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# FIRST PASSAGE PERCOLATION ON RANDOM GEOMETRIC GRAPHS AND AN APPLICATION TO SHORTEST-PATH TREES

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**Abstract**

We consider Euclidean first passage percolation on a large family of connected random geometric graphs in the $d$-dimensional Euclidean space encompassing various well-known models from stochastic geometry. In particular, we establish a strong linear growth property for shortest-path lengths on random geometric graphs which are generated by point processes. We consider the event that the growth of shortest-path lengths between two (end) points of the path does not admit a linear upper bound. Our linear growth property implies that the probability of this event tends to zero sub-exponentially fast if the direct (Euclidean) distance between the endpoints tends to infinity. Besides, for a wide class of stationary and isotropic random geometric graphs, our linear growth property implies a shape theorem for the Euclidean first passage model defined by such random geometric graphs. Finally, this shape theorem can be used to investigate a problem which is considered in structural analysis of fixed-access telecommunication networks, where we determine the limiting distribution of the length of the longest branch in the shortest-path tree extracted from a typical segment system if the intensity of network stations converges to 0.

*Keywords:* First passage percolation; shape theorem; shortest-path tree; longest shortest path; random geometric graph

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## 1. Introduction

We investigate a first passage percolation model on a large class of connected, stationary, and isotropic random geometric graphs, where the edge-passage times are given by the Euclidean lengths of the edges. The classical first passage percolation model due to Hammersley and Welsh [15] considers shortest-path lengths on a randomly weighted lattice, where the edge weights form a sequence of independent and identically distributed (i.i.d.) nonnegative random variables. More recently, the analysis of the asymptotic behaviour of such shortest-path lengths has been extended to geometrically irregular random geometric graphs, such as Poisson–Delaunay graphs; see Figure 1 for an illustration. While [28], [32], and [33] considered the classical scenario of i.i.d. edge weights, such connected random geometric graphs give rise also

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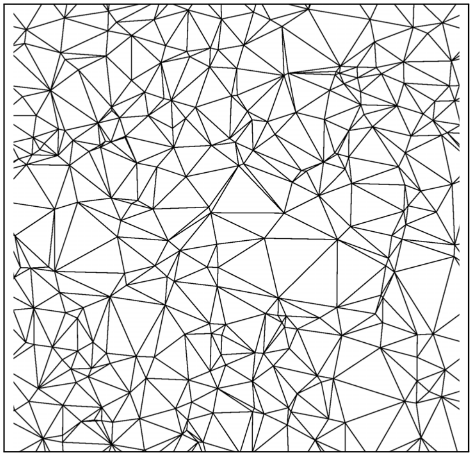

FIGURE 1: Realisation of a Delaunay graph generated by a homogeneous Poisson point process (cutout).

to another natural first passage percolation model, where the edge weights are determined by the Euclidean length of the edge [1]–[3], [5].

In this paper, building on the work of [1], [3], and [12], we show that not only the Poisson–Delaunay graph, but in fact a considerably larger class of connected random geometric graphs satisfies a strong linear growth property. To be more precise, considering the event that the growth of shortest-path lengths between two (end) points does not admit a linear upper bound in the Euclidean distance between the endpoints, we show in Theorem 1 that its probability tends to zero at least sub-exponentially fast as this distance tends to $\infty$.

We continue to elaborate a variety of further implications of the strong linear growth property stated in Theorem 1. First, for a rather general class of stationary and isotropic random geometric graphs in $\mathbb{R}^d$, we show that this growth property implies a shape theorem for the Euclidean first passage model defined by such random geometric graphs (Theorem 2). In particular, we extend the classical shape theorem, which has been derived in [19] for first passage percolation on the lattice $\mathbb{Z}^d$ (with i.i.d. edge weights), to a framework involving geometrically complex random graphs with nonindependent edge lengths. Furthermore, in a two-dimensional setting, the growth property stated in Theorem 1 can be used to deduce the almost sure (a.s.) boundedness of cells defined by planar random geometric graphs (Theorem 3). In this way, the a.s. boundedness of cells of the creek-crossing graphs $(G_n)_{n\geq 2}$ introduced in [16] can be shown.

We also show how Theorem 2 can be used to investigate a problem which is considered in structural analysis of (wired) fixed-access telecommunication networks. In those networks, access points are located along the roads of urban or rural regions and each access point is dedicated to providing a service to all users in some bounded region of the plane, which is referred to as its serving zone. Physical links from network users to access points are deployed along the shortest Euclidean path in the road graph, thus giving rise to a shortest-path connection tree representing the subgraph inside a serving zone; see, e.g. [14], [34]. Cost estimation of the telecommunication network requires knowledge of the structural properties of this tree. In this paper we show how our shape theorem stated in Theorem 2 (in conjunction with an asymptotic distributional result due to Calka [8]) can be used to determine the limiting distribution of the length of the longest branch in the shortest-path tree associated with a randomly chosen serving zone as the intensity of access points converges to 0 (Theorem 4).

This paper is organised as follows. First, in Section 2 we state our main results and introduce some general conditions on the considered random geometric graphs and the underlying point processes, which are used later in the proofs of our results. In Section 3 we show that these conditions are satisfied for various well-known classes of random geometric graphs and point processes from stochastic geometry. Then in Section 4 we provide a proof for the strong linear growth property stated in Theorem 1. Section 5 is devoted to the proof of the shape theorem and the boundedness of cells stated in Theorems 2 and 3, respectively. Finally, we conclude the paper with Section 6, where a proof of Theorem 4 is presented.

## 2. Main results

### 2.1. Random geometric graphs based on point processes

In classical models of first passage percolation, one considers shortest-path lengths in independently marked lattices, but more recently also first passage percolation on the Poisson–Delaunay graph has received considerable attention. While [28], [32], and [33] consider the scenario of independently marked edges, in [1] a spatially dependent marking using the Euclidean edge length is investigated not only for the Poisson–Delaunay graph, but in fact for a more general class of connected random geometric graphs whose vertices are given by a homogeneous Poisson point process in $\mathbb{R}^2$. However, even in two dimensions important examples of connected random geometric graphs, such as the Poisson–Voronoi tessellation, are not based on a Poisson point process of vertices. Therefore, in this paper, we state our results using the following general notion of random geometric graphs.

Denote by $\mathbb{M}$ the family of all line segments in $\mathbb{R}^d$. This family forms a topological space in the Fell topology [31] and we denote by $\mathcal{M}$ the Borel $\sigma$-algebra on $\mathbb{M}$ generated by this topology. We write $\mathbb{G}$ for the family of all simple counting measures $\varphi$ on $\mathbb{R}^d \times \mathbb{M}$ such that $\varphi(B \times \mathbb{M})$ is finite for every bounded Borel set $B \subset \mathbb{R}^d$. Furthermore, we denote by $\mathcal{G}$ the $\sigma$-algebra on $\mathbb{G}$ that is generated by the evaluation maps $\varphi \mapsto \varphi(B \times M)$, where $B \subset \mathbb{R}^d$ is a Borel set in $\mathbb{R}^d$ and $M \in \mathcal{M}$. Random variables with values in $\mathbb{G}$ are called *random segment processes* or *random geometric graphs*. It will be convenient to identify elements $\varphi \in \mathbb{G}$ with their support, so we can represent $\varphi$ as $\varphi = \{(x_n, u_n)\}_{n\geq 1}$ for some $x_n \in \mathbb{R}^d$ and $u_n \in \mathbb{M}$.

In order to deal with a large variety of commonly used connected random geometric graphs, we do not need the notion of random segment processes in its entire generality, but it is convenient to introduce a more specific and restricted subclass. To be more precise, we consider random geometric graphs that are obtained from point processes in a deterministic way. For instance, the edge set of the Delaunay tessellation forms a geometric graph in $\mathbb{R}^d$ whose vertices are given by a point process and whose edges are constructed by applying a deterministic connection rule. This observation also applies to the creek-crossing graphs $(G_n)_{n\geq 2}$ introduced in [16] which form a class of subgraphs of the Delaunay tessellation approximating the minimal spanning forest. Similarly, the Voronoi graph is defined as the edge set of a tessellation which is constructed from a given point process of cell centres by a deterministic rule.

All these random geometric graphs have two important attributes in common. On the one hand, local changes in the underlying point process typically lead to local changes in the structure of the random geometric graph and, on the other hand, the resulting random geometric graphs consist of a single connected component with probability 1. We show that for such random geometric graphs, shortest-path lengths along the edges grow at most linearly in the Euclidean distance of the endpoints of the paths.

Denote by $\mathbb{N}$ the family of all locally finite sets in $\mathbb{R}^d$. In the following, we consider random geometric graphs in $\mathbb{R}^d$ of the type $G = g(X)$, where $X$ denotes a point process in $\mathbb{R}^d$ which is stationary, isotropic, and $m$-dependent, and $g\colon \mathbb{N} \to \mathbb{G}$ is a measurable mapping which is motion-covariant. In other words, we have $g(\alpha(\psi)) = \alpha(g(\psi))$ for all $\psi \in \mathbb{N}$ and all rigid motions $\alpha\colon \mathbb{R}^d \to \mathbb{R}^d$. Note that since $g$ is motion-covariant, the random geometric graph $G$ inherits from $X$ the properties of stationarity and isotropy.

In the arguments used in this paper, we need to make further suitable assumptions on $G$. In the following, it will be convenient to think of an element $\{(x_n, u_n)\}_{n\geq 1}$ of $\mathbb{G}$ as the subset of $\mathbb{R}^d$ formed by the union $\bigcup_{n\geq 1}(u_n + x_n)$. First, we need a certain growth condition allowing us to control the total length of the random geometric graph $G$ inside cubic sampling windows (the total length in big windows should be positive and not too large, with high probability (w.h.p.). Formally, the standard length of a line segment can be measured using the one-dimensional Hausdorff measure $\nu_1$. Furthermore, the random geometric graph $G = g(X)$ should satisfy a suitable stability condition with respect to $X$ so that the configuration of $G$ inside a bounded sampling window $W \subset \mathbb{R}^d$ does not depend on the configuration of $X$ far away from the set $W$. Finally, we require a strong connectivity condition in the sense that any two points on $G \cap W$ can be connected by a path on $G$ which is contained in a suitable neighbourhood of the sampling window $W$. In order to state these additional assumptions on $G$ more precisely, we use the following notion of occurrence w.h.p. Let $(A_a)_{a>1}$ be a family of events in a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is assumed to be complete. We say that the events $A_a$ occur w.h.p. if

$$\liminf_{a\to\infty} \frac{\log(-\log(1 - \mathbb{P}\{A_a\}))}{\log a} > 0.$$

Note that the latter inequality is equivalent to the existence of constants $c_1, c_2 > 0$ such that $1 - \mathbb{P}\{A_a\} \leq c_1 \exp(-a^{c_2})$ for all $a > 1$. Furthermore, we use the following notation. For $x \in \mathbb{R}^d$ and $r \in (0, \infty]$ we denote by $Q_r(x) = [-r/2, r/2]^d + x$ the cube of side length $r$ and centre $x$.

In the following, the existence of a suitable radius of stabilisation for the construction rule $g$ will be crucial. Putting $\mathbb{Z}_{+,\infty} = ((0, \infty) \cap \mathbb{Z}) \cup \{\infty\}$ and denoting by $o$ the origin in $\mathbb{R}^d$, a radius of stabilisation is defined to be a measurable function $b\colon \mathbb{N} \to \mathbb{Z}_{+,\infty}$ such that with probability 1, it holds that

$$g(X) \cap Q_1(o) = g((X \cap Q_{b(X)}(o)) \cup \psi) \cap Q_1(o) \tag{1}$$

for all locally finite $\psi \subset \mathbb{R}^d \setminus Q_{b(X)}(o)$, and that

$$\min\{b(X), n + 1\} = \min\{b(X \cap Q_n(o) \cup \psi), n + 1\} \tag{2}$$

for all $n \in [1, \infty) \cap \mathbb{Z}$ and locally finite $\psi \subset \mathbb{R}^d \setminus Q_n(o)$. While (1) guarantees that the intersection $g(X) \cap Q_1(o)$ depends only on the point process $X$ in the window $Q_{b(X)}(o)$, we use (2) in order to ensure that for $n \geq 1$ the event $\{b(X) \leq n\}$ depends only on the point process $X$ in the window $Q_n(o)$. Now, assume that

(G1) the events $A_a^{(1)} = \{G \cap Q_a(o) \neq \varnothing\} \cap \{\nu_1(G \cap Q_1(o)) \leq a\}$ occur w.h.p. (growth condition),

(G2) the events $A_a^{(2)} = \{b(X) \leq a\}$ occur w.h.p. (stability condition),

(G3) the events $A_a^{(3)} = \{G \cap Q_{a/2}(o)$ is contained in a connected component of $G \cap Q_a(o)\}$ occur w.h.p. (connectivity condition).

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In Section 3, we verify that these conditions are satisfied for the previously discussed examples of random geometric graphs. Note that conditions (G2) and (G3) are modifications of the asymptotic essential connectedness property introduced by Aldous [1]. Furthermore, the Borel–Cantelli lemma shows that condition (G3) implies a.s. connectivity of $G$.

The asymptotic behaviour of shortest-path lengths is a recurring theme in first passage percolation and is also the content of the main result of this paper. This result deals with the sub-exponential decay of the probability that such path lengths along the edges of $G$ increase superlinearly in the Euclidean distance of their endpoints. To formulate it precisely, we put $e_1 = (1, 0, \dots, 0)^\top$ and $q(x) = \operatorname{argmin}_{y \in G}|x - y|$ for any $x \in \mathbb{R}^d$. If this is not unique, we take the lexicographically smallest point on $G$ with this property. Furthermore, for $x, y \in \mathbb{R}^d$ we denote by $\ell(x, y)$ the length of the shortest Euclidean path between $q(x)$ and $q(y)$ on $G$.

**Theorem 1.** *Let $G$ denote a random geometric graph in $\mathbb{R}^d$ of the form $G = g(X)$ satisfying conditions (G1)–(G3). Then there exists $u_0 \geq 1$ with*

$$\liminf_{\substack{ur\to\infty \\ u\geq u_0, r\geq 1}} \frac{\log(-\log \mathbb{P}\{\ell(o, re_1) \geq ur\})}{\log(ur)} > 0. \tag{3}$$

The proof of Theorem 1 is postponed to Section 4. Note that this theorem is an extension of a similar result which was derived by Aldous for a class of planar graphs; see [1] and [3]. Before we move on, we also remark that Theorem 1 contains two interesting special cases. On the one hand, we may fix $r$ and consider the asymptotic behaviour of the tail probabilities $\mathbb{P}\{\ell(o, re_1) \geq ur\}$ as $u \to \infty$. In this case, Theorem 1 yields sub-exponential decay of the tail function of the length of the shortest path between two points at predefined locations of distance $r$, i.e. there exist constants $c_1, c_2 > 0$ such that $\mathbb{P}\{\ell(o, re_1) \geq ur\} \leq c_1 \exp(-u^{c_2})$ for all $u > 1$. On the other hand, we can also fix $u \geq u_0$ and let $r \to \infty$. Then as $r \to \infty$ the shortest-path length between points at distance $r$ grows, at most, linearly in $r$ w.h.p.

We believe that Theorem 1 is useful for future research, since the issue of existence of short paths in random geometric graphs occurs in rather diverse contexts, such as the nontriviality of Bernoulli percolation on the Gabriel and relative neighbourhood graphs [6], [7] or the transience of random walks on random geometric graphs [30]. Furthermore, note that the framework of Theorem 1 can be extended to include also random geometric graphs generated by curved fibres, such as the dead leaves model or the Johnson–Mehl tessellation [18], [24]. However, as proving Theorem 1 only for random geometric graphs consisting of line segments simplifies the exposition, we restrict our attention to this special class of random fibre processes.

### 2.2. A general class of random geometric graphs

It turns out that the strong linear growth property (3) is satisfied not only by the random geometric graphs considered in Theorem 1, but also by the isotropic Poisson line tessellation in $\mathbb{R}^2$, which does not fit into the framework of condition (G2); see Section 3.3. Therefore, in this section we assume that $G$ is an arbitrary stationary, ergodic, and isotropic random geometric graph in $\mathbb{R}^d$ for which (3) holds and which satisfies conditions (G1) and (G3). Theorems 2 and 3, whose proofs are postponed to Sections 5.1 and 5.2, respectively, provide two implications of property (3). For $r > 0$ and $x \in \mathbb{R}^d$, we denote by $B_r(x)$ the $d$-dimensional Euclidean ball with centre $x$ and radius $r$ and, similarly, by $B_r^G(x) = \{y \in \mathbb{R}^d : \ell(x, y) \leq r\}$, we denote the ball of radius $r$ and centre $x$ in the metric induced by the shortest-path lengths $\ell(x, y)$.

Theorem 2 supports the intuition that the notion of distance defined by the shortest-path lengths $\ell(x, y)$ behaves asymptotically as a scalar multiple of the ordinary Euclidean metric.

This result can be regarded as a shape theorem for the first passage percolation model in which the passage time of an edge in $G$ is given by its length. We refer the reader to [19, Theorem 1.7] for the classical statement in the situation of i.i.d. weights on the lattice $\mathbb{Z}^d$.

**Theorem 2.** *There exists a constant (the time constant) $\xi \geq 1$ such that for all $\varepsilon > 0$*

$$\mathbb{P}\{B_{(1-\varepsilon)r}(o) \subset B^G_{\xi r}(o) \subset B_{(1+\varepsilon)r}(o) \text{ for all sufficiently large } r\} = 1.$$

The second implication of property (3) deals with the a.s. boundedness of cells defined by random geometric graphs in $\mathbb{R}^2$, where for a planar random geometric graph $G$ in $\mathbb{R}^2$ we call the connected components of $\mathbb{R}^2 \setminus G$ the *cells* of $G$.

**Theorem 3.** *Let $d = 2$. Then, with probability 1, all cells of $G$ are bounded.*

### 2.3. An application to shortest-path trees in spatial telecommunication networks

Last but not least, for $d = 2$ we provide an application of Theorem 2 to a problem which is considered in structural analysis of fixed-access telecommunication networks. In particular, we show how Theorem 2 can be used to determine the limiting distribution of the length of the longest branch in a typical shortest-path tree if the intensity of access points converges to zero.

We start by recalling some notation and definitions related with this kind of problem and refer the reader to [25] for details. Let $G$ be a stationary, isotropic, and ergodic random geometric graph in $\mathbb{R}^2$ satisfying (3) and conditions (G1) and (G3) of Section 2.1. We write $\gamma = \mathbb{E}\nu_1(G \cap [0, 1]^2) > 0$ for its *length intensity*, where $\mathbb{E}$ is the expectation, and $G^*$ stands for the *Palm version* of $G$ with respect to $\nu_1(\cdot \cap G)$. Recall that $G^*$ is a random geometric graph whose distribution is determined by

$$\mathbb{E}h(G^*) = \frac{1}{\gamma}\mathbb{E}\int_{G\cap[0,1]^2} h(G - x)\nu_1(\mathrm{d}x),$$

where $h\colon \mathbb{G} \to [0, \infty)$ is any $\mathcal{G}$-measurable function. By $X_\lambda$ we denote a Cox process on $G^*$ whose random intensity measure is given by $\lambda\nu_1(\cdot \cap G^*)$, for some linear intensity $\lambda > 0$. Denote by $\Xi_{0,\lambda}$ the zero-cell of the Voronoi tessellation on $X_\lambda \cup \{o\}$ and write $S^* = \Xi_{0,\lambda} \cap G^*$ for the *typical segment system* within $\Xi_{0,\lambda}$. Since shortest paths do not contain cycles, they induce a natural tree structure on the set of points of $G^*$ for which the shortest path to $o$ is unique. This tree is sometimes referred to as the *shortest-path tree*.

For practical applications to telecommunication networks it is desirable to have knowledge about a variety of distributional properties of typical segment systems. Such properties could help to find useful approximate simulation algorithms (deduced from limit theorems) that allow for a rapid creation of such graphs without having to implement concepts of stochastic geometry. In the following, we denote by $Z(\lambda) = \sup_{x\in S^*} \ell(o, x)$ the length of the longest shortest path among all shortest paths emanating from the origin $o$ and ending at an element of $S^*$. Although there is only little hope to obtain an explicit analytical equation for the distribution of $Z(\lambda)$ when considering general values of $\lambda \in (0, \infty)$, we show how Theorem 2 in conjunction with a distributional result due to Calka [8] on the circumradius of typical Poisson–Voronoi cells can be used to obtain an explicit asymptotic equation in the case when $\lambda \to 0$.

**Theorem 4.** *Let $R$ be the radius of the smallest circle centred at the origin and containing the zero-cell of the Voronoi tessellation on $Y \cup \{o\}$, where $Y$ is a homogeneous Poisson point process with intensity $\gamma$. Then $\sqrt{\lambda}Z(\lambda) \xrightarrow{\mathrm{D}} \xi R$ as $\lambda \to 0$, where $\xi = \lim_{n\to\infty} \mathbb{E}\ell(o, ne_1)/n$ is the time constant appearing in Theorem 2 and '$\xrightarrow{\mathrm{D}}$' denotes convergence in distribution.*

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The proof of Theorem 4 will be provided in Section 6. To go further in the analysis of connection trees, it seems promising to consider not only the longest shortest path in a typical serving zone, but, for example, also the joint distribution of the lengths of the main branches in each of the two subtrees rooted at the origin. They may be considered as backbones of the entire connection tree. This problem is considered in [17] and [26] in greater detail from a theoretical and practical point of view, respectively.

Moreover, for practical applications to telecommunication networks, it is important not only to know the behaviour of the longest shortest path as $\lambda \to 0$, but also to have some information about its length for arbitrary values of $\lambda > 0$. Since it seems rather unlikely that there exists an explicit analytical equation for the distribution of the longest shortest-path length $Z(\lambda)$, parametric density functions are fitted to simulated data [26]. However, these approximate densities rely on Monte Carlo simulations, which become increasingly time consuming as one approaches the asymptotic setting. Hence, our result is useful for certain parameter constellations, where standard Monte Carlo simulations are not feasible.

## 3. Examples of connected graphs

In this section, we show that many well-known connected random geometric graphs satisfy the growth, stability, and connectivity conditions (G1)–(G3) introduced in Section 2.1. We consider the Delaunay graph, the family of creek-crossing graphs $(G_n)_{n\geq 2}$ introduced in [16], and the Voronoi graph in arbitrary dimensions. Finally, we show that the (two-dimensional) isotropic Poisson line tessellation has property (3). Note, however, that the Poisson line tessellation does not fit naturally into the framework of point-process-based random geometric graphs described in Section 2.1 and also exhibits long-range dependencies which are incompatible with the stability condition (G2) of Section 2.1.

### 3.1. Delaunay graph Del and the creek-crossing graphs $G_n$, $n \geq 2$

For $\varphi \subset \mathbb{R}^d$ locally finite and $B \subset \mathbb{R}^d$ a Borel set, we denote the number of elements of $\varphi$ in $B$ by $\varphi(B) = \#(\varphi \cap B)$. In the following we assume that $X$ is a stationary, isotropic, and $m$-dependent point process in $\mathbb{R}^d$ satisfying the following additional conditions. Suppose that

(D1) for $a > 1$ the events $\{X \cap Q_a(o) \neq \varnothing\} \cap \{X(Q_1(o)) \leq a\}$ occur w.h.p., and

(D2) the second factorial moment measure of $X$ is absolutely continuous with respect to a $2d$-dimensional Lebesgue measure, and its density $\rho(x, y)$ is bounded from above by some constant $c > 0$.

The homogeneous Poisson point process in $\mathbb{R}^d$ with intensity $\lambda_{\mathrm{p}} > 0$ obviously satisfies the above conditions.

Recall that for any $\varphi \subset \mathbb{R}^d$ locally finite, the Delaunay graph $\mathrm{Del}(\varphi)$ denotes a graph with vertex set $\varphi$, where two vertices $x, y \in \varphi$ are connected by an edge in $\mathrm{Del}(\varphi)$ if there exists a ball $B \subset \mathbb{R}^d$ such that $x, y \in B$ but $\varphi \cap \operatorname{int} B = \varnothing$. Here $\operatorname{int} B$ denotes the topological interior of $B$. Furthermore, for any $\varphi \subset \mathbb{R}^d$ locally finite, $G_n(\varphi)$ denotes a graph with vertex set $\varphi$, where two vertices $x, y \in \varphi$ are connected by an edge in $G_n(\varphi)$ if there do not exist an integer $k \leq n$ and vertices $x_0 = x, x_1, \dots, x_k = y \in \varphi$ such that $|x_i - x_{i+1}| < |x - y|$ for all $i \in \{0, \dots, k-1\}$; see [16]. To begin with, we state an easy result for the maximum length of the edges in $\mathrm{Del}(X)$ that intersect a given bounded set. Since $G_n(X) \subset \mathrm{Del}(X)$, this also yields

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useful information for the creek-crossing graphs $G_n(X)$, $n \ge 2$. For $c_1, c_2 > 0$ we define

$$b_{c_1,c_2}(\varphi) = \inf\{n \ge c_2 \colon \varphi \cap Q_{n,i} \ne \varnothing \text{ for all } i \in \{1, \dots, \lfloor n^{c_1} \rfloor^d\}\}, \qquad \varphi \in \mathbb{N}, \tag{4}$$

where $Q_{n,1}, \dots, Q_{n,\lfloor n^{c_1} \rfloor^d}$ denotes a subdivision of $Q_n(o)$ into $\lfloor n^{c_1} \rfloor^d$ congruent subcubes. Note that $b_{c_1,c_2}$ satisfies (2).

**Lemma 1.** *Let $\beta \in (0, 1)$ be an arbitrary fixed number and write $b_\beta$ instead of $b_{1-\beta/2,c}$. Then there exists $c' > 1$ such that for all $c > c'$ for $a > 1$ the events $\{b_\beta(X) \le a\}$ occur w.h.p. Furthermore, with probability 1 it holds that $|X_1 - X_2| < b_\beta(X)^\beta$ or $[X_1, X_2] \cap Q_1(o) = \varnothing$ for all locally finite $\psi \subset \mathbb{R}^d \setminus Q_{b_\beta(X)}(o)$ and $X_1, X_2 \in (X \cap Q_{b_\beta(X)}(o)) \cup \psi$ such that $[X_1, X_2]$ forms an edge in* $\mathrm{Del}((X \cap Q_{b_\beta(X)}(o)) \cup \psi)$.

*Proof.* For $n > 1$ subdivide $Q_n(o)$ into $k = \lfloor n^{1-\beta/2} \rfloor^d$ congruent cubes $Q_{n,1}, \dots, Q_{n,k}$ of side length $n/\lfloor n^{1-\beta/2} \rfloor$. Next, observe that there exists $c > 1$ such that for all $n \ge c$, any ball which intersects $Q_1(o)$ and whose diameter is at least $n^\beta$ contains at least one of the cubes $Q_{n,1}, \dots, Q_{n,k}$. Furthermore, if $X_1, X_2 \in (X \cap Q_n(o)) \cup \psi$ are such that $[X_1, X_2] \cap Q_1(o) \ne \varnothing$, where $|X_1 - X_2| \ge n^\beta$ and $X_1, X_2$ are connected by an edge in $\mathrm{Del}(X \cap Q_n(o) \cup \psi)$, then there exists a ball $B$ of diameter at least $n^\beta$ satisfying $(X \cap Q_n(o) \cup \psi) \cap \mathrm{int}\, B = \varnothing$. Finally, note that for every $a > 1$ we have $\mathbb{P}\{b_\beta(X) > a\} \le \sum_{i=1}^{k} \mathbb{P}\{X \cap Q_{\lfloor a \rfloor,i} = \varnothing\}$, so condition (D1) implies that for $a > 1$ the events $\{b_\beta(X) \le a\}$ occur w.h.p.

We now verify conditions (G1)–(G3) of Section 2.1 for the graphs $\mathrm{Del}(X)$ and $G_n(X)$, respectively.

**Lemma 2.** *Let $n \ge 2$ be an arbitrary fixed number. Then for $a > 1$ the events $G_n(X) \cap Q_a(o) \ne \varnothing$ and, therefore, also the events $\mathrm{Del}(X) \cap Q_a(o) \ne \varnothing$ occur w.h.p. Moreover, for $a > 1$ the events $\nu_1(\mathrm{Del}(X) \cap Q_1(o)) \le a$ and, therefore, also the events $\nu_1(G_n(X) \cap Q_1(o)) \le a$ occur w.h.p. In other words, for $G = \mathrm{Del}(X)$ and $G = G_n(X)$ the events $A_a^{(1)}$ in condition (G1) occur w.h.p.*

*Proof.* As $X \subset G_n(X) \subset \mathrm{Del}(X)$, the first assertion follows from condition (D1). Due to the subgraph relation $G_n(X) \subset \mathrm{Del}(X)$, it suffices to prove the second claim when $G = \mathrm{Del}(X)$. Observe that by Lemma 1 w.h.p. the length of any edge intersecting $Q_1(o)$ is at most $a^{1/(2d+3)}$. Furthermore, by condition (D1) we have $X(Q_{3a^{1/(2d+3)}}(o)) \le a^{(d+1)/(2d+3)}$ w.h.p., so $\nu_1(\mathrm{Del}(X) \cap Q_1(o)) \le \sqrt{d} a^{(2d+2)/(2d+3)} \le a$ w.h.p.

**Lemma 3.** *Let $G = \mathrm{Del}(X)$. Then there exists $c > 1$ such that the function $b_{3/4,c} \colon \mathbb{N} \to \mathbb{Z}_{+,\infty}$ introduced in (4) satisfies (1) and such that for $a > 1$ the events $\{b_{3/4,c}(X) \le a\}$ occur w.h.p.*

*Proof.* We can use similar arguments as in the proof of Lemma 1. For $n > 1$, subdivide $Q_n(o)$ into $k = \lfloor n^{3/4} \rfloor^d$ congruent subcubes $Q_{n,1}, \dots, Q_{n,k}$ of side length $n/\lfloor n^{3/4} \rfloor$. Next, observe that there exists $c > 1$ such that for all $n \ge c$, any ball of diameter at least $\sqrt{n}$ intersecting $Q_1(o)$ contains at least one of these subcubes. Furthermore, if $\psi \subset \mathbb{R}^d \setminus Q_{b_{3/4,c}(X)}(o)$ and $X_1, X_2 \in (X \cap Q_n(o)) \cup \psi$ are such that $[X_1, X_2] \cap Q_1(o) \ne \varnothing$ and $[X_1, X_2]$ forms an edge in exactly one of the two graphs $\mathrm{Del}(X \cap Q_n(o))$ and $\mathrm{Del}(X \cap Q_n(o) \cup \psi)$, then there exists a ball $B \subset \mathbb{R}^d$ intersecting both $Q_1(o)$ and $\mathbb{R}^d \setminus Q_n(o)$ such that $X \cap B = \varnothing$. Finally, note that for every $a > 1$ we have $\mathbb{P}\{b_{3/4,c}(X) > a\} \le \sum_{i=1}^{k} \mathbb{P}\{X \cap Q_{\lfloor a \rfloor,i} = \varnothing\} \le \lfloor a^{3/4} \rfloor^d \mathbb{P}\{X \cap Q_{\lfloor a \rfloor,1} = \varnothing\}$, so condition (D1) implies that for $a > 1$ the events $\{b_{3/4,c}(X) \le a\}$ occur w.h.p.

**Lemma 4.** *Let $G = G_n(X)$ for some $n \ge 2$. Then there exists $c > 4(n+1)^2$ such that the measurable function $b^{(n)} = b_{3/4,c}$ satisfies (1) and for $a > 1$ the events $\{b^{(n)}(X) \le a\}$ occur w.h.p.*

*Proof.* Indeed, suppose we could find $\psi \subset \mathbb{R}^d \setminus Q_{b^{(n)}(X)}(o)$ locally finite and $X_1, X_2 \in X \cap Q_{\sqrt{b^{(n)}(X)}}(o)$ such that $|X_1 - X_2| \le \sqrt{b^{(n)}(X)}$ and $[X_1, X_2] \cap Q_1(o) \ne \varnothing$, where $[X_1, X_2]$ forms an edge in exactly one of the two graphs $G_n(X \cap Q_{b^{(n)}(X)}(o))$ and $G_n(X \cap Q_{b^{(n)}(X)}(o) \cup \psi)$. However, since $|X_1 - X_2| \le \sqrt{b^{(n)}(X)}$, the existence of an edge between $X_1$ and $X_2$ depends only on the $X \cap Q_{2(n+1)\sqrt{b^{(n)}(X)}}(o) \subset X \cap Q_{b^{(n)}(X)}(o)$. This contradiction implies that $b^{(n)}$ satisfies (1), so an application of Lemma 2 completes the proof.

Again let $G = G_n(X)$ for some $n \ge 2$. Our next goal is to show that for $a > 1$ the events $A_a^{(3)}$ occur w.h.p. To prove this claim, we need a result on generalised descending chains. This notion is introduced in [16] and is closely related to the concept of descending chains discussed in [9]. Let $b > 0$ and $\varphi \subset \mathbb{R}^d$ be locally finite. We say that a finite sequence $x_1, \dots, x_k \in \varphi$ forms a *finite b-bounded generalised descending chain* in $\varphi$ if there exists an ordered set $I = \{i_1, \dots, i_{k'}\} \subset \{1, \dots, k\}$ with the properties $|i_{j+1} - i_j| \le 2$ for all $j \in \{0, \dots, k'-1\}$, $0 < |x_i - x_{i+1}| \le b$ for all $i \in \{1, \dots, k-1\}$, and $|x_{i_j+1} - x_{i_j}| < |x_{i_{j-1}+1} - x_{i_{j-1}}|$ for all $j \in \{2, \dots, k'\}$, where we use the convention $i_0 = 0$.

**Lemma 5.** *Let $A\colon \mathbb{R}^d \times [0,\infty)^2 \times \mathbb{N} \to \{0,1\}$ denote the function with the property that for $b, r > 0$, $\varphi \subset \mathbb{R}^d$ locally finite, and $x \in \varphi$, it holds that $A(x, b, r, \varphi) = 1$ if and only if there exists a b-bounded generalised descending chain in $\varphi$ starting at $x$ and leaving the ball $B_r(x)$. Then for $b > 1$ the events $\{A(\eta, b, 4db^{2d+3}, X) = 0$ for all $\eta \in X \cap Q_1(o)\}$ occur w.h.p.*

*Proof.* Let $p \in (0,1)$ and consider Bernoulli site percolation on the lattice with a set of sites $\mathbb{Z}^d$ and edges given by $\{\{x,y\} \subset \mathbb{Z}^d : |x-y|_\infty \le 1\}$, where $|z|_\infty = \max_{i \in \{1,\dots,d\}} |z_i|$ for $z = (z_1, \dots, z_d) \in \mathbb{R}^d$. By Peierl's argument (see, e.g. [27, Lemma 9.3]) the probability that the open cluster at the origin contains at least $k$ sites can be bounded from above by $(2^{3^d-1}p)^k$. We choose $p$ such that $\gamma = 2^{3^d-1}p < 1$ and define suitable site percolation models. Let $b > 0$ be an arbitrary fixed number. For $k \ge 0$, $\varepsilon > 0$, and $w \in \mathbb{Z}^d$, we say that $w$ is $(k,\varepsilon)$-open if there exists $\eta \in Q_{4b}(4bw) \cap X$ with $X(B_{k,\varepsilon,\eta}) > 0$, where $B_{k,\varepsilon,\eta} = B_{(k+1)\varepsilon}(\eta) \setminus B_{k\varepsilon}(\eta)$. For $\varepsilon = b^{-2d}$, and $k \ge 1$ with $k\varepsilon \le b$, the probability that an arbitrary site $w \in \mathbb{Z}^d$ is $(k,\varepsilon)$-open can be bounded from above in the following way. Let $\nu_d$ be the Lebesgue measure in $\mathbb{R}^d$. Then the probability that $w$ is $(k,\varepsilon)$-open is at most

$$\mathbb{E} \sum_{\eta \in X} \sum_{\eta' \in X \setminus \{\eta\}} \mathbf{1}_{\{Q_{4b}(4bw)\}}(\eta)\, \mathbf{1}_{\{B_{k,\varepsilon,\eta}\}}(\eta') \le c \int_{Q_{4b}(4bw)} \nu_d(B_{k,\varepsilon,u})\, \mathrm{d}u,$$

and the latter expression is bounded from above by

$$c(4b)^d \kappa_d \varepsilon^d ((k+1)^d - k^d) \le 2^{3d} c \kappa_d \varepsilon b^d (k\varepsilon)^{d-1} \le 2^{3d} c \kappa_d b^{-1},$$

where $\mathbf{1}$ is the indicator fraction and $\kappa_d$ denotes the volume of the unit ball in $\mathbb{R}^d$. A similar upper bound can be deduced for $k = 0$. Thus, we see that by choosing $b$ sufficiently large (independent of $w$ and $k$), the probability of a site being open can be made as small as desired. Furthermore, for sufficiently large $b$ these site percolation models are 2-dependent. In particular, by [21, Theorem 0.0], if $b$ is chosen sufficiently large, then the site percolation model of $(k,\varepsilon)$-open sites can be dominated from above by an independent Bernoulli site percolation model at probability $p$ chosen as above. Now, assume the existence of $\eta_1 \in X \cap Q_1(o)$ such that $A(\eta_1, b, 4db^{2d+3}, X) = 1$. Then there exists a $b$-bounded generalised descending chain $\eta_1, \eta_2, \dots, \eta_n$ with $\eta_n \notin B_{4db^{2d+3}}(\eta_1)$. Define a map $f: \{1, \dots, n\} \to \mathbb{Z}^d \times \{0, \dots, \lceil b^{2d+1} \rceil\}$, $i \mapsto (z, j)$, where $z$ is determined by $\eta_i \in Q_{4b}(4bz)$ and $j = \lfloor b^{2d} |\eta_i - \eta_{i+1}| \rfloor$ if $i \in I$ and $j = \lfloor b^{2d} |\eta_{i+1} - \eta_{i+2}| \rfloor$ otherwise. Note that the composition of $f$ and the projection $\pi_2$ to the second argument defines a monotonously decreasing function. Furthermore, the image of the composition of $f$ with the projection $\pi_1$ to the first component is of size at least $b^{2d+2}$ (otherwise we could not reach $\mathbb{R}^d \setminus B_{4db^{2d+3}}(\eta_1)$ from $Q_1(o)$). Thus, we conclude the existence of at least one $j \in \{0, \dots, \lceil b^{2d+1} \rceil\}$ satisfying $\#\pi_1((\pi_2 \circ f)^{-1}(\{j\})) \geq b$, so that

$$\begin{aligned}
&\mathbb{P}\{A(\eta, b, 4db^{2d+3}, X) = 1 \text{ for some } \eta \in X \cap Q_1(o)\} \\
&\quad \leq \sum_{j=0}^{b^{2d+1}} \mathbb{P}\{Q_{db^{2d+2}}(o) \text{ contains a } (j, b^{-2d})\text{-open cluster of size at least } b\} \\
&\quad \leq \sum_{j=0}^{b^{2d+1}} \sum_{z \in Q_{db^{2d+2}}(o) \cap \mathbb{Z}^d} \mathbb{P}\{\text{the } (j, b^{-2d})\text{-open cluster at } z \text{ has size at least } b\}.
\end{aligned}$$

Since the latter expression is at most $(b^{2d+1} + 1)(db^{2d+2})^d \gamma^b$, this completes the proof.

Furthermore, we need the following auxiliary results; see also [1, Lemmas 10–12].

**Lemma 6.** *Let $a > 1$ and $\varphi \subset \mathbb{R}^d$ be locally finite. Furthermore, let $\eta, \eta' \in \varphi$ be such that $2n|\eta - \eta'| \leq a$, where $\eta, \eta'$ are contained in different connected components of $G_n(\varphi) \cap B_a(\eta)$. Then $A(\eta, n|\eta - \eta'|, a/2, \varphi) = 1$, i.e. there exists an $n|\eta - \eta'|$-bounded generalised descending chain starting at $\eta$ and leaving the ball $B_{a/2}(\eta)$.*

*Proof.* We construct the desired chain $x_0, x_0', x_1, x_1', \dots$ recursively, starting with $x_0 = \eta$ and $x_0' = \eta'$. This construction will ensure that for all $k \geq 0$ the sites $x_k, x_k'$ belong to different connected components of $G_n(\varphi) \cap B_a(\eta)$ and we stop the construction as soon as $|y - \eta| \geq a/2$ for $y = x_k$ or $y = x_{k'}$. Suppose that $x_k$ and $x_k'$ have been constructed. By assumption, we know that $\{x_k, x_k'\}$ does not constitute an edge in $G_n(\varphi) \cap B_a(\eta)$. Thus, there exist $z_0 = x_k, z_1, \dots, z_j = x_k' \in \varphi$ with $j \leq n$ and $|z_i - z_{i+1}| < |x_k - x_k'|$ for all $1 \leq i \leq j - 1$. As $2n|\eta - \eta'| \leq a$, we conclude that $z_i \in B_a(\eta) \cap \varphi$ for all $0 \leq i \leq j$. By assumption, there exists at least one index $i_0$ such that $z_{i_0}$ and $z_{i_0+1}$ belong to different connected components of $G_n(\varphi) \cap B_a(\eta)$. Then we define $x_{k+1} = z_{i_0}$ and $x_{k+1}' = z_{i_0+1}$.

For $r > 0$ and $\varphi \subset \mathbb{R}^d$ locally finite, we denote by $G(\varphi, r)$ the graph on the vertex set $\varphi$, where $x_1, x_2 \in \varphi$ are connected by an edge if and only if $|x_1 - x_2| < r$.

**Lemma 7.** *Let $\alpha \in (0, 1)$. Then for $a > 1$ the graphs $G(X \cap Q_a(o), a^\alpha)$ are connected w.h.p.*

*Proof.* Subdivide $Q_a(o)$ into $k = \lceil (d + 1)a^{1-\alpha} \rceil^d$ subcubes $Q_{a,1}, \dots, Q_{a,k}$ so that any points in two neighbouring cubes (i.e. cubes sharing a $(d - 1)$-dimensional face) are at a distance at most $a^\alpha$. Thus, if each of these cubes contains at least one element from $X$, then $G(X \cap Q_a(o), a^\alpha)$ is connected and we obtain

$$\mathbb{P}\{G(X \cap Q_a(o), a^\alpha) \text{ not connected}\} \leq \mathbb{P}\{X(Q_{a,i}) = 0 \text{ for some } i \in \{1, \dots, k\}\},$$

which is at most $k\mathbb{P}\{X(Q_{a,1}) = 0\}$. An application of (D1) now completes the proof.

**Lemma 8.** *Let $G = G_n(X)$ for some $n \geq 2$. Then for $a > 1$ the events $A_a^{(3)}$ occur w.h.p.*

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*Proof.* If $A_a^{(3)}$ does not occur, then we may assume by Lemmas 1 and 7 that there exist $X_1, X_2 \in X \cap Q_{a/2+\sqrt{a}}(o)$ such that $|X_1 - X_2| \le a^{1/(2d+4)}$, where $X_1, X_2$ are contained in different connected components of $G_n(X) \cap B_{a/4}(X_1) \subset G_n(X) \cap Q_a(o)$. In particular, Lemma 6 implies that $A(X_1, na^{1/(2d+4)}, a/8, X) = 1$. An application of Lemma 5 therefore completes the proof.

Due to the relation $G_n(X) \subset \mathrm{Del}(X)$, Lemma 8 is also true for $G = \mathrm{Del}(X)$.

### 3.2. Voronoi tessellation

Let $\varphi \subset \mathbb{R}^d$ be locally finite and define $\mathrm{Vor}(\varphi) \subset \mathbb{R}^d$ to be the geometric graph obtained by considering the edge set of the Voronoi tessellation with centres in $\varphi$. To any $x \in \varphi$ we can associate the cell $\{y \in \mathbb{R}^d : |x - y| \le \inf_{x' \in \varphi} |x' - y|\}$, and we define $\mathrm{Vor}(\varphi)$ as the union of the edges of all such cells. Let $X$ be a stationary, isotropic, and $m$-dependent point process in $\mathbb{R}^d$, and let $G = \mathrm{Vor}(X)$. Then, in contrast to the random geometric graph considered in Section 3.1, the point process $X$ does not describe the vertices of the graph $G = \mathrm{Vor}(X)$, but the locations of its cell centres. In this subsection, we make the following additional assumption on $X$. Suppose that

(V) for $a > 1$ the events $\{X \cap Q_a(o) \ne \varnothing\} \cap \{X(Q_1(o)) \le a\}$ occur w.h.p.

To begin with, we verify one part of condition (G1).

**Lemma 9.** *For $a > 1$ the events $\nu_1(\mathrm{Vor}(X) \cap Q_1(o)) \le a$ occur w.h.p.*

*Proof.* Subdivide $Q_{(4d+1)a}(o)$ into $k = (4d+1)^d$ congruent subcubes $Q_{a,1}, \dots, Q_{a,k}$ of side length $a$ and write $A_a = \bigcap_{i=1}^k \{X(Q_{a,i}) \ge 1\}$. Choosing an odd number of subcubes is convenient as it guarantees that the cube $Q_a(o)$ is a member of this decomposition. The dimension $d$ needs to enter the number of elements in this decomposition, since also the diameter of the unit cube increases in $d$. We conclude from condition (V) that for $a > 1$ the events $A_a$ occur w.h.p. Furthermore, provided that $A_a$ holds, the following is true:

(i) if a Voronoi cell has a nonempty intersection with $Q_1(o)$ then its centre is contained in $Q_{(4d+1)a}(o)$,

(ii) each edge intersecting $Q_1(o)$ is determined by a collection of $d$ adjacent cells.

Indeed, (ii) follows from basic linear algebra and to show (i), we proceed as follows. Let $\eta \in X$ be such that there exists $\eta' \in Q_1(o)$ contained in the Voronoi cell associated with $\eta$. By assumption, $X \cap Q_a(o) \ne \varnothing$, so $|\eta - \eta'| \le \sqrt{d}a$, implying that $\eta \in Q_{(4d+1)a}(o)$. Therefore, (provided that $A_a$ holds) the number of edges in $\mathrm{Vor}_1(X) = \mathrm{Vor}(X) \cap Q_1(o)$ is bounded from above by $X(Q_{(4d+1)a}(o))^d$, so $\nu_1(\mathrm{Vor}_1(X)) \le \sqrt{d} X(Q_{(4d+1)a}(o))^d$. Hence,

$$\begin{aligned}
&\mathbb{P}\{\nu_1(\mathrm{Vor}_1(X)) > a^{d^2+3d}\} \\
&\quad \le \mathbb{P}\{A_a^c\} + \mathbb{P}\{\{\nu_1(\mathrm{Vor}_1(X)) \le \sqrt{d} X(Q_{(4d+1)a}(o))^d\} \cap \{\nu_1(\mathrm{Vor}_1(X)) > a^{d^2+3d}\}\} \\
&\quad \le \mathbb{P}\{A_a^c\} + \mathbb{P}\{X(Q_{(4d+1)a}(o)) > a^{d+2}\},
\end{aligned}$$

so that an application of condition (V) completes the proof.

Next, we prove the sub-exponential decay of $\mathbb{P}\{\mathrm{Vor}(X) \cap Q_a(o) = \varnothing\}$, $\mathbb{P}\{b(X) > a\}$, and $1 - \mathbb{P}\{A_a^{(3)}\}$ as $a \to \infty$. For $a > 1$ subdivide $Q_a(o)$ into $k = (8d+1)^d$ congruent subcubes $Q_{a,1}, \dots, Q_{a,k}$ of side length $a/(8d+1)$ and for $\varphi \in \mathbb{N}$ let $b(\varphi)$ denote the smallest $n \ge 2$

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such that $\varphi \cap Q_{n,i} \neq \varnothing$ for all $i \in \{1, \dots, k\}$. First, note that $b$ satisfies (2). Moreover, as explained in Lemma 9, it is convenient to consider a subdivision into an odd number of cubes in the dimension $d$.

**Lemma 10.** *The function* $b \colon \mathbb{N} \to \mathbb{Z}_{+,\infty}$ *introduced above satisfies (1). Moreover, for* $a > 1$ *the events* $\mathrm{Vor}(X) \cap Q_a(o) \neq \varnothing$, $\{b(X) \leq a\}$ *and* $A_a^{(3)}$ *occur w.h.p.*

*Proof.* We write $A_a = \bigcap_{i=1}^{k} \{X(Q_{a,i}) \geq 1\}$ and conclude from condition (V) that the events $A_a$ and $\{b(X) \leq a\}$ occur w.h.p. If $A_a$ holds, then

(i) the centre of any Voronoi cell intersecting $Q_1(o)$ is contained in $Q_{a/2}(o)$,

(ii) the centre of any Voronoi cell intersecting $Q_{a/2}(o)$ is contained in $Q_{3a/4}(o)$,

(iii) the Voronoi cell associated with any $X_n \in X \cap Q_{3a/4}(o)$ is contained in $Q_a(o)$.

We provide a proof of (iii), noting that (i) and (ii) can be proven by similar arguments. Indeed, let $\eta \in \mathbb{R}^d \setminus Q_a(o)$ be arbitrary and denote by $P$ the intersection point of the line segment $[X_n, \eta]$ and $\partial Q_a(o)$. Let $i \in \{1, \dots, k\}$ be such that $P \in Q_{a,i}$ and choose an arbitrary $X_0 \in X \cap Q_{a,i}$. Then

$$|\eta - X_0| - |\eta - X_n| \leq |\eta - P| + |P - X_0| - |\eta - P| - |P - X_n| \leq a\left(\frac{\sqrt{d}+1}{8d+1} - \frac{1}{8}\right),$$

which is negative for $d \geq 2$, so $\eta$ is not contained in the cell associated with $X_n$. On the one hand, (i)–(iii) imply that $b$ satisfies (1). On the other hand, as $A_a^{(3)}$ and $\mathrm{Vor}(X) \cap Q_a(o) \neq \varnothing$ are implied by the joint occurrence of (ii) and (iii), we see that the events $A_a^{(3)}$ and $\mathrm{Vor}(X) \cap Q_a(o) \neq \varnothing$ also occur w.h.p.

### 3.3. Poisson line tessellation

In this subsection we show that the linear growth property (3) holds for the isotropic two-dimensional Poisson line tessellation. Although conditions (G1) and (G3) could be verified using similar arguments as in Lemma 11 below, we conjecture that condition (G2) (or some variant thereof) does not hold due to the long-range dependence inherent to the Poisson line model. However, as we will see, it is quite simple to check (3) directly. To be more precise, we consider the planar graph formed by the union of lines in an isotropic Poisson line process, which is defined as follows. Let $\{(R_n, U_n)\}_{n\geq 1} \subset \mathbb{R} \times [0, \pi)$ denote an independently marked Poisson point process in $\mathbb{R}$ with intensity $\lambda > 0$, where the marks are uniformly distributed on $[0, \pi)$. Then the system of random lines $\{\ell_n\}_{n\geq 1}$ defined by $\ell_n = \{(x, y) \in \mathbb{R}^2 \colon x \cos U_n + y \sin U_n = R_n\}$, $n \geq 1$ is called an *isotropic Poisson line process.*

**Lemma 11.** *Let $G$ be the edge set of the tessellation induced by an isotropic Poisson line process. Then property (3) holds.*

*Proof.* For $r, u > 1$ denote by $E_{r,u}$ the event that there exist four random lines $\{\ell_{n_i}\}_{1\leq i\leq 4}$ of the Poisson line process such that they form the extensions of the edges of a quadrilateral $\Xi$ satisfying $\{q(o), q(re_1)\} \subset \Xi \subset Q_{12ur}(o)$, where the notation $q(\cdot)$ was introduced in Section 2. Observe that if $E_{r,u}$ occurs, then $\ell(o, re_1) \leq 2\sqrt{2} \cdot 12ur + 4 \cdot 12ur$, where the notation $\ell(\cdot, \cdot)$ was also introduced in Section 2. Thus, it suffices to find sub-exponential bounds for $\mathbb{P}\{E_{r,u}^c\}$. We denote by $E_{r,u}^{(1)}$ the event that there exists a line $\ell_1$ of the Poisson line process whose angle is contained in $[\pi/2 - \pi/24, \pi/2 + \pi/24]$ and that intersects the ball $B_{\sqrt{ur}}((r + ur)e_1)$.

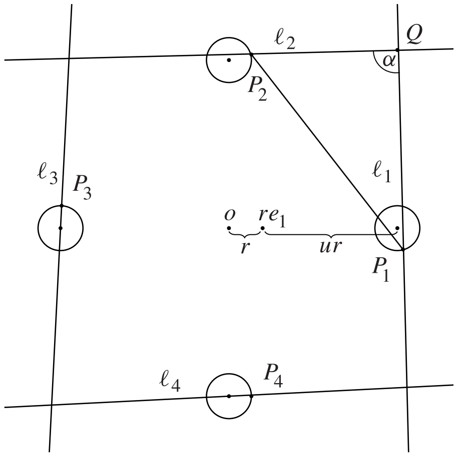


FIGURE 2: Occurrence of the event $\bigcap_{i=1}^4 E_{r,K}^{(i)}$.

Furthermore, we denote by $E_{r,u}^{(i)}$ the event obtained from $E_{r,u}^{(1)}$ by applying a rotation of angle $(i-1)\pi/2$ and centre $o$. See Figure 2 for an illustration of the event $\bigcap_{i=1}^4 E_{r,u}^{(i)}$.

We claim the existence of $u_0 > 1$ such that for all $u \geq u_0$ and for all sufficiently large $ur$ the event $\bigcap_{i=1}^4 E_{r,u}^{(i)}$ implies $E_{r,u}$. The inclusion $\{q(o), q(re_1)\} \subset \Xi$ is clear. To prove $\Xi \subset Q_{12ur}(o)$ choose an arbitrary intersection point $P_1 \in \ell_1 \cap \partial B_{\sqrt{ur}}((r+ur)e_1)$. Similarly, choose $P_2$ as an intersection point of $\ell_2$ with the circle $\partial B_{\sqrt{ur}}((r+ur)e_2)$ and put $\{Q\} = \ell_1 \cap \ell_2$. Using elementary geometry we see that for all sufficiently large $ur$ the angle $\alpha = \angle P_2 Q P_1$ forms the largest angle in the triangle $\triangle P_2 P_1 Q$ and that $|P_1 - P_2| \leq 4ur$; see Figure 2. In particular, the point $Q$ is contained in $Q_{12ur}(o)$. Since the same is true for $\ell_2 \cap \ell_3$, $\ell_3 \cap \ell_4$, and $\ell_4 \cap \ell_1$ we see that $E_{r,u}$ holds. Therefore, it remains to prove sub-exponential bounds for the complement of $E_{r,u}^{(1)}$. However, by the definition of a Poisson line process the number of lines with the properties described in $E_{r,u}^{(1)}$ is Poisson distributed with parameter $(\lambda/12)\sqrt{ur}$. In particular, $\mathbb{P}\{E_{r,u}^{(1)}\} = 1 - \exp(-(\lambda/12)\sqrt{ur})$.

## 4. Proof of Theorem 1

To prove Theorem 1, we proceed in three steps, where we use the general method of global and local paths that has already been successfully applied in the literature; see [1], [2], [4], [13], and [35]. First, in Section 4.1 we discretise $\mathbb{R}^d$ into boxes, allowing us to use results from percolation theory on lattices. Next, in Section 4.2 we explain how to construct a global path, i.e. a path that is used to move from a point on $G$ close to $o$ to a point on $G$ close to $re_1$. Finally, in Section 4.3 we provide a construction for local paths that are used to connect $q(o)$ and $q(re_1)$ to the global path constructed in the previous step.

### 4.1. Discretisation of the Euclidean space into boxes

In order to prove that the probability $\mathbb{P}\{\ell(o, re_1) \geq ur\}$ decreases as stated in (3) if $ur \to \infty$, it suffices to show that w.h.p. we can construct *some* path of length at most $ur$ connecting $q(o)$ and $q(re_1)$ (because then $\ell(o, re_1) \leq ur$). Here, we recall that $q(o)$ and $q(re_1)$ denote the closest points on the graph $G$ to the origin and to $re_1$, respectively. To construct such a path, we decompose the Euclidean space $\mathbb{R}^d$ into congruent $d$-dimensional subcubes with a certain

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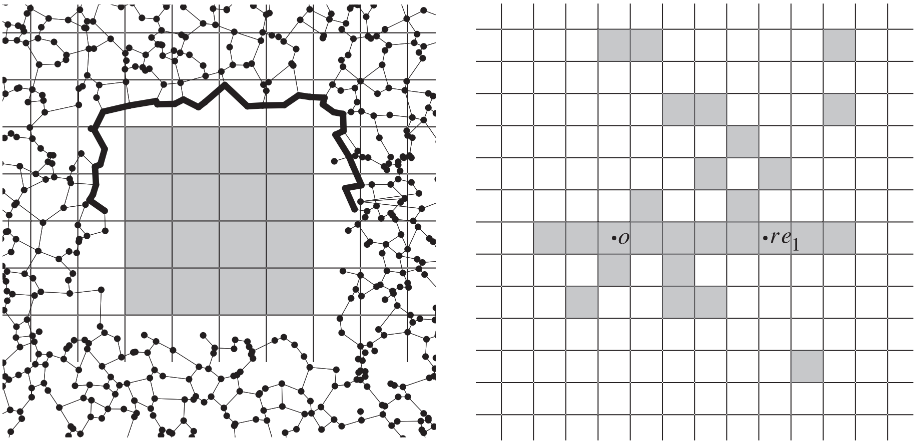


FIGURE 3: A path (thick) avoiding $L$-bad cubes (*left*) and $L$-bad connected component containing $o$ and $re_1$ (*right*).

side length $L > 0$. In particular, our goal is to move only along cubes for which the total length of the random geometric graph is bounded from above and for which it is possible to pass directly to neighbouring cubes along the random geometric graph. We call these cubes $L$-good cubes/sites in the following, other cubes are called $L$-bad cubes/sites, see also Definition 1 below. Loosely speaking, $L$-good cubes describe subregions of $\mathbb{R}^d$, where the graph $G$ has good properties, which are closely related to the growth, stability, and connectivity conditions (G1)–(G3) introduced in Section 2.1.

As in [4], we will show that this allows us to construct efficient paths inside connected components of $L$-good cubes and also that nonpercolation of $L$-bad cubes occurs for all sufficiently large $L > 0$. For illustrations of the situation in two configurations, see Figure 3, where $L$-bad cubes are shaded. For simplicity, only the left figure shows the underlying graph.

More precisely, we introduce the following definition.

**Definition 1.** Let $L > 0$, $z \in \mathbb{Z}^d$ and put $c_L = (5L)^d L$. Furthermore, let $b: \mathbb{N} \to \mathbb{Z}_{+,\infty}$ be the function considered in condition (G2) of Section 2.1. A site $z \in \mathbb{Z}^d$ is said to be *$L$-good* if the following items are satisfied:

(i) $q(x) \in B_{L/4}(x)$ for all $x \in \mathbb{Q}^d \cap Q_L(Lz)$ and $\nu_1(G \cap Q_{5L}(Lz)) \le c_L$,

(ii) $b((X - z') \cap Q_{L/2}(o)) \le L/2$ for all $z' \in \mathbb{Z}^d \cap Q_{5L+1}(Lz)$,

(iii) $G \cap Q_{3L}(Lz)$ is contained in a connected component of $G \cap Q_{5L}(Lz)$.

For a given $L > 0$, we consider the site percolation model $\{Y_z\}_{z\in\mathbb{Z}^d}$ of *$L$-good* sites, where for $z \in \mathbb{Z}^d$ the $\{0, 1\}$-valued random variable $Y_z$ takes the value 1 if and only if $z$ forms an $L$-good site. Observe that (due to (ii)) there exists $m \ge 1$ such that for all sufficiently large $L > 0$ the percolation process defined above is $m$-dependent. Furthermore, the following useful results hold, where we say that a subset of $\mathbb{Z}^d$ is *$*$-connected* if it is a connected set in the graph on $\mathbb{Z}^d$ with edges given by $\{\{x, y\} \subset \mathbb{Z}^d : |x - y|_\infty \le 1\}$.

**Lemma 12.** *Let $\Lambda \subset \mathbb{Z}^d$ be a finite and $*$-connected set of $L$-good sites. Then for all $z, z' \in \Lambda$ and all $\eta \in G \cap Q_{3L}(Lz)$, $\eta' \in G \cap Q_{3L}(Lz')$ the points $\eta, \eta'$ can be connected by a path in $G \cap (L\Lambda \oplus Q_{5L}(o))$ satisfying $\ell(\eta, \eta') \le (\#\Lambda + 1)c_L$.*

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*Proof.* Put $z = z_0$, $z' = z_k$ and let $\gamma = \langle z_0, z_1, \dots, z_k\rangle \subset \Lambda$ be a self-avoiding path of $*$-connected vertices connecting $z$ and $z'$. Geometrically, the path $\gamma$ corresponds to a sequence of vertex-adjacent cubes $Q_L(Lz_0), Q_L(Lz_1), \dots, Q_L(Lz_k)$. By condition (i) of Definition 1, we have $q(Lz_i) \in G \cap Q_L(Lz_i)$ for all $i \in \{0, \dots, k\}$. Furthermore, by condition (iii), we conclude that $q(Lz_i)$ and $q(Lz_{i+1})$ can be connected by a path in $G \cap Q_{5L}(Lz_i)$. By the same reasoning, we can find corresponding paths from $\eta$ to $q(Lz_0)$ and from $\eta'$ to $q(Lz_k)$. Finally, using condition (i), the assertion follows.

**Lemma 13.** *It holds that* $\lim_{L\to\infty} \mathbb{P}\{o \textit{ is } L\textit{-good}\} = 1$.

*Proof.* The growth and stability conditions (G1) and (G2) of Section 2.1 immediately imply conditions (i) and (ii) in Definition 1 of $L$-goodness. To deal with condition (iii), subdivide $Q_{3L}(o)$ into $k = 6^d$ congruent subcubes $Q_{L,1}, \dots, Q_{L,k}$ of side length $L/2$. The connectivity condition (G3) introduced in Section 2.1 implies that if $Q_{L,i}$ and $Q_{L,j}$ are neighbouring subcubes, then $G \cap Q_{L,i}$ and $G \cap Q_{L,j}$ are contained in the same connected component of $G \cap Q_{5L}(o)$ w.h.p. Since the growth condition (G1) implies that $G \cap Q_{L,i} \neq \varnothing$ for all $i \in \{1, \dots, k\}$ w.h.p., the proof is completed.

Using $m$-dependence in conjunction with Lemma 13 and stationarity allows us to apply [21, Theorem 0.0]. This means that the family of $L$-bad sites can be dominated from above by a Bernoulli site percolation model with arbitrarily small marginal probability, provided that $L$ is chosen sufficiently large. In particular, we henceforth fix a value of $L$ such that in the dominating Bernoulli site percolation model the size of the $*$-connected closed component at the origin (also called *cluster size*) admits a finite exponential moment.

### 4.2. Construction of global paths

In this section we elaborate on how to construct an efficient global path, i.e. a path that is used to move from a point on $G$ not too far from $o$ to a point on $G$ not too far from $re_1$. This is done by searching for $L$-good cubes close to $o$ and $re_1$ that are contained in a set of $L$-good sites surrounding the $L$-bad connected components intersecting some cube between $o$ and $re_1$.

For every finite set of sites $\Lambda \subset \mathbb{Z}^d$ we can decompose its complement $\Lambda^c$ into finitely many connected components, i.e. $\Lambda^c = \Lambda^c_1 \cup \dots \cup \Lambda^c_k$. Observe that precisely one of these components, say $\Lambda^c_1$, is infinite. We define the *external boundary* of $\Lambda$ as

$$\partial^{\mathrm{ext}}\Lambda = \{z \in \Lambda^c_1 \colon |z - z'|_1 = 1 \text{ for some } z' \in \Lambda\};$$

see Figure 4. Recall from [29, Lemma 2.1] that the external boundary of any $*$-connected set is again $*$-connected.

If we consider the site percolation model introduced in Section 4.1, then for any $z \in \mathbb{Z}^d$ we denote by $\boldsymbol{C}_z$ the $*$-connected $L$-bad component at $z$. Furthermore, for $n \geq 0$ we define $V_n = \partial^{\mathrm{ext}}(\bigcup_{i=0}^{n-1}(\boldsymbol{C}_{ie_1} \cup \{ie_1\}))$ as well as $z_{1,n} = \sup_{k\leq 0}\{ke_1 \in V_n\}$ and $z_{2,n} = \inf_{k\geq n}\{ke_1 \in V_n\}$. Note that $z_{1,n}$ and $z_{2,n}$ are contained in the same $*$-connected component of $L$-good sites; see Figure 5. First, we provide an upper bound on the length of the shortest path connecting $q(Lz_{1,n})$ and $q(Lz_{2,n})$.

**Lemma 14.** *There exists* $u_1 \geq 1$ *with*

$$\liminf_{\substack{un\to\infty \\ u\geq u_1, n\geq 1}} \frac{-\log \mathbb{P}\{\ell(Lz_{1,n}, Lz_{2,n}) \geq un\}}{un} > 0.$$

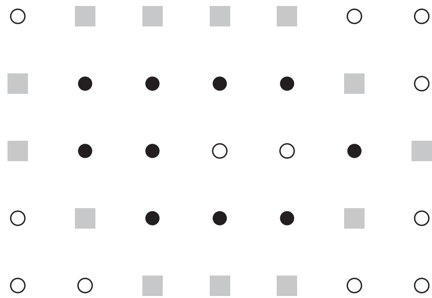

FIGURE 4: Grey squares form external boundary of the set of (filled) black disks.

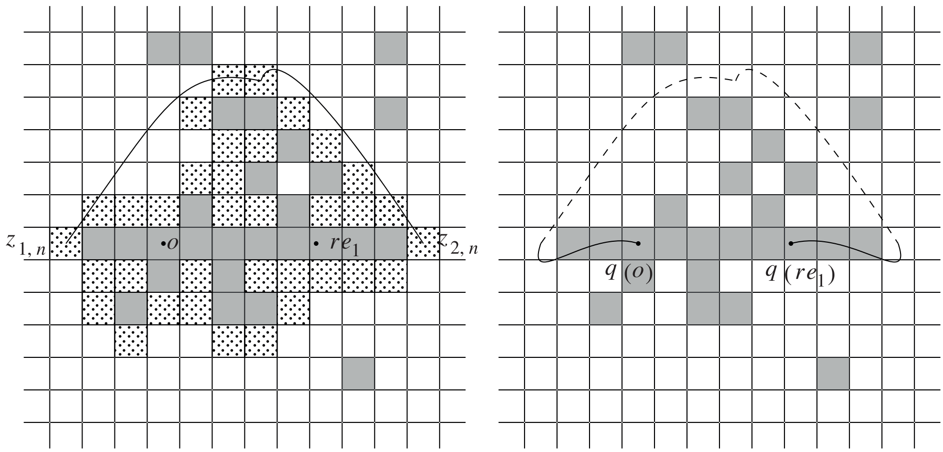


FIGURE 5: Global path and the set $V_n$ (dotted pattern, *left*) and local path (solid) connecting $q(o)$ respectively $q(re_1)$ to the global path (dashed, *right*).

*Proof.* First, Lemma 12 implies that $\ell(Lz_{1,n}, Lz_{2,n}) \le c_L(\#V_n + 1)$ for all $n \ge 1$. Furthermore, there exists $c > 0$ with $\#V_n + 1 \le c(n + \#A)$, where $A = \bigcup_{i=0}^{n-1} \boldsymbol{C}_{ie_1}$ denotes the union of the $*$-connected $L$-bad components at $ie_1$ for $i \in \{0, \dots, n-1\}$. In particular,

$$\mathbb{P}\{\ell(Lz_{1,n}, Lz_{2,n}) \ge un\} \le \mathbb{P}\left\{n + \#A \ge \frac{un}{cc_L}\right\}.$$

Recall that the percolation process of $L$-bad sites can be dominated from above by a sub-critical Bernoulli site percolation process, whose cluster size admits a finite exponential moment; see the remark at the end of Section 4.1. Furthermore, if we consider the union of the closed connected components at the sites $ie_1$ for $i \in \{0, \dots, n-1\}$, then it has been shown in [12, Lemma 2.3] that the size of this union is dominated from above by $\sum_{i=0}^{n-1} D_i$, where $\{D_i\}_{0\le i\le n-1}$ is a family of i.i.d random variables whose marginal distributions coincide with the distribution of the cluster size. By the choice of $L$, we have $\exp(a) = \mathbb{E}\exp(h(D_1 + 1)) < \infty$ for some $h > 0$. In particular, for all $u \ge 2acc_L/h$ the Markov inequality yields

$$\mathbb{P}\left\{n + \#A \ge \frac{un}{cc_L}\right\} \le \mathbb{P}\left\{\sum_{i=0}^{n-1}(D_i + 1) \ge \frac{un}{cc_L}\right\} \le \exp\left(n\left(a - \frac{uh}{cc_L}\right)\right).$$

Since the latter expression is at most $\exp(-nuh/(2cc_L))$, this completes the proof.

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### 4.3. Construction of local paths

Similar to the approach considered in [1], to prove Theorem 1 we need the existence of suitable local paths in addition to the global paths constructed in Section 4.2, see Figure 5.

The goal of this subsection is to provide the following bounds on the lengths of such local paths.

**Lemma 15.** *For $r \in \mathbb{R}$ let $n(r) \in \mathbb{Z}$ be the uniquely determined integer $n$ satisfying $-L/2 < L(n-1) - r \le L/2$. Then there exist $u_2, u_3 \ge 1$ such that*

$$\liminf_{\substack{ur\to\infty\\ u\ge u_2, r\ge 1}} \frac{\log(-\log \mathbb{P}\{\ell(o, Lz_{1,n(r)}) > ur\})}{\log(ur)} > 0, \tag{5}$$

*and*

$$\liminf_{\substack{ur\to\infty\\ u\ge u_3, r\ge 1}} \frac{\log(-\log \mathbb{P}\{\ell(Lz_{2,n(r)}, re_1) > ur\})}{\log(ur)} > 0.$$

Before we begin the proof of Lemma 15, we extend the definition of $V_n$ introduced in Section 4.2 to negative values of $n$ in a natural way. For $n \le -1$ and $\Lambda = \bigcup_{i=n+1}^{0} \boldsymbol{C}_{ie_1} \cup \{ie_1\}$ write $V_n = \partial^{\mathrm{ext}}\Lambda$ and $A_n = \mathbb{Z}^d \setminus \Lambda_1^c$, where we recall that $\Lambda_1^c$ denotes the uniquely determined infinite component of the complement of $\Lambda \subset \mathbb{Z}^d$. The idea is to consider the external boundary $V_{z_{1,n(r)}}$ of the union of $L$-bad components intersecting one of the sites $o, -e_1, \dots, (z_{1,n(r)}+1)e_1$, i.e. $V_{z_{1,n(r)}} = \partial^{\mathrm{ext}}(\bigcup_{i=z_{1,n(r)}+1}^{0}(\boldsymbol{C}_{ie_1} \cup \{ie_1\}))$. We first show in Lemma 16 that any two paths starting from the domain surrounded by this external boundary, intersect $V_{z_{1,n(r)}}$ in a particularly nice way that allows us to efficiently join these two paths together along the external boundary. In a second step, see Lemma 17 below, we derive for suitable values of $r$ an upper bound on $\#V_{-n(r)}$, which holds w.h.p.

**Lemma 16.** *The points $q(o)$ and $q(Lz_{1,n(r)})$ are connected by a path in*

$$G \cap ((LA_{z_{1,n(r)}} \cup LV_{z_{1,n(r)}}) \oplus Q_{5L}(o)).$$

*Proof.* As $G$ is connected and stationary, there exists a path $\gamma$ in $G$ starting at $q(o)$ and leaving $(LA_{z_{1,n(r)}} \cup LV_{z_{1,n(r)}}) \oplus Q_{3L}(o)$. Choose $z_0 \in V_{z_{1,n(r)}}$ such that $Lz_0 \oplus Q_{3L}(o)$ is the first cube of the form $\{Lz \oplus Q_{3L}(o)\}_{z\in V_{z_{1,n(r)}}}$ that is intersected by $\gamma$. Furthermore, we write $\eta \in G\cap(Lz_0 \oplus Q_{3L}(o))$ for the first intersection point of $\gamma$ and $Lz_0 \oplus Q_{3L}(o)$. Similarly, we can construct analogous objects $\gamma'$, $z_0'$, and $\eta'$ which are obtained when starting from $q(Lz_{1,n(r)})$. Since $V_{z_{1,n(r)}}$ is a $*$-connected set of $L$-good sites, we conclude from Lemma 12 that $\eta$ and $\eta'$ can be joined by a path inside $LV_{z_{1,n(r)}} \oplus Q_{5L}(o)$.

**Lemma 17.** *For each $\beta > 0$ there exists $u_4 = u_4(\beta) > 0$ with*

$$\liminf_{\substack{ur\to\infty\\ u\ge u_4, r\ge 1}} \frac{-\log \mathbb{P}\{\#V_{-n(u^\beta r^{2\beta})} > u^{2\beta} r^{2\beta}\}}{u^{2\beta} r^{2\beta}} > 0.$$

*Proof.* We can proceed similarly as in the proof of Lemma 14. In particular, there exist $u_4, c_1 \ge 1$ such that for all $u \ge u_4$ and $r \ge 1$ we have $\mathbb{P}\{\#V_{-n(u^\beta r^{2\beta})} > n(u^\beta r^{2\beta})u^\beta\} \le \exp(-c_1 n(u^\beta r^{2\beta})u^\beta)$. Furthermore, there exist constants $c_2(L), c_3(L) > 0$ such that for all $u \ge u_4$ and $r \ge 1$ we have $c_2(L)u^{2\beta}r^{2\beta} \le n(u^\beta r^{2\beta})u^\beta \le c_3(L)u^{2\beta}r^{2\beta}$, which proves the claim.

Finally, we show that $|z_{1,n(r)}|$ is rather small w.h.p.

**Lemma 18.** *It holds that*

$$\limsup_{a\to\infty} \sup_{r\geq 1} \frac{1}{a} \log \mathbb{P}\{z_{1,n(r)} < -a\} < 0.$$

*Proof.* Let $a > 0$ and note that if $z_{1,n(r)} < -a$, then there exists $k \geq 0$ such that $\boldsymbol{C}_{ke_1}$ has diameter larger than $k + a$. As discussed in Section 4.1, we choose $L$ sufficiently large so that this diameter has an exponentially bounded tail. In particular, there exists a constant $c > 0$ such that the probability $\mathbb{P}\{z_{1,n(r)} < -a\}$ is bounded above by

$$\sum_{k=0}^{\infty} \mathbb{P}\{\mathrm{diam}(\boldsymbol{C}_{ke_1}) > k + a\} \leq \sum_{k=0}^{\infty} \exp(-c(k + a)) \leq (1 - \exp(-c))^{-1} \exp(-ca).$$

*Proof of Lemma 15.* Using the notation $k = n(u^{1/(6d+12)} r^{1/(3d+6)})$, by means of Lemma 16 we obtain

$$\begin{aligned}\mathbb{P}\{\ell(o, Lz_{1,n(r)}) > ur\} &\leq \mathbb{P}\{\nu_1(G \cap ((LA_{z_{1,n(r)}} \cup LV_{z_{1,n(r)}}) \oplus Q_{5L}(o))) > ur\}\\ &\leq \mathbb{P}\{z_{1,n(r)} < -k\} + \mathbb{P}\{\nu_1(G \cap ((LA_{-k} \cup LV_{-k}) \oplus Q_{5L}(o))) > ur\}.\end{aligned}$$

Observe that by the discrete isoperimetric inequality (see, e.g. [12]), $\#V_{-k}$ is at least $d^{-1}(\#A_{-k})^{(d-1)/d}$. In particular, if $t > 0$ is sufficiently large and $\#V_{-k} \leq t$, then $A_{-k} \cup V_{-k} \subset Q_{t^3}(o)$. Hence,

$$\begin{aligned}\mathbb{P}\{\ell(o, Lz_{1,n(r)}) > ur\} &\leq \mathbb{P}\{z_{1,n(r)} < -k\} + \mathbb{P}\{\#V_{-k} > t\}\\ &\quad + \mathbb{P}\{\nu_1(G \cap Q_{Lt^3+5L}(o)) > ur\}\\ &= \mathbb{P}\{z_{1,n(r)} < -k\} + \mathbb{P}\{\#V_{-k} > t\}\\ &\quad + \mathbb{P}\{\nu_1(G \cap Q_{Lt^3+5L}(o)) > t^{3d+6}\},\end{aligned}$$

where $t = (ur)^{1/(3d+6)}$. Applying the sub-exponential bounds of Lemmas 17 and 18, condition (G1), therefore, yields (5). The second assertion of Lemma 15 can be deduced by very similar arguments.

### 4.4. Combining paths

Finally, we patch together the global and local paths constructed in Sections 4.2 and 4.3, respectively. For all $r \geq 1$ and $u \geq u_0 = 3\max\{u_1, u_2, u_3\}$,

$$\begin{aligned}\mathbb{P}\{\ell(o, re_1) > ur\} \leq \mathbb{P}\left\{\ell(o, Lz_{1,n(r)}e_1) > \frac{ur}{3}\right\} + \mathbb{P}\left\{\ell(Lz_{1,n(r)}e_1, Lz_{2,n(r)}e_1) > \frac{ur}{3}\right\}\\ + \mathbb{P}\left\{\ell(Lz_{2,n(r)}e_1, re_1) > \frac{ur}{3}\right\}.\end{aligned}$$

The first and third expressions on the right-hand side of this inequality exhibit sub-exponential decay by Lemma 15 (local path), whereas the second expression exhibits exponential decay by Lemma 14 (global paths). In this way, we can deduce the desired sub-exponential bound stated in Theorem 1.

## 5. Proofs of Theorems 2 and 3

In this section we provide proofs of Theorems 2 and 3, which can be seen as applications of Theorem 1.

### 5.1. Shape theorem

Recall that Theorem 2 can be considered as a shape theorem (in the sense of [19, Theorem 1.7]) for Euclidean first passage percolation on random geometric graphs. Let $G$ be a stationary, ergodic, and isotropic random geometric graph in $\mathbb{R}^d$ for which (3) holds and which satisfies conditions (G1) and (G3). First, we derive the following preliminary results stated in Lemmas 19–21 below.

**Lemma 19.** *Let $\xi \geq 1$ be an arbitrary fixed number. Then*

$$\mathbb{P}\{B_{(1-\varepsilon)r}(o) \subset B^G_{\xi r}(o) \subset B_{(1+\varepsilon)r}(o) \text{ for all sufficiently large } r > 0\} = 1 \tag{6}$$

*for all $\varepsilon > 0$ if and only if for all $\varepsilon > 0$ it holds that*

$$\mathbb{P}\{(\xi - \varepsilon)|x| \leq \ell(o, x) \leq (\xi + \varepsilon)|x| \text{ for all } x \in \mathbb{R}^d \text{ with } |x| \text{ sufficiently large}\} = 1. \tag{7}$$

*Proof.* Let $\varepsilon \in (0, 1)$ be arbitrary and choose a (random) threshold $r_0 > 0$ such that the inclusions in (6) hold for all $r > r_0$ when using $\varepsilon' = \varepsilon/(2\xi)$ instead of $\varepsilon$. Then for $x \in \mathbb{R}^d$ with $(1 + \varepsilon/\xi)|x| > r_0$, we obtain $x \in B_{|x|}(o) \subset B_{(1-\varepsilon')(1+\varepsilon/\xi)|x|}(o) \subset B^G_{\xi(1+\varepsilon/\xi)|x|}(o)$, which means that $\ell(o, x) \leq (\xi + \varepsilon)|x|$. Similarly, choosing $r_0 > 0$ as above, the inequality $\ell(o, x) < (\xi - \varepsilon)|x|$ for $|x| > r_0$ would imply that

$$x \in B^G_{\xi(1-\varepsilon/\xi)|x|}(o) \subset B_{(1+\varepsilon/(2\xi))(1-\varepsilon/\xi)|x|}(o) \subset B_{(1-\varepsilon/(2\xi))|x|}(o),$$

which is a contradiction to $|y| < |x|$ for all $y \in B_{(1-\varepsilon/(2\xi))|x|}(o)$. Thus, assuming that (6) holds for all $\varepsilon > 0$, it follows that (7) holds for all $\varepsilon > 0$. The reverse implication can be shown similarly.

**Lemma 20.** *Let $E^{(1)}$ denote the event that $|y_n - q(y_n)|/|y_n| \to 0$ for all sequences $(y_n)_{n\geq 1}$ with $y_n \in \mathbb{R}^d$ and $|y_n| \to \infty$. Then $\mathbb{P}\{E^{(1)}\} = 1$.*

*Proof.* Let $z \in \mathbb{Z}^d$ and $\varepsilon \in (0, 1/d)$ be arbitrary. Subdivide the cube $Q_{\varepsilon|z|}(z)$ into $k = (4d + 1)^d$ congruent subcubes $Q_{z,1}, \dots, Q_{z,k}$ with side length $\varepsilon|z|/(4d + 1)$. We say that $z$ is *$\varepsilon$-good* if each of the $k$ subcubes has a nonempty intersection with $G$. It is easy to check that if $z$ is $\varepsilon$-good, then $q(y) \in Q_{(2\sqrt{d}+1)\varepsilon|z|/(4d+1)}(z)$ for all $y \in Q_{\varepsilon|z|/(4d+1)}(z)$. In particular,

$$\frac{|y - q(y)|}{|y|} \leq \frac{\sqrt{d}(2\sqrt{d} + 1)\varepsilon(|z|/(4d + 1))}{|z| - \sqrt{d}\varepsilon(|z|/(4d + 1))} \leq \frac{3d\varepsilon}{4d + 1 - \sqrt{d}\varepsilon}.$$

Furthermore, using stationarity, for any $r \geq 1$ we compute

$$\begin{aligned}\mathbb{P}\Bigg\{\bigcup_{z\in\mathbb{Z}^d:\,|z|\geq r} \{z \text{ is } \varepsilon\text{-bad}\}\Bigg\} &\leq \sum_{z\in\mathbb{Z}^d:\,|z|\geq r} \mathbb{P}\{z \text{ is } \varepsilon\text{-bad}\} \\ &\leq (4d + 1)^d \sum_{z\in\mathbb{Z}^d:\,|z|\geq r} \mathbb{P}\{Q_{\varepsilon|z|/(4d+1)}(o) \cap G = \varnothing\}.\end{aligned}$$

Now, using condition (G1) in conjunction with the Borel–Cantelli lemma shows that with probability 1 we have only a finite number of $\varepsilon$-bad lattice points and, therefore,

$$\limsup_{n\to\infty} \frac{|y_n - q(y_n)|}{|y_n|} \leq \frac{3d\varepsilon}{4d + 1 - \sqrt{d}\varepsilon}.$$

Since $\varepsilon > 0$ was arbitrary this proves the claim.

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**Lemma 21.** *Let $\alpha > 0$ be arbitrary. If $X$ is a point process satisfying condition (G1), then for $a > 1$ the events $q(o) \in Q_{a^\alpha}(o)$ occur w.h.p.*

*Proof.* Clearly, $q(o) \in Q_{a^\alpha}(o)$ if $Q_{a^\alpha/\sqrt{d}}(o) \cap G \neq \varnothing$. Hence, the proof is completed by recalling that condition (G1) implies the existence of $c_1, c_2 > 0$ such that

$$\mathbb{P}\{Q_{a^\alpha/\sqrt{d}}(o) \cap G = \varnothing\} \le c_1 \exp\Big(-\Big(\frac{a^\alpha}{\sqrt{d}}\Big)^{c_2}\Big).$$

Next, note the following two results.

**Lemma 22.** *Let $\widetilde{S} \subset \partial B_1(o)$ be a fixed countable subset of the unit sphere in $\mathbb{R}^d$. Then there exists $\xi \ge 1$ such that*

$$\mathbb{P}\Big\{\frac{\lim_{n\to\infty} \ell(o, ns)}{n} = \xi \text{ for all } s \in \widetilde{S}\Big\} = 1.$$

*Proof.* By isotropy and the countability assumption, it suffices to prove the assertion for fixed $s = e_1$. It is easy to check that the family of random variables $\{\ell_{k,n} = \ell(ke_1, ne_1)\}_{k,n\ge 0}$ is sub-additive. Moreover, it is stationary with respect to the mappings $\{\ell_{k,n}\}_{k,n\ge 0} \mapsto \{\ell_{k+m,n+m}\}_{k,n\ge 0}$, $m \ge 1$. Thus, to apply Kingman's sub-additive ergodic theorem [20] it suffices to verify $\mathbb{E}\ell(o, e_1) < \infty$. To prove this claim write $\mathbb{E}\ell(o, e_1) = \int_0^\infty \mathbb{P}\{\ell(o, e_1) > \rho\}\,\mathrm{d}\rho$. Relation (3) implies that the integrand decays sub-exponentially fast in $\rho$, so $\mathbb{E}\ell(o, e_1) < \infty$. Finally, the ergodicity of $G$ implies that $\lim_{n\to\infty} \ell(o, ne_1)/n$ is a.s. constant.

**Lemma 23.** *Let $\delta \in (0, 1)$ be arbitrary. For $a > 1$ and $\eta \in G \cap Q_{a^\delta}(o)$ denote by $E^{(2)}_{\eta,a}$ the event that $\ell(o, \eta) \le a^{2d\delta}$. Then there exists a family of events $(E^{(2)}_a)_{a>1}$ such that the occurrence of $E^{(2)}_a$ implies the occurrence of $E^{(2)}_{\eta,a}$ for all $\eta \in G \cap Q_{a^\delta}(o)$ and such that for $a > 1$ the events $E^{(2)}_a$ occur w.h.p.*

*Proof.* By Lemma 21, we have $q(o) \in Q_{3a^\delta}(o)$ w.h.p., and by condition (G3) for any $\eta \in Q_{3a^\delta}(o)$ we know that $q(o)$ and $\eta$ can be connected by a path in $G \cap Q_{5a^\delta}(o)$ w.h.p. In particular, it suffices to show that for $a > 1$ the events $\nu_1(Q_{5a^\delta}(o) \cap G) \le a^{2d\delta}$ occur w.h.p. To show this, we may subdivide $Q_{5a^\delta}(o)$ into $k = \lceil 5a^\delta \rceil^d$ congruent subcubes of side length at most 1 and apply condition (G1) to obtain that $\nu_1(Q_{5a^\delta}(o) \cap G) \le ka^\delta \le a^{2d\delta}$ holds w.h.p.

Finally, we need one further preliminary lemma. A similar result is also the key ingredient in Kesten's original proof; see [19, Lemma 3.6].

**Lemma 24.** *For $0 < \varepsilon < \frac{1}{4}$ write $E^{(3)}_\varepsilon$ for the following event. There exists a random $K > 0$ such that $\ell(\eta, \eta') \le 4u_0|\eta - \eta'|$ for all $\eta, \eta' \in G$ with $|\eta| \ge K$ and $\varepsilon|\eta|/2 \le |\eta - \eta'| \le 2\varepsilon|\eta|$. Then $\mathbb{P}\{E^{(3)}_\varepsilon\} = 1$.*

*Proof.* First, put $B_{\varepsilon|\eta|,\eta} = \{\eta' \in \mathbb{R}^d : \varepsilon|\eta|/2 \le |\eta' - \eta| \le 2\varepsilon|\eta|\}$. For every $u > 1$ and $z \in \mathbb{Z}^d$ with $|z|$ sufficiently large, we then consider the probability

$$\mathbb{P}\{\text{there exist } \eta \in G \cap Q_1(z) \text{ and } \eta' \in G \cap B_{\varepsilon|\eta|,\eta} \text{ with } \ell(\eta, \eta') \ge u|\eta - \eta'|\}.$$

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Putting $D(z,\varepsilon) = \{z' \in \mathbb{Z}^d : \varepsilon|z|/4 \le |z-z'| \le 4\varepsilon|z|\}$, we note that it is at most

$$\sum_{z'\in D(z,\varepsilon)} \mathbb{P}\Big\{\text{there exist } \eta,\eta' \in G \text{ with } \ell(\eta,\eta') \ge \frac{u|z-z'|}{2},\ \eta \in Q_1(z) \text{ and } \eta' \in Q_1(z')\Big\}$$
$$\le \sum_{z'\in D(z,\varepsilon)} \mathbb{P}\Big\{\ell(z,z') \ge \frac{u|z-z'|}{4}\Big\}$$
$$+ \sum_{z'\in D(z,\varepsilon)} \mathbb{P}\{\text{there exists } \eta \in G \cap Q_{|z|^{1/(4d)}}(z) \text{ with } \ell(z,\eta) \ge \sqrt{|z|}\}$$
$$+ \sum_{z'\in D(z,\varepsilon)} \mathbb{P}\{\text{there exists } \eta' \in G \cap Q_{|z|^{1/(4d)}}(z') \text{ with } \ell(z',\eta') \ge \sqrt{|z|}\}.$$

Choosing $u = 4u_0$, and applying Lemma 23 and Theorem 1 in conjunction with the Borel–Cantelli lemma then completes the proof.

Using these auxiliary results, we may now proceed similarly to [19, Theorem 1.7] to deduce Theorem 2.

*Proof of Theorem 2.* Let $\varepsilon > 0$ be arbitrary. Our goal is to show that

$$-\varepsilon + \xi \le \frac{\ell(o,y)}{|y|} \le \xi + \varepsilon \tag{8}$$

for all $y \in \mathbb{R}^d$ with $|y|$ sufficiently large. We assume that we are given a realisation where the event $E^{(3)}_{m^{-1}}$ in Lemma 24 occurs for all $m \ge 1$ and where, additionally, the event $E^{(1)}$ of Lemma 20 occurs. For the sake of deriving a contradiction, we assume that there exists a sequence $y_n$ with $|y_n| \to \infty$, $y_n|y_n|^{-1} \to z \in \partial B_1(o)$, so (8) is violated for these $y_n \in \mathbb{R}^d$ when $\varepsilon$ is replaced by $8u_0\varepsilon$. Since $|q(y_n) - y_n|/|y_n| \to 0$, we may assume $y_n = q(y_n)$. Now, choose an arbitrary countable dense subset $\widetilde{S} \subset \partial B_1(o)$ and an element $s \in \widetilde{S}$ so $\varepsilon \le |s-z| \le \frac{5}{4}\varepsilon$. Then

$$\Big|\frac{\ell(o,y_n)}{|y_n|} - \xi\Big| \le \Big|\frac{\ell(o,y_n)}{|y_n|} - \frac{\ell(o,\lfloor|y_n|\rfloor s)}{|y_n|}\Big| + \Big|\frac{\ell(o,\lfloor|y_n|\rfloor s)}{|y_n|} - \xi\Big|$$
$$\le \frac{\ell(y_n,\lfloor|y_n|\rfloor s)}{|y_n|} + \Big|\frac{\ell(o,\lfloor|y_n|\rfloor s)}{|y_n|} - \xi\Big|.$$

By Lemma 22, the second expression tends to 0 as $n \to \infty$. Thus, it remains to consider the behaviour of the first expression. For simplicity write $k_n = \lfloor|y_n|\rfloor$ and $x_n = q(k_n s)$, so that $\ell(y_n, k_n s) = \ell(y_n, x_n)$. Now,

$$|y_n - x_n| \le k_n\Big(\Big|\frac{y_n}{k_n} - z\Big| + |z-s| + \Big|\frac{s-x_n}{k_n}\Big|\Big).$$

By Lemma 20, the third summand is less than $\varepsilon/8$ for all sufficiently large $n$. Furthermore, by the definition of $z$, the same holds for the first summand. As the second summand is at most $5\varepsilon/4$, we conclude that $|y_n - x_n| \le 3k_n\varepsilon/2 \le 2|y_n|\varepsilon$. Similarly, as $|z-s| > \varepsilon$, it follows that $|y_n - x_n| \ge |y_n|\varepsilon/2$. An application of Lemma 24 thus yields $\ell(y_n,x_n)/|y_n| \le 4u_0|y_n - x_n|/|y_n| \le 8u_0\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this completes the proof of Theorem 2.

We conclude this section by showing that if (7) is satisfied for $G$ then it is satisfied for the Palm version $G^*$ of $G$ with respect to $\nu_1(\cdot \cap G)$.

**Proposition 1.** *Let $G$ be a stationary and isotropic random geometric graph in $\mathbb{R}^d$. If (7) is satisfied for $G$ then it is also satisfied for $G^*$.*

*Proof.* For enhanced readability, we write $\ell_G$ for the lengths of the shortest paths on $G$ and $\ell_{G^*}$ for the lengths of the shortest paths on $G^*$, respectively. We prove only the statement for the first inequality in (7), the proof of the second one being very similar. First, note that for all $\varepsilon > 0$ the definition of the Palm version yields

$$\mathbb{P}\{(\xi-\varepsilon)|x| \le \ell_{G^*}(o,x) \text{ for all } x \in \mathbb{R}^d \text{ with } |x| \text{ sufficiently large}\}$$
$$= \frac{1}{\gamma}\mathbb{E}\int_{G\cap Q_1(o)} \mathbf{1}_{\{(\xi-\varepsilon)|x-y|\le \ell_G(y,x) \text{ for all } x\in\mathbb{R}^d \text{ with } |x-y| \text{ sufficiently large}\}}\,\mathrm{d}y.$$

In particular, it suffices to prove that for all $\varepsilon > 0$ there exists a (random) threshold $r_0 > 0$ such that for all $y \in Q_1(o) \cap G$ and all $x \in \mathbb{R}^d$ with $|x-y| \ge r_0$ we have $(\xi-\varepsilon)|x-y| \le \ell_G(y,x)$. Therefore, let $\varepsilon > 0$ be arbitrary and choose a (random) threshold $r_0' > 0$ such that the inequalities in (7) hold for $G$ with $\varepsilon/4$ instead of $\varepsilon$ and all $x \in \mathbb{R}^d$ with $|x| \ge r_0'$. Then we choose $r_0 > 0$ sufficiently large such that $r_0 \ge \max\{2\sqrt{d}\xi\varepsilon^{-1}, r_0' + \sqrt{d}\}$ and $\ell_G(y,o) \le (r_0 - \sqrt{d})\varepsilon/4$ hold for all $y \in G \cap Q_1(o)$. In particular, for all $y \in Q_1(o) \cap G$ and all $x \in \mathbb{R}^d$ with $|x-y| \ge r_0$, we compute $\ell_G(y,x) \ge \ell_G(o,x) - \ell_G(o,y) \ge (\xi - \varepsilon/4)|x| - \varepsilon|x|/4 \ge (\xi-\varepsilon)|x-y|$. This completes the proof of $(\xi-\varepsilon)|x-y| \le \ell_G(y,x)$. The second inequality of (7) can be obtained by a similar reasoning.

### 5.2. Boundedness of cells

For the convenience of the reader, we recall the statement of Theorem 3. Let $G$ be a stationary and isotropic random geometric graph in $\mathbb{R}^2$ for which relation (3) holds and which satisfies conditions (G1) and (G3). Then with probability 1, all cells of $G$ are bounded.

*Proof of Theorem 3.* For $r > 0$ we write $S_r = \{z \in \mathbb{Z}^d : |z|_\infty = \lceil r \rceil\}$ for the discrete $d_\infty$-sphere in $\mathbb{Z}^d$ of radius $\lceil r \rceil$ centred at $o$. Moreover, for $z \in S_r$ we denote by $z^{\mathrm{ccw}} \in \mathbb{Z}^2$ the counterclockwise successor of $z$ in $S_r$. We also denote by $C_r$ the event that $q(rz) \in Q_{\sqrt{r}}(rz)$ for all $z \in S_r$ and that $\ell(rz, rz^{\mathrm{ccw}}) \le 2u_0 r$ for all $z \in S_r$. Then, by stationarity, it suffices to prove the a.s. boundedness of the zero-cell of $G$. Moreover, the occurrence of $C_r$ implies the boundedness of the zero-cell, provided that $r > 0$ is sufficiently large. Hence, it suffices to prove that the probability that $C_r$ fails for infinitely many integer values of $r$ is 0. Note that this probability is at most $\sum_{z\in S_r}(\mathbb{P}\{q(rz) \notin Q_{\sqrt{r}}(rz)\} + \mathbb{P}\{\ell(rz, rz^{\mathrm{ccw}}) \ge 2u_0 r\})$. The sub-exponential decay of this sum follows from Lemma 21 and Theorem 1, so the proof is completed by an application of the Borel–Cantelli lemma.

## 6. Proof of Theorem 4

The proof of Theorem 4 is subdivided into several steps. First, we recall from [23, Theorem 6.5] that if $\Xi$, $\Xi_1$, and $\Xi_2, \ldots$ are random closed sets in $\mathbb{R}^2$ with $\Xi_n \xrightarrow{\mathrm{D}} \Xi$. Then $\mathbb{P}\{\Xi_n \cap K = \varnothing\} \to \mathbb{P}\{\Xi \cap K = \varnothing\}$ for all compact $K \subset \mathbb{R}^2$ with $\mathbb{P}\{\Xi \cap K = \varnothing\} = \mathbb{P}\{\Xi \cap \operatorname{int} K = \varnothing\}$. We begin by considering an elementary convergence property.

**Lemma 25.** *Let $\Xi, \Xi_1, \Xi_2, \ldots$ be random compact convex sets in $\mathbb{R}^2$ with $\Xi_n \xrightarrow{\mathrm{D}} \Xi$. Furthermore, suppose that $o \in \operatorname{int}\Xi$ and $\mathbb{P}\{\{\Xi_i = \varnothing\} \cup \{o \in \Xi_i\}\} = 1$ for all $i \ge 1$.*

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*Then* $\mathbb{P}\{\Xi_n \subset \operatorname{int} B\} \to \mathbb{P}\{\Xi \subset B\}$ *for all compact, convex* $B \subset \mathbb{R}^2$ *with* $o \in B$ *and* $\mathbb{P}\{\Xi \subset B\} = \mathbb{P}\{\Xi \subset \operatorname{int} B\}$.

*Proof.* Observe that $\Xi \subset B$ if and only if $\Xi \cap (B \oplus \operatorname{int} B_1(o)) \setminus B = \varnothing$ and similarly $\Xi \subset \operatorname{int} B$ if and only if $\Xi \cap (B \oplus B_1(o)) \setminus \operatorname{int} B = \varnothing$. In particular,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}\{\Xi_n \subset \operatorname{int} B\} &= \lim_{n\to\infty} \mathbb{P}\{\Xi_n \cap (B \oplus B_1(o)) \setminus \operatorname{int} B = \varnothing\} \\ &= \mathbb{P}\{\Xi \cap (B \oplus B_1(o)) \setminus \operatorname{int} B = \varnothing\},\end{aligned}$$

which completes the proof, since the latter expression equals $\mathbb{P}\{\Xi \subset B\}$.

Next, we identify the distributional limit of the scaled Voronoi cells $\sqrt{\lambda}\,\Xi_{0,\lambda}$. In the following, we denote by $\Xi_0$ the zero-cell of the Voronoi tessellation on $Y \cup \{o\}$, where $Y$ is a homogeneous Poisson point process with intensity $\gamma = \mathbb{E}\nu_1(G \cap Q_1(o))$.

**Lemma 26.** *As* $\lambda \to 0$ *the scaled Voronoi cells* $\sqrt{\lambda}\,\Xi_{0,\lambda}$ *converge in distribution to* $\Xi_0$.

*Proof.* First, we claim that $\sqrt{\lambda}X_\lambda \xrightarrow{\mathrm{D}} Y$ as $\lambda \to 0$. Indeed, observe that for any $\lambda \in (0, 1)$ the point process $X_\lambda$ can be obtained from $X_1$ by applying an independent thinning with survival probability $\lambda$. In particular, the claim follows from [10, Exercise 11.3.4] or [22, Theorem 7.3.1]. By the continuous mapping theorem, it suffices to show that the map which assigns to a point process the zero-cell of its associated Voronoi tessellation has discontinuities only in a null set with respect to the distribution of $Y \cup \{o\}$. So let $\varphi \subset \mathbb{R}^2$ be a locally finite set such that the interior of each of the four quadrants contains at least one point. For such locally finite $\varphi$ we define $\mathrm{Vor}_0(\varphi)$ to be the unique cell of the Voronoi tessellation induced by $\varphi \cup \{o\}$ that contains the origin. Now, let $(\varphi_n)_{n\ge1}$ be a sequence of locally finite sets with $\varphi_n \to \varphi$. We make use of the characterisation [10, Theorem A2.6.II], where it is shown that this convergence is equivalent to $\varphi_n(A) \to \varphi(A)$ for all $A \in \mathcal{B}_0(\mathbb{R}^2)$ with the property $\varphi(\partial A) = 0$. Our goal is to deduce $\mathrm{Vor}_0(\varphi_n) \to \mathrm{Vor}_0(\varphi)$. First, choose some fixed $r \ge 4$ such that $B_{r/4}(o)$ contains $\mathrm{Vor}_0(\varphi)$ and such that $\partial B_r(o) \cap \varphi = \varnothing$. Then there exists $k > 0$ with $\varphi_n(B_r(o)) = \varphi(B_r(o)) = k$ for all sufficiently large $n$. Write $\varphi \cap B_r(o) = \{P_1, \dots, P_k\}$. Furthermore, choose $\varepsilon_0 > 0$ such that $B_{2\varepsilon_0}(P_i) \subset B_r(o)$ for all $i \in \{1, \dots, k\}$ and such that $B_{\varepsilon_0}(P_i) \cap B_{\varepsilon_0}(P_j) = \varnothing$ for all distinct $i, j \in \{1, \dots, k\}$. Then again for all sufficiently large $n$, we have $\varphi_n(B_{\varepsilon_0}(P_i)) = 1$ for all $i \in \{1, \dots, k\}$ so that for all such $n$ there exist unique $P_1^{(n)}, \dots, P_k^{(n)} \in \varphi_n$ with $|P_i - P_i^{(n)}| \le \varepsilon_0$ for all $i \in \{1, \dots, k\}$. To prove the convergence $\mathrm{Vor}_0(\varphi_n) \to \mathrm{Vor}_0(\varphi)$ we use criterion (c) of [31, Theorem 12.2.2]. So let $x \in \mathrm{Vor}_0(\varphi)$ and $\varepsilon \in (0, \varepsilon_0)$ be arbitrary. It is easy to see that there exist $\delta > 0$ and $y_0 \in B_\varepsilon(x)$ with $|y_0 - P_j| \ge |y_0| + 2\delta$ for all $j \in \{1, \dots, k\}$. We prove $B_\varepsilon(x) \cap \mathrm{Vor}_0(\varphi_n) \ne \varnothing$ eventually by showing that for all sufficiently large values of $n \ge 1$ we have $\inf_{P\in\varphi_n} |y_0 - P| \ge |y_0|$. To prove this claim, we distinguish two cases. If $P \in \varphi_n \setminus B_r(o)$ then $|y_0 - P| \ge r/2 \ge |y_0|$. On the other hand, suppose we are given $P = P_j^{(n)}$ for some $j \in \{1, \dots, k\}$. Note that $|P_j - P_j^{(n)}| < \delta$ for all $j \in \{1, \dots, k\}$ provided $n$ is sufficiently large. In particular, $|y_0 - P_j^{(n)}| \ge |y_0 - P_j| - |P_j - P_j^{(n)}| \ge |y_0| + \delta$. This completes the proof of the first item of criterion (c) of [31, Theorem 12.2.2]. Next, suppose we are given a sequence $(n_i)_{i\ge1}$ and $x_{n_i} \in \mathrm{Vor}_0(\varphi_{n_i})$ with $x_{n_i} \to x \in \mathbb{R}^2$. We want to prove $x \in \mathrm{Vor}_0(\varphi)$. If this claim was false, we could find $\delta > 0$ and $j \in \{1, \dots, k\}$ with $|x| \ge |x - P_j| + \delta$. But this implies that $|x_{n_i}| \ge |x_{n_i} - P_j^{(n_i)}| + \delta/2$ for all sufficiently large values of $i$, thereby contradicting the assumption $x_{n_i} \in \mathrm{Vor}_0(\varphi_{n_i})$.

Next, we note that for small $\lambda$ and large $K$ the cell $\Xi_{0,\lambda}$ is likely to be contained in $Q_{K/\sqrt{\lambda}}(o)$.

**Lemma 27.** *It holds that* $\lim_{K\to\infty}\lim_{\lambda\to 0}\mathbb{P}\{\Xi_{0,\lambda}\subset Q_{K/\sqrt{\lambda}}(o)\}=1$.

*Proof.* First, observe that $\mathbb{P}\{\Xi_{0,\lambda}\subset Q_{K/\sqrt{\lambda}}(o)\}=\mathbb{P}\{\sqrt{\lambda}\Xi_{0,\lambda}\subset Q_K(o)\}$ and that Lemma 26 implies the convergence $\sqrt{\lambda}\Xi_{0,\lambda}\xrightarrow{D}\Xi_0$ as $\lambda\to 0$. Furthermore, it is easy to see that $\mathbb{P}\{\Xi_0\subset Q_K(o)\}=\mathbb{P}\{\Xi_0\subset \operatorname{int} Q_K(o)\}$, so that Lemma 25 yields $\lim_{\lambda\to 0}\mathbb{P}\{\Xi_{0,\lambda}\subset \operatorname{int} Q_{K/\sqrt{\lambda}}(o)\}=\mathbb{P}\{\Xi_0\subset Q_K(o)\}$. We conclude by observing that $\lim_{K\to\infty}\mathbb{P}\{\Xi_0\subset Q_K(o)\}=1$.

**Lemma 28.** *Let* $\alpha\in(0,1)$ *and* $K>0$ *be arbitrary. Then*

$$\lim_{s\to\infty}\mathbb{P}\{\sup_{x\in Q_{Ks}(o)}|x-q(x)|>s^{\alpha}\}=0.$$

*Proof.* Subdivide $Q_{Ks}(o)$ into $k=\lceil\sqrt{2}Ks^{1-\alpha}\rceil^2$ congruent subsquares $Q_{s,1},\dots,Q_{s,k}$ satisfying $\operatorname{diam}(Q_{s,i})\le s^{\alpha}$ for all $i\in\{1,\dots,k\}$. Then $\mathbb{P}\{\sup_{x\in Q_{Ks}(o)}|x-q(x)|>s^{\alpha}\}\le\sum_{i=1}^{k}\mathbb{P}\{Q_{s,i}\cap G^*=\varnothing\}$ which by condition (G1) tends to 0 as $s\to\infty$.

We write $\mathcal{K}$ for the family of convex compact sets in $\mathbb{R}^2$.

**Lemma 29.** *Let* $r>0$ *and* $A$ *be a convex polygon with no two parallel sides and such that no circle of radius* $r$ *touches three (or more) sides of* $A$*. Then the erosion operation* $h\colon\mathcal{K}\to\mathcal{K}$, $A'\mapsto h(A')=A'\ominus B_r(o)$ *is continuous at* $A$.

*Proof.* Suppose that $A_n\to A$ as $n\to\infty$. To prove the convergence

$$A_n\ominus B_r(o)\to A\ominus B_r(o)$$

we use criterion (c) of [31, Theorem 12.2.2]. Hence, we first suppose that $x\in A\ominus B_r(o)$, i.e. $B_r(x)\subset A$. By assumption $B_r(x)$ is tangent to at most two sides of $A$ and we suppose that it is tangent to *exactly* two sides (the other cases are similar, but easier). Write $\boldsymbol{u}$, $\boldsymbol{v}$ for the two unit vectors pointing from $x$ in the direction of the two tangent points. Furthermore, define $\boldsymbol{w}=\boldsymbol{u}+\boldsymbol{v}$ (observe that $\boldsymbol{w}\ne 0$ due to the nonparallelity assumption). It is easy to check that for all sufficiently small $\delta>0$ the ball $B_r(x-\delta\boldsymbol{w})$ has positive distance, say at least $\varepsilon=\varepsilon(\delta)>0$, from all sides of $A$. Denote by $\{P_1,\dots,P_k\}$ the vertices of the polygon $A$, see Figure 6. Then by condition (b1) of [31, Theorem 12.2.2] for all sufficiently large $j\ge 1$ we have $A_j\cap\operatorname{int}B_{\varepsilon/2}(P_i)\ne\varnothing$ for all $i\in\{1,\dots,k\}$. Since the convex hull of $\{y_1,\dots,y_k\}$ contains $B_r(x-\delta\boldsymbol{w})$ for all choices of points $y_i\in\operatorname{int}B_{\varepsilon/2}(P_i)$, we obtain that $B_r(x-\delta\boldsymbol{w})\subset A_j$. But this is simply a reformulation of $x-\delta\boldsymbol{w}\in A_j\ominus B_r(o)$. This verifies condition (c1) of [31, Theorem 12.2.2].

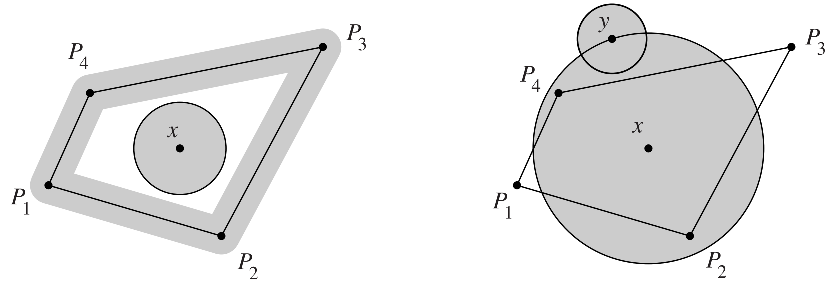
To check condition (c2) of [31, Theorem 12.2.2], we start from a given a sequence $(n_i)_{i\ge 1}$ and points $x_{n_i}\in A_{n_i}\ominus B_r(o)$ with $x_{n_i}\to x$ for some $x\in\mathbb{R}^2$. Our goal is to deduce $B_r(x)\subset A$. Suppose we could find $y\in B_r(x)\setminus A$. Then there exists $\varepsilon>0$ with $B_\varepsilon(y)\cap A=\varnothing$, see Figure 6(b). By criterion (b2) of [31, Theorem 12.2.2], we then also have $B_\varepsilon(y)\cap A_{n_i}=\varnothing$ for all sufficiently large $i\ge 1$. Now choose $y'\in B_\varepsilon(y)$ with $|y'-x|\le r-\varepsilon$. Then we compute

$$|y'-x_{n_i}|\le|y'-x|+|x-x_{n_i}|\le r-\varepsilon+|x-x_{n_i}|.$$

In particular, $y'\in B_r(x_{n_i})$ for all sufficiently large values of $i$ thereby contradicting

$$x_{n_i}\in A_{n_i}\ominus B_r(o).$$

We now have collected all necessary preliminaries to prove Theorem 4.



(a) Polygon $A$ and ball $B_r(x)$ (b) Polygon $A$ and balls $B_r(x)$, $B_\varepsilon(y)$

FIGURE 6: Configurations in the proof of Lemma 29.

*Proof of Theorem 4.* Let $\delta, \varepsilon \in (0, 1)$ be arbitrary. Using Lemmas 27 and 28, for all sufficiently small $\lambda > 0$, we obtain

$$\mathbb{P}\{\sqrt{\lambda} Z(\lambda) \le x\} = \mathbb{P}\bigg\{\max_{P_1 \in \Xi_{0,\lambda} \cap G^*} \ell(o, P_1) \le \frac{x}{\sqrt{\lambda}}\bigg\}$$
$$\le \mathbb{P}\bigg\{\max_{P \in \Xi_{0,\lambda} \ominus B_{\delta/\sqrt{\lambda}}(o)} \ell(o, q(P)) \le \frac{x}{\sqrt{\lambda}}\bigg\} + \varepsilon,$$

which is equal to $\mathbb{P}\{\Xi_{0,\lambda} \ominus B_{\delta/\sqrt{\lambda}}(o) \subset B^{G^*}_{x/\sqrt{\lambda}}(o)\} + \varepsilon$. Furthermore, Theorem 2 yields

$$\mathbb{P}\{\Xi_{0,\lambda} \ominus B_{\delta/\sqrt{\lambda}}(o) \subset B^{G^*}_{x/\sqrt{\lambda}}(o)\} \le \mathbb{P}\{\Xi_{0,\lambda} \ominus B_{\delta/\sqrt{\lambda}}(o) \subset \operatorname{int} B_{(x+\delta)/(\xi\sqrt{\lambda})}(o)\} + \varepsilon$$
$$= \mathbb{P}\{\sqrt{\lambda}\,\Xi_{0,\lambda} \ominus B_\delta(o) \subset \operatorname{int} B_{(x+\delta)/\xi}(o)\} + \varepsilon$$

for all sufficiently small $\lambda > 0$. By Lemma 29, we obtain the operation $\cdot \ominus B_\delta(o)$ is a.s. continuous at $\Xi_0$. In particular, from $\sqrt{\lambda}\,\Xi_{0,\lambda} \xrightarrow{\text{D}} \Xi_0$ we deduce that $\sqrt{\lambda}\,\Xi_{0,\lambda} \ominus B_\delta(o) \xrightarrow{\text{D}} \Xi_0 \ominus B_\delta(o)$. Using $\mathbb{P}\{\Xi_0 \ominus B_\delta(o) \subset B_{(x+\delta)/\xi}(o)\} = \mathbb{P}\{\Xi_0 \ominus B_\delta(o) \subset \operatorname{int} B_{(x+\delta)/\xi}(o)\}$ and Lemma 25, we conclude that

$$\limsup_{\lambda \to 0} \mathbb{P}\{\sqrt{\lambda} Z(\lambda) \le x\} \le \mathbb{P}\{\Xi_0 \ominus B_\delta(o) \subset B_{(x+\delta)/\xi}(o)\} + 2\varepsilon,$$

so that letting $\delta \to 0$ yields

$$\limsup_{\lambda \to 0} \mathbb{P}\{\sqrt{\lambda} Z(\lambda) \le x\} \le \mathbb{P}\{\operatorname{int} \Xi_0 \subset B_{x/\xi}(o)\} + 2\varepsilon \le \mathbb{P}\{\xi R \le x\} + 2\varepsilon.$$

In the next step, we prove a similar inequality in the other direction. Let $\delta, \varepsilon > 0$ be arbitrary. Then for all sufficiently small $\lambda > 0$, we obtain

$$\mathbb{P}\{\sqrt{\lambda} Z(\lambda) > x\} = \mathbb{P}\bigg\{\max_{Q_1 \in \Xi_{0,\lambda} \cap G^*} \ell(o, Q_1) > \frac{x}{\sqrt{\lambda}}\bigg\} \le \mathbb{P}\bigg\{\max_{Q \in \Xi_{0,\lambda}} \ell(o, q(Q)) > \frac{x}{\sqrt{\lambda}}\bigg\},$$

which equals $\mathbb{P}\{\Xi_{0,\lambda} \not\subset B^{G^*}_{x/\sqrt{\lambda}}(o)\}$. Using Theorem 2 we obtain

$$\mathbb{P}\{\Xi_{0,\lambda} \not\subset B^{G^*}_{x/\sqrt{\lambda}}(o)\} \le \mathbb{P}\{\sqrt{\lambda}\,\Xi_{0,\lambda} \not\subset \operatorname{int} B_{(x-\delta)/\xi}(o)\} + \varepsilon$$

for all sufficiently small $\lambda > 0$. Hence, by $\mathbb{P}\{\Xi_0 \subset B_{(x-\delta)/\xi}(o)\} = \mathbb{P}\{\Xi_0 \subset \text{int}\, B_{(x-\delta)/\xi}(o)\}$ and Lemma 25,

$$\limsup_{\lambda\to\infty} \mathbb{P}\{\sqrt{\lambda}Z(\lambda) > x\} \le \mathbb{P}\{\Xi_0 \not\subset B_{(x-\delta)/\xi}(o)\} + \varepsilon = \mathbb{P}\{\xi R > x - \delta\} + \varepsilon,$$

so letting $\delta \to 0$ yields $\limsup_{\lambda\to 0} \mathbb{P}\{\sqrt{\lambda}Z(\lambda) > x\} \le \mathbb{P}\{\xi R \ge x\} + \varepsilon$. As the distribution function of $R$ is continuous, this proves the claim.

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