desired result.

Step 4 – Interpolation

By interpolation in q , we get the desired result (see Figure 1). Note that one cannot interpolate between Triebel–Lizorkin spaces when $p = \infty$. □

Proof of Theorem 3.9, part i). Write $a(x, \xi) = \sigma(x, \xi) \langle \xi \rangle^{m_2(p)}$ for $\sigma \in S^0_{0,0}(\mathbb{R}^n)$, and let $b(D)$ be any pseudodifferential operator of order zero. Then [1, Theorem 6.1] asserts that $[b(tD), T^\varphi_\sigma] = T^\varphi_r$, for some $r(t, x, \xi) \in S^0_{0,0}(\mathbb{R}^n)$ uniformly in t if $t \in (0, 1]$. Therefore, we have that

$$\begin{aligned} b(tD)T^\varphi_a f(x) &= b(tD)T^\varphi_\sigma (1 - \Delta)^{m_2(p)/2} f(x) \\ &= T^\varphi_\sigma b(tD) (1 - \Delta)^{m_2(p)/2} f(x) + T^\varphi_r (1 - \Delta)^{m_2(p)/2} f(x), \end{aligned}$$

with $r(x, \xi, t) \in S^0_{0,0}(\mathbb{R}^n)$ uniformly in $t \in (0, 1]$. Then since $b \in S^0_{1,0}(\mathbb{R}^n)$, we have that $\sigma(x, \xi) b(t\xi) \langle \xi \rangle^{m_2(p)} \in S^{m_2(p)}_{0,0}(\mathbb{R}^n)$ uniformly in $t \in (0, 1]$ and also $r(t, x, \xi) \langle \xi \rangle^{m_2(p)} \in S^0_{0,0}(\mathbb{R}^n)$ uniformly in $t \in (0, 1]$. Therefore, we can apply Theorem 10.1 to conclude the desired result. □

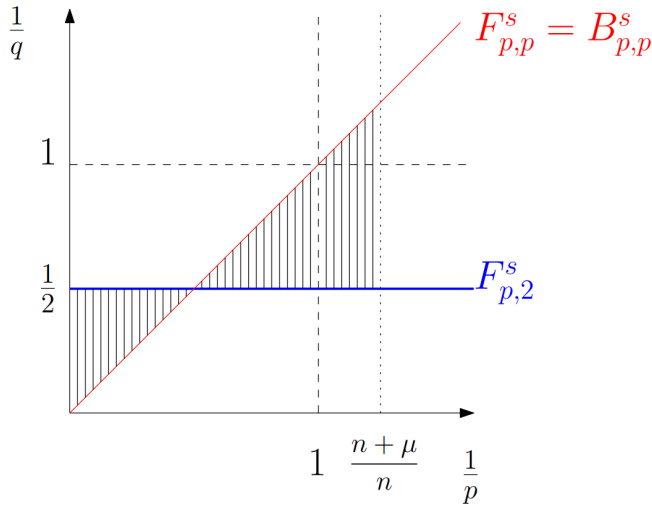


Figure 1. Boundedness and interpolation scheme in Triebel–Lizorkin scale.

14. Sharpness of the results

Let us start from a naive approach to the regularity problem of oscillatory integral operators, by considering a concrete case of an oscillatory integral operator, namely

$$Tf(x) := \int_{\mathbb{R}^n} |\xi|^m (1 - \psi_0(\xi)) e^{ix \cdot \xi + i|\xi|^k} \widehat{f}(\xi) \, d\xi,$$

with $k > 1$ and ψ_0 as in Definition 2.1.

Now, if we look upon T as a Ψ DO with symbol

$$a_{k,m}(\xi) := e^{i|\xi|^k} (1 - \psi_0(\xi)) |\xi|^m,$$

then we see that this symbol does not belong to any Hörmander class $S_{\rho,\delta}^m(\mathbb{R}^n)$ for any $\rho \in [0, 1]$, since $|\partial^\alpha a_{k,m}(\xi)| \lesssim \langle \xi \rangle^{m+(k-1)|\alpha|}$. Therefore, the appeal to the boundedness theory of pseudodifferential operators fails in a rather drastic way.

To understand the significance of the order $m_k(p) = -kn|1/p - 1/2|$, let

$$K_{k,m}(x) := \int_{\mathbb{R}^n} (1 - \psi_0(\xi)) |\xi|^m e^{ix \cdot \xi + i|\xi|^k} \, d\xi.$$

Let $1 < p < \infty$ and

$$f_\lambda(x) := \int_{\mathbb{R}^n} (1 - \psi_0(\xi)) |\xi|^{-\lambda} e^{ix \cdot \xi} \, d\xi.$$

It was shown in [23] that $f_\lambda \in L^p(\mathbb{R}^n)$ iff $-\lambda < n/p - n$. Now, if $m > m_k(p)$ and if λ is such that $-\lambda < n/p - n$ and $-m + \lambda - n + nk/2 < n(k - 1)/p$, then $f_\lambda \in L^p(\mathbb{R}^n)$, but $Tf_\lambda(x) = (K_{k,m} * f_\lambda)(x) \notin L^p(\mathbb{R}^n)$; see [23, p. 301, (I-ii)].

This shows that, if we regard the operator T above as an oscillatory integral operator with the amplitude $(1 - \psi_0(\xi)) |\xi|^m \in S_{1,0}^m(\mathbb{R}^n)$ and the phase function $x \cdot \xi + |\xi|^k$, then one cannot in general expect any L^p -boundedness, unless $m \leq m_k(p)$ and thus this order of the amplitude is sharp for the L^p -regularity of T .

Acknowledgments. The authors are grateful to Jorge Betancor for reading through the first draft of the manuscript and for his comments. We are indebted to Malabika Pramanik for many useful suggestions that have led to an overall improvement of the presentation of the paper. Finally, we would also like to thank the referee for useful suggestions.

Competing interest. The authors have no competing interest to declare.

Financial support. A. J. Castro is supported by the Nazarbayev University Faculty Development Competitive Research Grants Program, grant number 11022021FD2921. During the initial phases of development of this paper, W. Staubach was partially supported by a grant from the Crafoord foundation and by a grant from G. S. Magnusons fond, grant number MG2015-0077.

References

- [1] K. Asada and D. Fujiwara, ‘On some oscillatory integral transformations in $L^2(\mathbb{R}^n)$ ’, *Japan. J. Math. (N.S.)* **4** (1978), 299–361.
- [2] R. M. Beals, ‘ L^p boundedness of Fourier integral operators’, *Mem. Amer. Math. Soc.* **38** (1982), viii+57.
- [3] A. J. Castro, S. Rodríguez-López and W. Staubach, ‘Transference of local to global L^2 maximal estimates for dispersive partial differential equations’, *J. Math. Anal. Appl.* **471** (2019), 411–422.
- [4] R. R. Coifman and G. Weiss, ‘Extensions of Hardy spaces and their use in analysis’, *Bull. Amer. Math. Soc.* **83** (1977), 569–645.
- [5] E. Cordero, K. Gröchenig, F. Nicola and L. Rodino, ‘Wiener algebras of Fourier integral operators’, *J. Math. Pures Appl.* (9) **99** (2013), 219–233.
- [6] E. Cordero, F. Nicola and L. Rodino, ‘Time-frequency analysis of Fourier integral operators’, *Commun. Pure Appl. Anal.* **9** (2010), 1–21.
- [7] E. Cordero, F. Nicola and L. Rodino, ‘Gabor analysis for Schrödinger equations and propagation of singularities’, in *Recent Trends in Operator Theory and Partial Differential Equations, Oper. Theory Adv. Appl.*, vol. 258 (Birkhäuser/Springer, Cham, 2017), 257–274.
- [8] P. D’Ancona and F. Nicola, ‘Sharp L^p estimates for Schrödinger groups’, *Rev. Mat. Iberoam.* **32** (2016), 1019–1038.
- [9] D. Dos Santos Ferreira and W. Staubach, ‘Global and local regularity of Fourier integral operators on weighted and unweighted spaces’, *Mem. Amer. Math. Soc.* **229** (2014), xiv+65.
- [10] G. I. Eskin, ‘Degenerate elliptic pseudodifferential equations of principal type’, *Mat. Sb. (N.S.)* **82**(124) (1970), 585–628.
- [11] C. Fefferman, ‘A note on spherical summation multipliers’, *Israel J. Math.* **15** (1973), 44–52.
- [12] C. Fefferman and E. M. Stein, ‘ H^p spaces of several variables’, *Acta Math.* **129** (1972), 137–193.
- [13] D. Fujiwara, ‘On the boundedness of integral transformations with highly oscillatory kernels’, *Proc. Japan Acad.* **51** (1975), 96–99.
- [14] D. Goldberg, ‘A local version of real Hardy spaces’, *Duke Math. J.* **46** (1979), 27–42.
- [15] B. Helffer, *Théorie spectrale pour des opérateurs globalement elliptiques*, Astérisque, vol. 112 (Société Mathématique de France, Paris, 1984).
- [16] B. Helffer and D. Robert, *Comportement asymptotique précise du spectre d’opérateurs globalement elliptiques dans \mathbb{R}^n . Goulaouic–Meyer–Schwartz Seminar, Exp. No. II, 23, École Polytech., Palaiseau, (École Polytech., Palaiseau, 1981)*.
- [17] L. Hörmander, *Pseudo-Differential Operators and Hypoelliptic Equations. Singular Integrals: (Proc. Sympos. Pure Math., Vol. X)* (Amer. Math. Soc., Providence, RI, 1967), 138–183.
- [18] L. Hörmander, ‘Fourier integral operators. I’, *Acta Math.* **127** (1971), 79–183.
- [19] H. Ishii, ‘On some Fourier multipliers and partial differential equations’, *Math. Japon.* **19** (1974), 139–163.
- [20] A. Israelsson, S. Rodríguez-López and W. Staubach, ‘Local and global estimates for hyperbolic equations in Besov–Lipschitz and Triebel–Lizorkin spaces’, *Analysis & PDE* **14**(1) (2021), 1–44.
- [21] A. Israelsson, T. Mattsson and W. Staubach, ‘Boundedness of Fourier integral operators on classical function spaces’, *J. Funct. Anal.* **285**(5) (2023).
- [22] A. Miyachi, ‘On some estimates for the wave equation in L^p and H^p ’, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), 331–354.
- [23] A. Israelsson, T. Mattsson and W. Staubach, ‘On some singular Fourier multipliers’, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **28** (1981), 267–315.
- [24] J. Peetre, ‘On spaces of Triebel–Lizorkin type’, *Ark. Mat.* **13** (1975), 123–130.
- [25] J. C. Peral, ‘ L^p estimates for the wave equation’, *J. Funct. Anal.* **36** (1980), 114–145.
- [26] S. Rodríguez-López, D. Rule and W. Staubach, ‘A Seeger–Sogge–Stein theorem for bilinear Fourier integral operators’, *Adv. Math.* **264** (2014), 1–54.
- [27] S. Rodríguez-López, D. Rule and W. Staubach, ‘On the boundedness of certain bilinear oscillatory integral operators’, *Trans. Amer. Math. Soc.* **367** (2015), 6971–6995.
- [28] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, De Gruyter Series in Nonlinear Analysis and Applications*, vol. 3 (Walter de Gruyter & Co., Berlin, 1996).

- [29] A. Seeger, C. D. Sogge and E. M. Stein, 'Regularity properties of Fourier integral operators', *Ann. of Math. (2)* **134** (1991), 231–251.
- [30] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, *Princeton Mathematical Series*, vol. 43 (Princeton University Press, Princeton, NJ, 1993). With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.
- [31] H. Triebel, *Theory of Function Spaces, Mathematik und ihre Anwendungen in Physik und Technik [Mathematics and its Applications in Physics and Technology]*, vol. 38 (Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1983).