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MORE ABOUT METRIC SPACES ON WHICH CONTINUOUS FUNCTIONS ARE UNIFORMLY CONTINUOUS

GERALD BEER

An Atsuji space is a metric space X such that each continuous function from X to an arbitrary metric space Y is uniformly continuous. We here present (i) characterizations of metric spaces with Atsuji completions; (ii) Cantor-type theorems for Atsuji spaces; (iii) a fixed point theorem for self-maps of an Atsuji space.

1. Introduction.

Each continuous function from a compact metric space to an arbitrary metric space is uniformly continuous. However, this property does not characterize the compact spaces. The first extensive list of internal descriptions of the spaces so characterized was compiled by Atsuji [1], and in the sequel we shall call such spaces Atsuji spaces. Some of the more important ones are these: (i) each pair of disjoint closed sets in the metric space X is a positive distance apart; (ii) each open cover of X has a Lebesque number; (iii) the set X' of limit points of X is compact, and for each $\varepsilon > 0$ the set of points in X whose distance from X' exceeds ε is uniformly discrete; (iv) each sequence

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$\{x_n\}$ in X with $\lim_{n\to\infty} d(x_n, X - \{x_n\}) = 0$ has a cluster point [8]. We find it convenient to use a variant of (iii) presented in [3]: X' is compact, and each sequence $\{x_n\}$ of distinct isolated points in X with $\lim_{n\to\infty} d(x_{2n-1}, x_{2n}) = 0$ has a cluster point. Again, we call such a sequence a <u>sequence of paired isolated points</u>. Further information on Atsuji spaces can be found in Rainwater [11], Waterhouse [14], Hadjiivanov [7], and Toader [13].

Our purpose here is to present several characterizations of those spaces whose completion is an Atsuji space and to characterize Atsuji spaces in terms of intersection properties of families of closed sets. We also present a fixed point theorem for Atsuji spaces.

2. Metric Spaces with an Atsuji Completion.

Atsuji spaces have numerous sequential characterizations, asserting that certain kinds of sequences in the space have cluster points. The following rule of thumb holds: any such description can be modified to characterize those spaces with an Atsuji completion, provided we replace the phrase "has a cluster point" by "has a Cauchy subsequence". For example, (X,d) has an Atsuji completion (X^*,d) if and only if each sequence $\{x_n\}$ in X with $\lim_{n\to\infty} d(x_n, X - \{x_n\}) = 0$ has a Cauchy subsequence. From (iv) above, the necessity is obvious. For the sufficiency, let $\{x_n^*\}$ be a sequence in X^* with $\lim_{n\to\infty} d(x_n^*, X^* - \{x_n^*\}) = 0$. For each n choose $x_n \in X$ for which $d(x_n, x_n^*) < 1/n$. Since X is dense in X^*, we obtain $d(x_n, X - \{x_n\}) \le d(x_n^*, X^* - \{x_n^*\}) + 1/n$. By assumption, $\{x_n\}$ has a Cauchy subsequence, whence $\{x_n^*\}$ has a cluster point.

Next, we present a somewhat less obvious functional characterization. Each uniformly continuous function from a dense subset of a metric space X to a complete metric space Y can be extended to a continuous function from X to Y. But a standard proof (see, for example, [2], [6], or [12]) shows more, namely:

LEMMA 1. *Let X be a metric space and let Y be a complete metric space. If A is a dense subset of X, and $f:A \to Y$ maps Cauchy sequences to Cauchy sequences, then f has a continuous extension to X.*

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THEOREM 1. *Let* (X,d) *be a metric space, and let* (X^*,d) *be its completion. The following are equivalent:*

(1) *For each metric space* Y*, whenever* $f:X \to Y$ *maps Cauchy sequences to Cauchy sequences, then* f *is uniformly continuous;*

(2) *Whenever* $f:X \to R$ *maps Cauchy sequences to Cauchy sequences, then* f *is uniformly continuous;*

(3) X^* *is an Atsuji space.*

Proof. (1) → (2). Obvious.

(2) → (3). Suppose X^* is not an Atsuji space. Then there exists a continuous real valued function g on X^* that is not uniformly continuous [1]. Since X^* is complete, g maps Cauchy sequences to Cauchy sequences. Thus, $g|X$ has the same property. However, X is dense in X^* ; so, $g|X$ cannot be uniformly continuous, in violation of condition (2).

(3) → (1). Let $f:X \to Y$ map Cauchy sequences to Cauchy sequences. By Lemma 1, f has a continuous extension $g:X^* \to Y^*$ which, by assumption, must be uniformly continuous. Thus, $f = g|X$ is uniformly continuous.

Our next goal is to present a nontrivial characterization of spaces with an Atsuji completion in terms of a relationship between two geometric functionals defined on the nonempty subsets of X . Our first functional $\bar{d}$ is described as follows; for each nonempty subset A of X , let

$$\bar{d}(A) = \sup \{d(a, X - \{a\}): a \in A\} .$$

This functional measures the maximal degree to which points of A are isolated in X . It is quite possible that $\bar{d}(A) = \infty$ (for example, let $A = X = \{n^2: n \in Z^+\}$). Clearly, $\bar{d}$ has these properties: (i) $\bar{d}(A) \leq \bar{d}(B)$ whenever $A \subset B$; (ii) $\bar{d}(cl(A)) = \bar{d}(A)$; (ii) $\bar{d}(A) = 0$ if and only if $A \subset X'$. Actually, one can show that $\bar{d}$ is a regular capacity in the sense of Choquet [4]. For each $\delta > 0$ we let $E[\delta,X] = \{x \in X: d(x, X - \{x\}) < \delta \}$.

LEMMA 2. *Let* (X^*,d) *be the completion of* (X,d)*, and let* δ *be positive. Then* $E[\delta,X^*] \subset cl_*(E[\delta,X])$.

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Proof. Let $x^* \in E[\delta,X^*]$, and choose $y^* \in X^*$ with $0 < d(x^*,y^*) < \delta$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X convergent to x^* and y^*, respectively. Eventually, $0 < d(x_n,y_n) < \delta$ holds, that is, $x_n \in E[\delta,X]$. Thus, $x^* \in cl_*(E[\delta,X])$.

Our second functional is the Hausdorff measure of noncompactness functional [9] (which really measures nontotal boundedness): for each nonempty subset A of X, let $\alpha(A) = \inf\{\varepsilon: A$ has a finite ε-dense subset$\}$. Although $\bar{d}(A)$ depends on the space in which A sits, $\alpha(A)$ does not. Clearly, $\alpha(A) = \infty$ if and only if A is unbounded. The functional acts as follows: (i) $\alpha(A) \le 2\alpha(B)$ whenever $A \subset B$: (ii) $\alpha(cl(A)) = \alpha(A)$; (iii) $\alpha(A) = 0$ if and only if A is totally bounded. We require an analog of Cantor's Theorem, due to Kuratowski [10].

KURATOWSKI'S THEOREM. *The metric space (X,d) is complete if and only if whenever $\{A_n\}$ is a decreasing sequence of nonempty closed subsets of X with $\lim_{n\to\infty} \alpha(A_n) = 0$, then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.*

THEOREM 2. *Let (X,d) be a metric space with completion (X^*,d). The following are equivalent:*

(1) X^* is an Atsuji space:

(2) For each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever A is a nonempty subset of X with $\bar{d}(A) < \delta$, then $\alpha(A) < \varepsilon$.

Proof. (1) → (2). Suppose X^* is an Atsuji space, but for some $\varepsilon > 0$ and all $n \in Z^+$, there exists a subset A_n of X for which $\bar{d}(A_n) < 1/n$, but $\alpha(A_n) > \varepsilon$. Clearly, $(X^*)' = \emptyset$ is ruled out, for X^* cannot be uniformly discrete. Let F be a finite $\frac{\varepsilon}{2}$- dense subset of the compact set $(X^*)'$, and let V^* be the open subset of X^* defined by

$$V^* = \{x^* \in X^*: d(x^*,y) < \tfrac{\varepsilon}{2} \text{ for some } y \text{ in } F\}$$

No set A_n can lie in V^*, or else A_n would have an ε-dense subset. For each $n \in Z^+$ choose $x_n \in A_n - V^*$. Since

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$$\limsup_{n\to\infty} d(x_n, X^* - \{x_n\}) = \limsup_{n\to\infty} d(x_n, X - \{x_n\})$$
$$\leq \lim_{n\to\infty} \overline{d}(A_n) = 0 ,$$

the sequential characterization of Atsuji spaces in [8] yields a cluster point p^* for $\{x_n\}$ in X^*. The condition $\lim_{n\to\infty} d(x_n, X^* - \{x_n\}) = 0$ ensures that $p^* \in (X^*)'$. On the other hand, $\{x_n\}$ is a sequence in the closed set $X^* - V^*$; so $p^* \notin (X^*)'$. This contradiction establishes the necessity.

(2) → (1). We show that $(X^*)'$ is compact, and each sequence of paired isolated points in X^* has a cluster point in X^*. Evidently, $(X^*)' = \bigcap_{n=1}^{\infty} E[\frac{1}{n}, X^*]$. Let $\varepsilon > 0$ be arbitrary, and choose $\delta > 0$ such that $\alpha(A) < \frac{\varepsilon}{2}$ whenever A is a nonempty subset of X and $\overline{d}(A) < \delta$. Note that for each n, $\overline{d}(E[\frac{1}{n}, X]) \leq \frac{1}{n}$; so, if $\frac{1}{n} < \delta$, Lemma 2 gives

$$\alpha(E[\tfrac{1}{n}, X^*]) \leq 2\alpha(cl_*(E[\tfrac{1}{n}, X])) = 2\alpha(E[\tfrac{1}{n}, X]) < \varepsilon .$$

By our property (i) of the α functional, we obtain $\alpha((X^*)') = 0$. Thus, $(X^*)'$ is totally bounded, and since X^* is complete, the set of limit points is compact. Finally, suppose $\{x_n\}$ is a sequence of paired isolated points in X^* without a cluster point. Since X is dense in X^*, each x_n lies in X. For each index n let $A_n = \{x_j : j \geq n\}$. Since $\{x_n\}$ has no cluster point in X^*, each set A_n is closed in both X and X^*. Since the terms of $\{x_n\}$ are distinct, we obtain $\lim_{n\to\infty} \overline{d}(A_n) = 0$; so, $\lim_{n\to\infty} \alpha(A_n) = 0$. By Kuratowski's Theorem $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$. However, each point in the intersection appears infinitely often in $\{x_n\}$, contradicting $x_j \neq x_n$ for $j \neq n$.

3. Cantor-type Theorems for Atsuji Spaces.

Although the Atsuji spaces sit between the complete spaces and the compact ones, most results on Atsuji spaces relate them to the smaller

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class of spaces. An exception is the work of Toader [13], in which he described them sequentially in a manner analogous to complete metric spaces.

DEFINITION. *A sequence* $\{x_n\}$ *in a metric space* X *is called* *pseudo-Cauchy* *if for each* $\varepsilon > 0$ *and* $N \in Z^+$, *there exist distinct indices* j *and* n *exceeding* N *for which* $d(x_j, x_n) < \varepsilon$.

Intuitively, a pseudo-Cauchy sequence is one in which pairs of terms are arbitrarily close frequently, rather than eventually. Complete spaces are those in which each Cauchy sequence with distinct terms has a cluster point. Toader essentially showed that the Atsuji spaces are those in which each pseudo-Cauchy sequence with distinct terms has a cluster point.

Kuratowski's Theorem and Cantor's Theorem characterize complete spaces in terms of certain decreasing sequences of closed sets in the space. Equally well, complete spaces can be described in terms of certain families of closed sets in the space with the finite intersection property (see, for example, Theorem 4.3.10 of [5]). We now present analogous results for Atsuji spaces. For the purposes of the next theorem, we introduce yet another geometric functional: if A is a nonempty subset of X, let $\underline{d}(A) = \inf\{d(a, X - \{a\}) : a \in A\}$. Of course, $\underline{d}(A)$ gives the minimal degree to which any point of A can be isolated. Although $\underline{d}(A) = \underline{d}(cl(A))$, the functional is not monotone.

THEOREM 3. *Let* (X,d) *be a metric space. The following are equivalent:*

(a) *X is an Atsuji space;*

(b) *Whenever Ω is a family of closed subsets of X with the finite intersection property such that for some sequence $\{A_n\}$ of elements of Ω we have $\lim_{n\to\infty} \overline{d}(A_n) = 0$, then $\cap\Omega \neq \emptyset$;*

(c) *Whenever $\{A_n\}$ is a decreasing sequence of closed nonempty subsets of X with $\lim_{n\to\infty} \overline{d}(A_n) = 0$, then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$;*

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(d) *Each pseudo-Cauchy sequence in X with distinct terms has a cluster point;*

(e) *Whenever $\{A_n\}$ is a decreasing sequence of closed nonempty subsets of X with $\lim_{n\to\infty} \underline{d}(A_n) = 0$, then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.*

Proof. (a) → (b). Suppose X is an Atsuji space. Let Σ be a finite subfamily of Ω. For each $n \in Z^+$ let $E_n = (\cap \Sigma) \cap (\bigcap_{j=1}^{n} A_j)$, and for each n select x_n in E_n. Since $E_n \subset A_n$ and $\overline{d}$ is monotone, we obtain $\lim_{n\to\infty} d(x_n, X - \{x_n\}) = 0$. Since X is an Atsuji space, $\{x_n\}$ has a cluster point which evidently lies in $(\bigcap_{n=1}^{\infty} E_n) \cap X' \subset (\cap \Sigma) \cap X'$. Thus $\{A \cap X' : A \in \Omega\}$ has the finite intersection property. Since X' is compact, $\cap \Omega$ is nonempty.

(b) → (c). Trivial.

(c) → (d). Let $\{x_n\}$ be a pseudo-Cauchy sequence in X with distinct terms. By passing to a subsequence we can assume that for each n $d(x_{2n-1}, x_{2n}) < 1/n$. For each n let $A_n = cl(\{x_j : j \geq n\})$. Since the terms of $\{x_n\}$ are distinct, $\lim_{n\to\infty} \overline{d}(A_n) = 0$; so, $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$. Each point in the intersection is a cluster point of $\{x_n\}$.

(d) → (a). Suppose X is not a Atsuji space. We can then find disjoint nonempty closed subsets E and F of X such that $\inf\{d(x,y) : x \in E, y \in F\} = 0$. For each $n \in Z^+$ there exist points $x_n \in E$ and $y_n \in F$ such that $d(x_n, y_n) < 1/n$. Since for each n both $d(x_n, F) > 0$ and $d(y_n, E) > 0$, there is no loss of generality in assuming that the terms of $x_1, y_1, x_2, y_2, \ldots$ are distinct. This pseudo-Cauchy sequence can have no cluster point, or else E and F would not be disjoint.

(a) → (e). Let $\{A_n\}$ be a decreasing sequence of closed sets in X with

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$\lim_{n\to\infty} \underline{d}(A_n) = 0$. For each $n \in Z^+$ choose $x_n \in A_n$ satisfying $d(x_n, X - \{x_n\}) < \underline{d}(A_n) + \frac{1}{n}$. Since $\lim_{n\to\infty} d(x_n, X - \{x_n\}) = 0$, the sequence $\{x_n\}$ has a cluster point, which clearly lies in $\bigcap_{n=1}^{\infty} A_n$.

(e) → (c). Trivial.

EXAMPLE. Condition (b) above fails to characterize Atsuji spaces if we replace $\overline{d}$ by $\underline{d}$ in its statement. For example, let X be the following Atsuji subspace of the line: $Z \cup \{\frac{1}{n}: n \in Z^+\}$. For each $n \in Z^+$ let $A_n = \{k: k \geq n\} \cup \{\frac{1}{n}\}$. Although $\{A_n: n \in Z^+\}$ has the finite intersection property and $\lim_{n\to\infty} \underline{d}(A_n) = 0$, we have $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

4. A Fixed Point Theorem.

Let X be a complete metric space, and let $f:X \to X$ be continuous. Suppose for some $x \in X$ we have $\lim_{n\to\infty} d(f^n(x), f^{n+1}(x)) = 0$. Can we conclude that f has a fixed point? The answer is negative. For example, let x_n be the nth partial sum of the harmonic series, and let $X = \{x_n: n \in Z^+\}$ as a subspace of the line. The assignment $x_n \to x_{n+1}$ fails to have a fixed point. However, if X is an Atsuji space, a stronger result holds.

THEOREM 4. *Let $f:X \to X$ be a continuous function on an Atsuji space. Suppose for some x in X, we have $\liminf_{n\to\infty} d(f^n(x), f^{n+1}(x)) = 0$. Then f has a fixed point.*

Proof: For each $n \in Z^+$ let $x_n = f^n(x)$. By assumption, there exists a subsequence $\{x_{g(n)}\}$ of $\{x_n\}$ such that $\lim_{n\to\infty} d(x_{g(n)}, x_{g(n)+1}) = 0$. If for some n, $x_{g(n)} = x_{g(n)+1}$, then f has a fixed point. Otherwise, $\lim_{n\to\infty} d(x_{g(n)}, X - \{x_{g(n)}\}) = 0$; so, by our

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primary sequential characterization of Atsuji spaces, we conclude $\{x_{g(n)}\}$ has a cluster point p . Let $\{x_{h(n)}\}$ be a subsequence of $\{x_{g(n)}\}$ convergent to p . By the choice of g , $\lim_{n\to\infty} x_{h(n)+1} = p$. Also, the continuity of f yields

$$f(p) = \lim_{n\to\infty} f(x_{h(n)}) = \lim_{n\to\infty} x_{h(n)+1} .$$

Thus, f fixes p .

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Department of Mathematics
California State University
Los Angeles, CA 90032
U.S.A.

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